Survivability in hierarchical telecommunications networks

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Abstract

We consider the problem of designing a two level telecommunications network at minimum cost. The decisions involved are the locations of concentrators, the assignments of user nodes to concentrators and the installation of links connecting concentrators in a reliable backbone network. We define a reliable backbone network as one where there exist at least 2-edge disjoint paths between any pair of concentrator nodes. We formulate this problem as an integer program and propose a facial study of the associated polytope. We describe valid inequalities and give sufficient conditions for these inequalities to be facet defining. We also propose some reduction operations in order to speed up the separation procedures for these inequalities. Using these results, we devise a branch-and-cut algorithm and present some computational results.

Keywords: *Hierarchical network, survivability, integer programming, facets, branch-and-cut*

1 Introduction

Network design problems arising in telecommunications applications have originated new challenges in the field of optimization. Within the scope of this study is a two-layer telecommunications network infrastructure. In such a network, the traffic originating from the user nodes (terminals) is communicated through access networks to concentrators interconnected by a backbone network. The traffic then traverses the backbone network and finally reaches the access network of its destination terminal through a multiplexing at the backbone (the interested reader is referred to the surveys of [12] and [17] and the references therein). Since the backbone network is the primary means of providing communication between end-users, a reliable topological design is essential. A graph is called "2-edge connected" if it contains at least two paths that do not share any edge (edge-disjoint) between any pair of nodes.

[17] provides a classification of the underlying network design problems based on the topology of the access and the backbone networks. The current study focuses on a star access network and a 2-edge

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connected backbone network and introduces an in depth analysis of the *2-edge connected star subgraph problem* (2ECSSP for short) to the telecommunications network design literature. Figure 1 depicts such a network with squares representing concentrators and circles representing terminals.

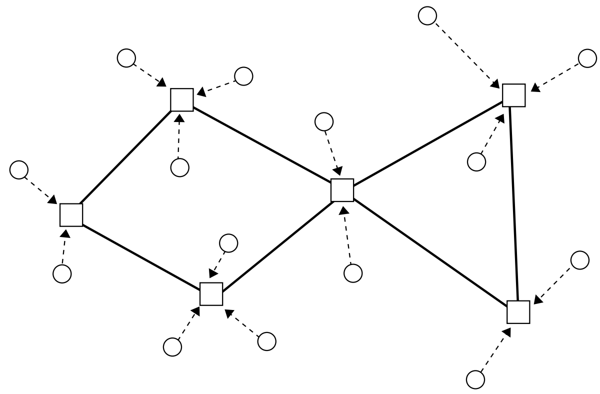

Figure 1: *An example of 2-edge connected/star network*

In particular, we seek the most cost effective way of designing a backbone survivable telecommunications network by partitioning a given set of nodes into terminals and concentrators and establishing edges linking concentrators such that each terminal gets assigned to a single concentrator and the edges connecting the concentrators forming up the backbone network becomes 2-edge connected.

Our problem inherently has two subproblems, namely the survivable network design problem and the concentrator location problem. Individually, both problems have been widely studied in the literature. Note that 2ECSSP is NP-hard since it possesses as a special case the 2-edge connected subgraph problem, which is NP-hard [10].

Gourdin et al. [12] review the studies within the telecommunications context which include many variations of the concentrator location problems. Labbé et al. [20] consider the fully connected/star network design problem. Pirkul and Nagarajan [24] and Lee et al. [21] analyze the tree/star network design problem while Gavish studies [11] the star/tree variant. Chardaire [3] analyzes the star/star network design problem. A path/path network design problem studied by Current and Pirkul [4] is another example to the two level network design approaches existing in the literature. However, except for the fully connected/star topology design, none of these designs guarantee survivability.

Survivable network design problems have been extensively studied, see for example [2, 13, 14, 15, 22, 23, 25] for surveys. In particular, the 2-edge connected subgraph problem, which is of close interest to 2ECSSP is thoroughly investigated in the literature. Several interesting extensions of the 2-edge connected subgraph problem have been proposed in the literature, see [1, 5, 7, 8, 9, 16, 26].

Labbe et al. [18] study the ring/star network design problem which is also of close kinship to the 2ECSSP. They design a network where the backbone network is a ring and the access nodes are connected directly to concentrators forming star networks. It can be seen that this problem is a restriction of the 2ECSSP as a ring is a 2-edge connected network.

The current work contributes to the two level network design literature by expanding it with the 2ECSSP, a special survivable infrastructure that has not yet been analyzed in the literature. The contribution includes a 0-1 model development and a detailed polyhedral analysis of the polytope associated with the 2ECSSP. For the families of facet defining inequalities, the computational complexity status of the separation problems are established and exact and/or heuristic separation algorithms are designed. To make the sizes of the separation problems more manageable, reduction operations are proposed. Finally, a branch and cut algorithm which assembles all this theoretical development is designed for the 2ESSP.

2 Notation and mathematical model

We now proceed with a formal description of the 2ECSSP to be followed with a 0-1 model. We assume that $V = \{0, 1, \ldots, n\}$ is a given set of terminals. Node 0 is a special concentrator corresponding to the root node in the two level network infrastructure. Let $E = \{\{i, j\} : i \in V, j \in V \setminus \{i\}\}$ represent the set of potential backbone links. Associated with installing a backbone link $e \in E$ is a nonnegative fixed setup cost c_e. Similarly, there is a nonnegative assignment cost of d_{ij} units associated with assigning terminal $i \in V$ to concentrator $j \in V$. In particular, d_{ii} corresponds to the cost of installing a concentrator at node $i \in V$.

Given V, 2ECSSP seeks for a partition of V into C and T such that $0 \in C$, a set of backbone links, say, $E' \subseteq E$ among nodes in C, an assignment of each node in T to one in C such that the graph $G = (C, E')$ is two edge connected and the total cost of installing backbone links and concentrators and assigning terminals to concentrators is minimum.

Before we proceed with the model development, we provide some preliminary notation. We represent an edge with endpoints i and j with ij or $\{i, j\}$. Let V_1 and V_2 be two subsets of V such that $V_1 \cap V_2 = \emptyset$. $[V_1, V_2]$ is the set of edges with one endpoint in V_1 and the other endpoint in V_2. For $S \subset V$, let $\delta(S) = [S, V \setminus S]$ and $E(S)$ be the set of edges with both endpoints in S. We use $G(S)$ to denote the subgraph induced by S, *i.e.*, $G(S) = (S, E(S))$. For simplicity, we use $\delta(i)$ instead of $\delta(\{i\})$.

We define x_e to be 1 if edge $e \in E$ is used in the backbone network and 0 otherwise and y_{ij} to be 1 if node $i \in V$ is assigned to node $j \in V$ and 0 otherwise. If a concentrator is installed at node $i \in V$ then node i is assigned to itself, *i.e.*, $y_{ii} = 1$.

Using these two sets of binary variables, we can model the *2ECSSP* as follows.

$$\min \sum_{e \in E} c_e x_e + \sum_{i \in V} \sum_{j \in V} d_{ij} y_{ij} \tag{1}$$

$$\text{s.t.} \sum_{j \in V} y_{ij} = 1 \qquad \forall i \in V \tag{2}$$

$$y_{ij} + x_{ij} \leq y_{jj} \qquad \forall i, j \in V : i \neq j \tag{3}$$

$$y_{00} = 1 \tag{4}$$

$$x(\delta(S)) \geq 2 \sum_{j \in S} y_{ij} \qquad \forall S \subseteq V \setminus \{0\}, i \in S \tag{5}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E \tag{6}$$

$$y_{ij} \in \{0, 1\} \qquad \forall i, j \in V. \tag{7}$$

Constraints (2), (3), and (7) ensure that either a concentrator is installed at a given node or this node is assigned to exactly one other node where a concentrator is installed. If an edge becomes a backbone edge, then concentrators are installed at both endpoints of this edge due to constraints (3). Constraint (4) fixes the value of y_{00} to one and hence a concentrator is installed at the root node 0. Constraints (5) ensure 2-edge connectivity in the backbone network. Consider a node subset $S \subseteq V \setminus \{0\}$ and a node $i \in S$. If i is assigned to some node in set S, *i.e.*, if $\sum_{j \in S} y_{ij} = 1$, then there is at least one concentrator in S implying that at least two edges from $\delta(S)$ have to be included in the backbone network to make it 2-edge connected. Finally, the objective function is the sum of the cost of installing the backbone edges and the concentrators and the cost of assigning the remaining nodes to concentrators.

Remark that the *2ECSSP* is a relaxation of the ring/star network design problem obtained by dropping the requirement that each concentrator is adjacent to exactly two backbone edges. We obtained the above formulation by removing the degree constraints from the formulation of the ring/star network design problem given in Labbé et al. [18].

To devise a polyhedral analysis for the convex hull of solutions to the *2ECSSP*, we first project out some variables to make the analysis easier. For $i \in V \setminus \{0\}$, we can eliminate variable y_{ii} by substituting $y_{ii} = 1 - \sum_{j \in V \setminus \{i\}} y_{ij}$. Additionally, the variables related to the assignment of the root node can also be dropped since their values are known ($y_{00} = 1$). Let $A = \{(i,j) : i \in V \setminus \{0\}, j \in V \setminus \{i\}\}$ and define $d'_{ij} = d_{ij} - d_{ii}$ for each $(i,j) \in A$. Now we obtain the following formulation:

$$\sum_{i\in V} d_{ii} + \min \sum_{e\in E} c_e x_e + \sum_{(i,j)\in A} d'_{ij} y_{ij}$$

$$\text{s.t. } x_{ij} + y_{ij} + \sum_{k\in V\setminus\{j\}} y_{jk} \le 1 \qquad \forall (i,j) \in A : j \neq 0 \tag{8}$$

$$x_{0i} + \sum_{k\in V\setminus\{i\}} y_{ik} \le 1 \qquad \forall i \in V \setminus \{0\} \tag{9}$$

$$x(\delta(S)) + 2 \sum_{j\in V\setminus S} y_{ij} \ge 2 \qquad \forall S \subseteq V \setminus \{0\}, i \in S \tag{10}$$

$$x_e \in \{0,1\} \qquad \forall e \in E \tag{11}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in A. \tag{12}$$

Let $X = \{(x,y) \in R^{|E|+|A|} : (x,y)$ satisfies (8)-(12)$\}$ and $\mathcal{P} = conv(X)$. In the sequel, we assume that $|V| \ge 5$.

3 Polyhedral analysis

We first investigate the dimension of $\mathcal{P}$ and study its trivial facet defining inequalities. We show that $\mathcal{P}$ is full dimensional and that inequalities $x_e \ge 0$ for $e \in E$ and $y_{ij} \ge 0$ for $(i,j) \in A$ are facet defining for $\mathcal{P}$. Inequalities $x_{ij} \le 1$ and $y_{ij} \le 1$ are not facet defining as they are implied by constraints (8) and (9).

Next, we give necessary and sufficient conditions for the constraints of the model to be facet defining for $\mathcal{P}$.

Theorem 1 *i. Let $(i,j) \in A$ with $j \neq 0$. The inequality (8) is facet defining for $\mathcal{P}$.*
ii. Let $i \in V \setminus \{0\}$. The inequality (9) is facet defining for $\mathcal{P}$.
iii. Let $S \subseteq V \setminus \{0\}$ such that $S \neq \emptyset$ and $i \in S$. Inequality (10) defines a facet of $\mathcal{P}$ if and only if $|S| \neq 2$ and $|V \setminus S| \neq 2$.

In the sequel, we present two other families of facet defining inequalities.

An important class of valid inequalities for the 2-edge connected subgraph problem is the class of *F-partition inequalities* (see [22]). We deeply modify these inequalities to be valid for the *2ECSSP* polytope $\mathcal{P}$, show that they are facet defining under some conditions, and investigate the complexity of the associated separation problem.

Theorem 2 *Let $V_0, \dots, V_p$ be a partition of V such that $V_l \neq \emptyset$, for $l = 0, \dots, p$ and $0 \in V_0$. Let $i_l \in V_l$ be fixed nodes for $l = 1, \dots, p$ and $F \subseteq \delta(V_0)$ such that $|F| = 2k+1$ for some $k \ge 0$ and integer. Let $\delta(V_0, \dots, V_p) = \{e \in E :$ endpoints of e are not in the same subset $\}$. The F-partition inequality*

$$x(\delta(V_0, \dots, V_p) \setminus F) + \sum_{l=1}^{p} \sum_{j\in V\setminus V_l} y_{i_l j} \ge p - k \tag{13}$$

is valid for $\mathcal{P}$.

Next we give sufficient conditions for the *F-partition inequalities* to be facet defining for $\mathcal{P}$.

Theorem 3 *Let $V_0, \dots, V_p$ be a partition of V such that $V_l \neq \emptyset$ for $l = 0, \dots, p$, $0 \in V_0$, $G(V_l)$ is 3-edge connected for $l = 0, \dots, p$, $i_l \in V_l$ for $l = 1, \dots, p$ be fixed nodes, $F \subseteq \delta(V_0)$ such that $|F| = 2k+1$ for some $k \ge 1$ and integer, $|F \cap \delta(V_l)| \le 1$ and $F \cap \delta(j) = \emptyset$ for $j \in V_l \setminus \{i_l\}$ and $l = 1, \dots, p$, and $|F \cap \delta(j)| \le 1$ for $j \in V_0 \setminus \{0\}$. Then the F-partition inequality (13) is facet defining for $\mathcal{P}$.*

We also prove that the separation problem associated with the *F-partition inequalities* is NP-hard by a reduction from the uncapacitated concentrator location problem.

Next, we introduce a new family of facet defining inequalities called the $s$tar-path inequalities. These inequalities generalize constraints (8).

Theorem 4 *Let $m \geq 1$ be an integer, $i_0, i_1, \ldots, i_m$ be distinct nodes in $V \setminus \{0\}$, and $P_m = \cup_{l=0}^{m-1}\{i_l, i_{l+1}\}$. The star-path inequality*

$$x(P_m) + \sum_{l=1}^{m} \sum_{j \in V \setminus \{i_l\}} y_{i_l j} + \sum_{l=1}^{m} y_{i_0 i_l} \leq m \tag{14}$$

is valid for $\mathcal{P}$.

We know that the constraints (8) are facet defining for $\mathcal{P}$ and these are special cases of $s$tar-path inequalities. In the following theorem, we give sufficient conditions for the $s$tar-path inequalities to be facet defining.

Theorem 5 *Let $m \geq 1$ be an integer, $i_0, i_1, \ldots, i_m$ be distinct nodes in $V \setminus \{0\}$, and $P = \cup_{l=0}^{m-1}\{i_l, i_{l+1}\}$. If $|V \setminus \{i_0, \ldots, i_m\}| \geq 3$, then the inequality (14) is facet defining for $\mathcal{P}$.*

We prove that the separation problem associated with the *star-path inequalities* (14) is NP-hard by a reduction from the Hamiltonian path problem.

4 Reduction operations

In this section we introduce some reduction operations to reduce the sizes of the separation problems. These operations use ideas developed by Fonlupt and Mahjoub [6] for the 2-edge connected subgraph polytope.

For $e \in E$, $G \setminus e$ denotes the graph obtained by deleting e, and for $v \in V$, $G \setminus v$ denotes the graph obtained by removing v, the edges and arcs incident to it. Given $e = uv \in E$, contracting e means deleting e, identifying u and v, deleting the resulting loops, keeping the new parallel edges and arcs. If $F \subseteq E$, then G/F denotes the graph obtained by contracting F, that is by contracting all edges in F.

Let $(\overline{x}, \overline{y})$ be an optimal solution to the linear programming relaxation. Consider the following reduction operations with respect to $(\overline{x}, \overline{y})$:

θ_1:Delete an edge e with $\overline{x}_e = 0$.

θ_2:Delete an arc (i,j) with $\overline{y}_{ij} = 0$.

θ_3:Delete a node i as well as all the edges and arcs incident to i, if $\exists\ j$ such that $\overline{y}_{ij} = 1$.

θ_4:Contract a node set W inducing a two edge connected graph and $\overline{x}_e = 1\ \forall\ e \in E(W)$.

θ_5:Contract an edge if at least one of the endpoints of e is incident to exactly two edges, and these two edges have value 1 w.r.t. $\overline{x}$.

Note that the edges with fractional values are preserved by all the reduction operations.

Let $G = (V, E \cup A)$ be the initial graph and $G' = (V', E' \cup A')$ and $(\overline{x}', \overline{y}')$ be the reduced graph and the reduced solution obtained after applying operations $\theta_1, \ldots, \theta_5$.

Theorem 6 *There exists a cut (resp. F-partition, star-path) inequality violated by $(\overline{x}, \overline{y})$ in G if and only if there exists a cut (res. F-partition, star-path) inequality violated by $(\overline{x}', \overline{y}')$ in G'.*

5 Branch-and-cut algorithm and computational results

Using the results presented above we devise a branch-and-cut algorithm. The algorithm has been implemented in C++ using the ABACUS framework and CPLEX LP solver. The algorithm starts with a linear relaxation including trivial inequalities, constraints (8) and (9), and the cut inequalities for single node cutsets. The separation of the inequalities is performed in the following order: clique inequalities, cut inequalities, star-path inequalities, and F-partition inequalities. For cut inequalities, we use an exact separation algorithm. For the other classes of inequalities, we propose efficient heuristic separation algorithms. Before running the separation algorithms, we apply our reduction operations in order to speed up the computation.

We test our algorithm on instances from the TSPLIB. For each instance, we set l_{ij} as the euclidean distance between tthe nodes i and j and we compute several cost coefficient vectors in order to have different ratios, denoted by α, between the costs of installing backbone links and the assignment costs. In fact, we use $\alpha \in \{3, 5, 7, 9\}$ with $c_{ij} = \lceil \alpha l_{ij} \rceil$ and $d_{ij} = \lceil (10 - \alpha) l_{ij} \rceil$. The following table summarizes our first preliminary computational results. The two first columns of the tables gives the instances TSPlib names and α. The other entries of the table are as follows.

$\lvert V \rvert$	:	Number of nodes,
Opt	:	Optimal value,
Gap	:	Relative error between the optimal value and the upper bound achieved before branching,
Clq	:	Number of generated clique inequalities,
Cut	:	Number of generated cut inequalities,
Fpart	:	Number of generated F-partition inequalities,
Spath	:	Number of generated star-path inequalities,
BB	:	Number of generated nodes in the branch-and-cut tree,
CPU	:	Total CPU time in seconds to solve problem instance to optimality.

Instances	α	$\lvert V \rvert$	Opt	Gap	Clq	Cut	Fpart	Spath	BB	CPU
kroA150	3	150	79572	0,58	64	686	1380	0	215	58
kroA150	5	150	125435	0	228	268	12	0	1	0
kroA150	7	150	140961	0,24	745	5131	106	13	17	45
kroA150	9	150	113080	0	1922	3932	9	0	1	322
d198	3	198	47340	0,29	46	399	302	0	197	123
d198	5	198	76945	0,06	390	9759	332	4	7	77
d198	7	198	94300	0,13	1047	11251	34	47	43	579
d198	9	198	96088	0	2933	12829	18	1	1	1256
rat195	3	195	6957	0,6	58	489	1157	0	213	172
rat195	5	195	11320	0,1	358	5331	980	0	49	106
rat195	7	195	12319	0	1004	7180	5	0	1	81
rat195	9	195	8977	0	2849	8607	2	4	1	1135
kroA200	3	200	87951	0,6	668	4425	4512	0	1569	706
kroA200	5	200	138885	0,19	369	17519	1176	0	51	335
kroA200	7	200	158227	0,18	972	9896	113	34	29	455
kroA200	9	200	122594	0	2696	6495	10	3	1	1106

Our preliminary results reach to solve exactly instances till 200 nodes in less that 21 minutes. The instances computed with $\alpha = 9$ seem to need more CPU time to be solved than the ones computed with lower α values. Conversely, they are solved before the branching phase thanks to our separation algorithms which succeed to find more star-path inequalities in this case. For lower α values, the corresponding instances induce a numerous number of subproblems in the branching phase and we can remark that, for $\alpha = 3$, the integrality gap is very high. We consequently expect to solve exactly instances computed with $\alpha > 5$ for more than 400 nodes.

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