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Resolution Method for Mixed Integer Linear Multiplicative-Linear Bilevel Problems Based on Decomposition Technique

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Abstract. I In this paper, we propose an algorithm base on decomposition technique for solving the mixed integer linear multiplicative-linear bilevel problems. In fact, this algorithm is an application of the algorithm given by G. K. Saharidis et al for the case in which the first level objective function is linear multiplicative. We use properties of quasi-concave of bilevel programming problems and decompose the initial problem into two subproblems named RMP and SP. The lower and upper bound provided from the RMP and SP are updated in each iteration. The algorithm converges when the difference between the upper and lower bound is less than an arbitrary tolerance. In conclusion, some numerical examples are presented in order to show the efficiency of algorithm.

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1. Introduction

Bilevel programming problems are a special kind of decision processes involving two decision makers with a hierarchical structure in which the upper level decision maker is called the leader and the lower level decision maker is called the follower; additionally, the constraint region of the upper level optimization problem is implicitly determined by the lower level optimization problem. In general, Bilevel problems consist of determining a vector $x = (x_1, x_2) \in \mathbb{R}^n$ such that [10]:

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$$\min_{(x_1,x_2)\in S} f_1(x_1,x_2)$$
$$s.t \quad x_2 \in arg \min_{\nu\in S(x_1)} f_2(x_1,\nu) \tag{1}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the variables controlled by the upper level and the lower level decision makers, respectively; f_1,$f_2 : \mathbb{R}^n \Rightarrow \mathbb{R}$, $n = n_1 + n_2$; $S \subset \mathbb{R}^n$ defines the common constraint region and $S(x_1) = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in S\}$. The most approaches and algorithms proposed for solving bilevel problems have focused on the linear case of this problem such as the KuhnTucker approach [3, 13], the $Kth-best$ algorithm [9], the penalty function approach [31], the branch-and-bound algorithm [4, 13], then, some scholars such as Bilias and Karwan [8] and Bard [1, 2] extended them. Important decision making problems may involve decisions in both discrete and continuous variables. For example, a chemical engineering design problem may involve discrete decisions regarding the existence of chemical process units in addition to decisions in continuous variables, such as temperatures or pressures. Problems in this class, involving both discrete and continuous decision variables are mixed integer BLPP. Note that real world problems often involve nonlinear terms, resulting in mixed integer nonlinear BLPP [19]. Bilevel programming problems are NP hard [16], and discrete nonlinear bilevel problems are harder to solve, so despite numerous applications of them in real world, there have been alittle attention in bilevel problems involving discrete decisions and even less attention in case both nonlinear and discrete. For the solution of the integer linear BLPP, a branch and bound method has been developed by Moore and Bard [22] and For the solution of the mixed integer BLPP, another branch and bound technique is developed by Wen and Yang [30]. Cutting plane and parametric solution approaches have been developed by Dempe [15].Global optimization for solving mixed integer bilevel problems has been developed by Floudas [19]. Also, the few algorithms have been published for the nonlinear problem by Shi et al.[26], Bialas and Karwan [8] and Hansen et al.[20]. One of techniques for solving bilevel problems, is the reformulation techniques that have been developed for solving the nonlinear or quadratic bilevel problems; it transform the bilevel problem into a single level problem, for example by replacing the KarushKuhnTucker optimality conditions instead the inner problem and considering them as constraints of the upper level problem, the bilevel problem is transformed into a single level problem. Therefore, instead of solving the bilevel problem directly, it is transformed into a single level problem to be solved. Shi et al.[26], Bialas and Karwan [8] use the KKT optimality conditions for replacing the inner problem and hence they use different forms of branch and bound algorithm for solving of the reformulated problem. In this paper, that is an application from the algorithm presented in reference [25], we consider the class of bilevel problems in which the upper level objective function f_1 is linear multiplicative, the lower level one f_2 is linear and the common constraint region S is a bounded polyhedron thus feasible conditions such as conditions expressed in reference [25] but optimality conditions change. The algorithm is based on Benders decomposition technique which uses Karush-Kuhn-Tucker optimality conditions. The only assumption of the proposed algorithm is that although integer variables could appear in both levels of problem, they should be controlled by the upper optimization problem. Then at each iteration of the algorithm, the SP gives a new valid cut to the RMP which converges to the optimal solution. If the RMP optimality condition is not satisfied by the solution obtained by SP, the RMP sends the updated information to the SP which produces another cut for RMP

and the algorithm continues until the RMP optimality condition is satisfied [25]. The paper is organized as follows. After having defined the problem in section 2, an algorithm for solving mixed integer linear multiplicative-linear bilevel problems is offered in section 3. Further, in section 4, the numerical experiments are presented.

2. Linear Multiplicative-Linear Bilevel Problem [7]

Using the common notation in bilevel programming, the LMLB problem can be stated as follow:

$$\begin{aligned}
&\min_{x_1} \quad (\alpha + c_1^t x)(\beta + c_2^t x) \\
&\qquad = (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\
&\min_{x_2} \quad d_2 x_2 \\
&S.t. \begin{cases} A_1x_1 + A_2x_2 \leqslant b \\ x_1 \geqslant 0, \; x_2 \geqslant 0 \end{cases}
\end{aligned} \tag{2}$$

where $A_1 \in M_{m\times n_1}$; $A_2 \in M_{m\times n_2}$; b, d_2, c_{22}, c_{21}, c_{12}, c_{11} are vectors of conformal dimension; $c_1 = (c_{11}, c_{12})$, $c_2 = (c_{21}, c_{22})$; α, β are scalars and $(\beta + c_{21}x_1 + c_{22}x_2) > 0$, $(\alpha + c_{11}x_1 + c_{12}x_2) > 0$, for all $x = (x_1, x_2) \in S$, the polyhedron defined by the constraints. We assume that S is a nonempty and bounded polyhedron of full dimension in $\mathbb{R}^{n_1}$. Let S_1 be the projection of S onto $\mathbb{R}^{n1}$. For each $x_1 \in S_1$, a feasible solution to the $LMLB$ problem is obtained by solving the following linear programming problem [10]:

$$\begin{aligned}
&\min_{x_2} \quad d_2 x_2 \\
&S.t. \begin{cases} A_1x_1 + A_2x_2 \leqslant b \\ x_1 \geqslant 0, \; x_2 \geqslant 0 \end{cases}
\end{aligned} \tag{3}$$

Let $P(x_1)$ be the set of optimal solutions to (3), we assume that $P(x_1)$ is nonempty and singleton. Ref.[2] show the difficulties which may arise when $P(x_1)$ is not single-valued. The feasible region of problem (2) , called inducible region, is implicitly defined as follows:

$$IR = \{(\bar{x_1}, \bar{x_2}) : \; \bar{x_1} \geqslant 0, \; \bar{x_2} \in P(\bar{x_1})\}$$

Let us consider the following linear multiplicative problem:

$$\begin{aligned}
&\min_{x_1, x_2} \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\
&S.t. \begin{cases} A_1x_1 + A_2x_2 \leqslant b \\ x_1 \geqslant 0, \; x_2 \geqslant 0 \end{cases}
\end{aligned} \tag{4}$$

If an optimal solution $(\tilde{x_1}, \tilde{x_2})$ of (4) pertains to IR, then it is an optimal solution of the $LMLB$ problem. We used the following results.

Lemma 2.1 *[11] The inducible region of the quasiconcave bilevel programming problem (1) is a piecewise linear.*

Theorem 2.2 *[11] There is an extreme point of the constraint region S which is an optimal solution to the quasiconcave bilevel programming problem (1).*

3. Theoretical background of the proposed algorithm

In the sequel, we used the following notations:
Index:

- l denotes the number of extreme rays of the dual slave problem;
- p denotes the number of extreme rays of the dual slave problem with $z = z_1$;
- k denotes the number of extreme rays of the dual slave problem with $z = z_2$;
- r denotes the number of extreme rays of the dual slave problem with $z = z_n$;
- i denotes the number of extreme points of the dual slave problem with $z = z_1$;
- j denotes the number of extreme points of the dual slave problem with $z = z_2$;
- t denotes the number of extreme points of the dual slave problem with $z = z_n$.

Auxiliary decision variables:

- $v_m = 1$ if the *mth* constraint is active otherwise takes the value of zero;
- u_m dual value of the *mth* constraint;
- w_m Lagrangian multiplier *mth* constraint.

Parameter:

- M big value number.

Vector symboles:

$$\mathbb{R}^{f_1+f_2+f_3+\cdots} = \mathbb{R}^{f_1} \cup \mathbb{R}^{f_2} \cup \mathbb{R}^{f_3} \cup \mathbb{R}^{\cdots}$$

$$b_1 \in \mathbb{R}^{b_1},\ b_2 \in \mathbb{R}^{b_2},\ x \in X \subset \mathbb{R}^{x},\ y \in Y \subset \mathbb{R}^{y},\ s \in S \subset \mathbb{R}^{b_1+b_2},$$

$$u \in U \subset \mathbb{R}^{b_1+b_2},\ v \in V \subset \mathbb{R}^{b_1+b_2}$$

$$z \in Z\{0,1\} \subset \mathbb{R}^{z},\ bv \in Z\{0,1\} \subset \mathbb{R}^{v},\ h \in H \subset \mathbb{R}^{h}$$

$$w_1 \in W_1 \subset \mathbb{R}^{w_1},\ w_2 \in W_2 \subset \mathbb{R}^{w_2},\ w_3 \in W_3 \subset \mathbb{R}^{w_3},\ w_4 \in W_4 \subset \mathbb{R}^{w_4}$$

$$c_1 \in \mathbb{R}^{x},\ c_2 \in \mathbb{R}^{y},\ c_3 \in \mathbb{R}^{z}$$

$$F_1, F_2 : X \times Y \times Z \Rightarrow \mathbb{R}^1$$

$$A_1 \in \mathbb{R}^{b_1\times x},\ B_1 \in \mathbb{R}^{b_1\times y},\ E_1 \in \mathbb{R}^{b_1\times z},\ Q_1 \in \mathbb{R}^{b_1\times b_1},\ C_1 = I \in \mathbb{R}^{h\times h}$$

$$A_2 \in \mathbb{R}^{b_2\times x},\ B_2 \in \mathbb{R}^{b_2\times y},\ E_2 \in \mathbb{R}^{b_2\times z},\ Q_2 \in \mathbb{R}^{b_2\times b_2},\ C_2 = I \in \mathbb{R}^{h\times h}$$

$$c_T = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix},\ c^T \in \mathbb{R}^{c} = \mathbb{R}^{x+y+z},\ D = \begin{bmatrix} A_1\ B_1\ Q_1 \\ A_2\ B_2\ Q_2 \end{bmatrix},\ D \in \mathbb{R}^{D} = \mathbb{R}^{(b_1+b_2)\times(x+y+b_1+b_2)}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},\ b \in \mathbb{R}^{b} = \mathbb{R}^{b_1+b_2},\ g = \begin{bmatrix} x \\ y \\ s \end{bmatrix},\ g \in \mathbb{R}^{g} = \mathbb{R}^{x+y+b_1+b_2},\ E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},\ E \in \mathbb{R}^{E} = \mathbb{R}^{b_1+b_2}$$

$$c'^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ c'^T \in \mathbb{R}^{c'} = \mathbb{R}^{x+y+z+h},\ D' = \begin{bmatrix} A_1\ B_1\ Q_1\ C_1 \\ A_2\ B_2\ Q_2\ C_2 \end{bmatrix}$$

$$D' \in \mathbb{R}^{D'} = \mathbb{R}^{(b_1+b_2)\times(x+y+h+b_1+b_2)}$$

$$g' = \begin{bmatrix} x \\ y \\ s \\ h \end{bmatrix},\ g' \in \mathbb{R}^{g'} = \mathbb{R}^{x+y+h+b_1+b_2}$$

Next, we consider the following problem for the mixed integer linear multiplicative-linear bilevel problem:

$$P_1 : \begin{cases} \min_{x_1,x_2,z} \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ St. \\ \quad \min_{x_2} \quad d_2x_2 \\ \quad St. \\ \qquad\qquad A_1x_1 + B_1x_2 + E_1z \leqslant b_1 \\ \qquad\qquad A_2x_1 + B_2x_2 + E_2z \leqslant b_2 \\ \qquad\qquad\quad x_1, x_2 \geqslant 0 \quad z \in \{0,1\} \end{cases}$$

Note that constraints can be considered in the upper level and also, binary variable can be considered in the objective function of the upper level problem.
We get the following problem by fixing binary variables:

$$P_2: \begin{cases} \min_{x_1} \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ St. \\ \quad \min_{x_2} \quad d_2x_2 \\ \quad St. \\ \qquad A_1x_1 + B_1x_2 \leqslant b_1 - E_1\bar{z} \Rightarrow w_1 \\ \qquad A_2x_1 + B_2x_2 \leqslant b_2 - E_2\bar{z} \Rightarrow w_2 \\ \qquad -x_1 \leqslant 0 \Rightarrow w_3 \\ \qquad -x_2 \leqslant 0 \Rightarrow w_4 \end{cases}$$

The problem P_2 can be transformed to a mixed integer multiplicative problem using KKT optimality conditions and the active constraints strategy [18] as following:

$$P_3: \begin{cases} \min \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ St. \\ \quad A_1x_1 + B_1x_2 \leqslant b_1 - E_1\bar{z} \\ \quad A_2x_1 + B_2x_2 \leqslant b_2 - E_2\bar{z} \\ \quad w_1 - Mbv_1 \leqslant 0, \quad w_2 - Mbv_2 \leqslant 0 \\ \quad w_3 - Mbv_3 \leqslant 0, \quad w_4 - Mbv_4 \leqslant 0 \\ \quad (b_1 - E_1\bar{z}) - A_1x_1 - B_1x_2 - M(1 - bv_1) \leqslant 0 \\ \quad (b_2 - E_2\bar{z}) - A_2x_1 - B_2x_2 - M(1 - bv_2) \leqslant 0 \\ \quad x_2 - M(1 - bv_3) \leqslant 0, x_1 - M(1 - bv_4) \leqslant 0 \\ \quad w_1A_1 + w_2A_2 - w_4 = 0 \\ \quad w_1B_1 + w_2B_2 - w_3 = -d_2 \\ \quad x, y, w_1, w_2, w_3, w_4 \geqslant 0 \quad bv_1, bv_2, bv_3, bv_4 \in \{0, 1\} \end{cases}$$

From the solution of P_3, we find active constraints, so we can replace $LMLB$ problem with active constraints plus remainder constraints and the first level objective function, and transform the problem P_1 to the following linear multiplicative problem:

$$P_4: \begin{cases} \min \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ St. \\ \quad A_1x_1 + B_1x_2 = b_1 - E_1\bar{z} \\ \quad A_2x_1 + B_2x_2 \leqslant b_2 - E_2\bar{z} \\ \quad x_1, x_2 \geqslant 0 \end{cases}$$

From reformulated the problem P_4, we have:

$$P_5: \begin{cases} \min \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ St. \\ \quad A_1x_1 + B_1x_2 + Q_1s = b_1 - E_1\bar{z} \\ \quad A_2x_1 + B_2x_2 + Q_2s = b_2 - E_2\bar{z} \\ \quad x_1, x_2, s \geqslant 0 \end{cases}$$

where Q_1, Q_2 are matrices that all the their elements are equal to zero except the elements in the diagonal that correspond to non-active constraints and are equal to 1. We reformulate the problem P_5 to P_5' using the notations of nomenclature:

$$P_5' : \begin{cases} \min \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ St. \\ \quad Dg = b - E\bar{z} \\ \quad g \geqslant 0 \end{cases}$$

The problem P_2 (and P_3) should have at least a non-empty solution set for z, so we cannot choose the variable z arbitrarily, thus we add $-c_1h$ and $-c_2h$ to active constraints and non-active constraints of the P_2 problem, respectively. Also we use KKT conditions and the active constraints strategy that due to problem P_3' which gives the best feasible solution minimizing the auxiliary variables h.

$$P_3' : \begin{cases} \min \quad h \\ St. \\ \quad A_1x_1 + B_1x_2 - C_1h \leqslant b_1 - E_1\bar{z} \\ \quad A_2x_1 + B_2x_2 - C_2h \leqslant b_2 - E_2\bar{z} \\ \quad w_1 - Mbv_1 \leqslant 0, \quad w_2 - Mbv_2 \leqslant 0 \\ \quad w_3 - Mbv_3 \leqslant 0, \quad w_4 - Mbv_4 \leqslant 0 \\ \quad (b_1 - E_1\bar{z}) - A_1x_1 - B_1x_2 + C_1h - M(1 - bv_1) \leqslant 0 \\ \quad (b_2 - E_2\bar{z}) - A_2x_1 - B_2x_2 + C_2h - M(1 - bv_2) \leqslant 0 \\ \quad x_2 - M(1 - bv_3) \leqslant 0, x_1 - M(1 - bv_4) \leqslant 0, h - M(1 - bv_5) \leqslant 0 \\ \quad w_1A_1 + w_2A_2 - w_4 = 0 \\ \quad -w_1C_1 - w_2C_2 - w_5 = 0 \\ \quad w_1B_1 + w_2B_2 - w_3 = -c_2 \\ \quad x_1, x_2, w_1, w_2, w_3, w_4, w_5 \geqslant 0 \quad bv_1, bv_2, bv_3, bv_4, bv_5 \in \{0, 1\} \end{cases}$$

We transform P_3' to P_5'' like transformed process P_3 to P_5', so we have:

$$P_5'' : \begin{cases} \max \quad c'g' \\ St. \\ D'g' = b - E\bar{z} \\ \quad g' \geqslant 0 \end{cases}$$

The necessary and sufficient condition for existence at least a non-empty solution for z is given by lemma of Farkas and Minkowski:

LEMMA 3.1 *[14] [Farkas and Minkowski] Let $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. Then there exists a vector $x = (x_1, \ldots, x_n) \geqslant 0$ satisfying $Ax = b$ if and only if for each $u \in \mathbb{R}^m$ with $uA \geqslant 0$ it also holds that $ub \geqslant 0$.*

If we correspond to each constraint i of P_5'' a dual variable u_i, then the Lemma of Farkas and Minkowski states:
The problem P_5'' has a solution $g' \geqslant 0$ if and only if $u(b - E\bar{z}) \leqslant 0$ for all u which $uD' \leqslant 0$ holds.
We should notice that for each z the matrix D' is different based on the values of matrix Q. That means that the matrix D' is related to the value of z, so it can be stated as $D'(z)$ [25].

THEOREM 3.2 *[20] [Minkowski and Weyl] A convex cone C is polyhedral if and only if it is finitely generated, that is, the cone is generated by a finite number of vectors $b_1, \ldots, b_m$.*

For each z the cone $U(z) = \{u : uD'(z) \leqslant 0\}$ is a polytope, so, in view of theorem (3.1), the cone $U(z)$ has a finite number of generators as $u_1^z, \dots, u_l^z$. First, we considered the problem for $z \in \{0,1\}$. Now we extend the problem and consider it for nonnegative integer values $z_1, \dots, z_n$. The necessary and sufficient conditions of the Farkas and Minkowski Lemma for each $z_1, \dots, z_n$, then, is equivalent to the following system of inequalities:

$$\text{if } z = z_1 \Longrightarrow \begin{cases} u_1^{z_1}(b - Ez_1) \leqslant 0 \\ u_2^{z_1}(b - Ez_1) \leqslant 0 \\ \quad \cdots\cdots\cdots \\ u_p^{z_1}(b - Ez_1) \leqslant 0 \end{cases}$$

$$\text{if } z = z_2 \Longrightarrow \begin{cases} u_1^{z_2}(b - Ez_2) \leqslant 0 \\ u_2^{z_2}(b - Ez_2) \leqslant 0 \\ \quad \cdots\cdots\cdots \\ u_K^{z_2}(b - Ez_2) \leqslant 0 \end{cases} \tag{5}$$

$$\text{if } z = z_n \Longrightarrow \begin{cases} u_1^{z_n}(b - Ez_n) \leqslant 0 \\ u_2^{z_n}(b - Ez_n) \leqslant 0 \\ \quad \cdots\cdots\cdots \\ u_R^{z_n}(b - Ez_n) \leqslant 0 \end{cases}$$

Now, consider the following quadratic programming problem:

$$\begin{aligned} \min \quad & \frac{1}{2}x^t H x + d^t x \\ S.t. \quad & \\ & Ax \leqslant b \end{aligned}$$

where H is symmetric and positive semidefinite, so that the objective function is convex. Then, the dual problem be written as follows[5]:

$$\begin{aligned} \max \quad & -\frac{1}{2}x^t H x - b^t u \\ S.t. \quad & \\ & Hx + A^t u = -d \\ & u \geqslant 0 \end{aligned}$$

It is clear that linear multiplicative programming problem is a special type of quadratic programming problem, so we can reformulate P_5' as follows:

$$P_6 : \begin{cases} \min \quad (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \qquad = \frac{1}{2}(x_1 x_2) \begin{pmatrix} 2c_{11}c_{21} & c_{11}c_{22} + c_{12}c_{21} \\ c_{11}c_{22} + c_{12}c_{21} & 2c_{12}c_{22} \end{pmatrix} \binom{x_1}{x_2} \\ \qquad +(\alpha c_{21} + \beta c_{11} \alpha c_{22} + \beta c_{12}) \binom{x_1}{x_2} + \alpha\beta \\ St. \\ \qquad Dg = b - E\bar{z} \\ \qquad g \geqslant 0 \end{cases}$$

where,

$$H = \begin{pmatrix} 2c_{11}c_{21} & c_{11}c_{22} + c_{12}c_{21} \\ c_{11}c_{22} + c_{12}c_{21} & 2c_{12}c_{22} \end{pmatrix}$$

and $d^t = (\alpha c_{21} + \beta c_{11} \alpha c_{22} + \beta c_{12})$,
With suppose existence at least a non-empty solution for the problem P_3 for given $z = \bar{z}$, the dual of P_6 can be written as folows:

$$P_6 \text{ dual} : \begin{cases} \max & -\frac{1}{2}x^t Hx - (b - E\bar{z})^t v + \alpha\beta \\ S.t. & \\ & Hx + A^t v \leqslant -d \\ & v \text{ of any sign, } x \geqslant 0 \end{cases}$$

where v is the vector of dual variables of the constraints of the problem P_6. Constraints of P_6 make a convex polytope, so by using the duality theorem, problem P_6 can be rewritten as:

$$\max\{-\frac{1}{2}x^t Hx - (b - Ez)^t v + \alpha\beta : (Hx + A^t v)(z) \leqslant -d\}$$

$$z \in \mathbb{R}^Z \text{ where } Z = \{z_1, z_2, \ldots, z_n\}$$

This maximum is obtained at the vertex of each polytope
$V(z) = \{(x, v) : Hx + A^t v(z) \leqslant -d\}$.
Assuming that $V(z)$ is non-empty for all z, we denote by $((x_1^{z_1}, v_1^{z_1}), \ldots, (x_I^{z_1}, v_I^{z_1})), \ldots, ((x_1^{z_n}, v_1^{z_n}), \ldots, (x_T^{z_n}, v_T^{z_n}))$ the vertices of the polytope $V(z)$ where $z \in \{z_1, z_2, \ldots, z_n\}$ [25]. Then P_6 can be written:

$$\begin{cases} \max_{i=1,\ldots,I}\{-\frac{1}{2}(x_i^{z_1})^t Hx_i^{z_1}(b - Ez_1)^t v_i^{z_1} + \alpha\beta\} \\ \ldots \\ \max_{k=1,\ldots,K}\{-\frac{1}{2}(x_k^{z_n})^t Hx_k^{z_n}(b - Ez_n)^t v_k^{z_n} + \alpha\beta\} \end{cases}$$

These maximization is equivalent to the following nonlinear problem:

$$\begin{cases} \min \quad \xi \\ St. \\ -\frac{1}{2}(x_1^{z_1})^t Hx_1^{z_1}(b - Ez_1)^t v_1^{z_1} + \alpha\beta \leqslant \xi \\ \ldots \\ -\frac{1}{2}(x_I^{z_1})^t Hx_I^{z_1}(b - Ez_1)^t v_I^{z_1} + \alpha\beta \leqslant \xi \\ -\frac{1}{2}(x_1^{z_2})^t Hx_1^{z_2}(b - Ez_2)^t v_1^{z_2} + \alpha\beta \leqslant \xi \\ \ldots \\ -\frac{1}{2}(x_J^{z_2})^t Hx_J^{z_2}(b - Ez_2)^t v_J^{z_2} + \alpha\beta \leqslant \xi \\ \ldots \\ -\frac{1}{2}(x_1^{z_n})^t Hx_1^{z_n}(b - Ez_n)^t v_1^{z_n} + \alpha\beta \leqslant \xi \\ \ldots \\ -\frac{1}{2}(x_T^{z_n})^t Hx_T^{z_n}(b - Ez_n)^t v_T^{z_n} + \alpha\beta \leqslant \xi \end{cases}$$

Considering these inequalities and the inequalities have been obtained for existence at least solution set, the following formulation is acquireed:

$$\begin{cases} \min \quad \xi \\ \ St. \\ -\frac{1}{2}(x_1^{z_1})^t H x_1^{z_1}(b-Ez_1)^t v_1^{z_1} + \alpha\beta \leqslant \xi \\ \qquad\qquad \cdots\cdots \\ -\frac{1}{2}(x_I^{z_1})^t H x_I^{z_1}(b-Ez_1)^t v_I^{z_1} + \alpha\beta \leqslant \xi \\ \qquad\qquad \cdots\cdots \\ -\frac{1}{2}(x_1^{z_n})^t H x_1^{z_n}(b-Ez_n)^t v_1^{z_n} + \alpha\beta \leqslant \xi \\ \qquad\qquad \cdots\cdots \\ -\frac{1}{2}(x_T^{z_n})^t H x_T^{z_n}(b-Ez_n)^t v_T^{z_n} + \alpha\beta \leqslant \xi \\ u_1^{z_1}(b-Ez_1) \leqslant 0 \\ \qquad\qquad \cdots\cdots \\ u_p^{z_1}(b-Ez_1) \leqslant 0 \\ \qquad\qquad \cdots\cdots \\ u_1^{z_n}(b-Ez_n) \leqslant 0 \\ \qquad\qquad \cdots\cdots \\ u_R^{z_n}(b-Ez_n) \leqslant 0 \\ \qquad \xi \in (-\infty, +\infty) \end{cases}$$

Which is equivalent to the following problem (P_7) and is named master problem (MP):

$$MP : P_7 \begin{cases} \min \quad \xi \\ \ St. \\ -\frac{1}{2}(x_1^{z_1})^t H x_1^{z_1}(b-Ez)^t v_1^{z_1} + \alpha\beta \leqslant \xi \\ \qquad\qquad \cdots\cdots \\ -\frac{1}{2}(x_I^{z_1})^t H x_I^{z_1}(b-Ez)^t v_I^{z_1} + \alpha\beta \leqslant \xi \\ \qquad\qquad \cdots\cdots \\ -\frac{1}{2}(x_1^{z_n})^t H x_1^{z_n}(b-Ez)^t v_1^{z_n} + \alpha\beta \leqslant \xi \\ \qquad\qquad \cdots\cdots \\ -\frac{1}{2}(x_T^{z_n})^t H x_T^{z_n}(b-Ez)^t v_T^{z_n} + \alpha\beta \leqslant \xi \\ u_1^{z_1}(b-Ez) \leqslant 0 \\ \qquad\qquad \cdots\cdots \\ u_p^{z_1}(b-Ez) \leqslant 0 \\ \qquad\qquad \cdots\cdots \\ u_1^{z_n}(b-Ez) \leqslant 0 \\ \qquad\qquad \cdots\cdots \\ u_R^{z_n}(b-Ez) \leqslant 0 \\ z \in \{0,1\},\ \xi \in (-\infty, +\infty) \end{cases}$$

At each stage of the algorithm, only some constraints of P_7 are known explicitly which gives a problem named Restricted Master Problem (RMP) and involves a subset of the constraints of P_7 (Master Problem). Let $(\bar{z}, \bar{\xi})$ be an optimal solution of RMP, $\bar{\xi}$ is a lower bound of optimal ξ^* such that $\bar{\xi} \leqslant \xi^*$. An upper bound can be obtained by the resolution of P_4 or P_2 (which is a restricted form of the initial $MIBLP$). The upper bound (UB) is updated when a lower bound is obtained of the current P_4 (or P_2) compared with the current UB. Sufficient condition for the $(\bar{z}, \bar{\xi})$ to be an optimal solution of P_7 is that the $UB - LB \leqslant \epsilon$ because the RMP is a relaxation of the original problem whereas the SP represent a restriction [25]. In each iteration of the algorithm, three cases can be arised:

(1) (Production of feasibility cut)
The optimal value of dual P_5 is unbounded. The Simplex algorithm is applied to dual P_5'' and produces an extreme ray $\bar{u}$ such that $\bar{u}(b - E\bar{z}) > 0$ and $\bar{u}D' \leqslant 0$. Thus the constraint $\bar{u}(b - E\bar{z}) \leqslant 0$ does not hold for the current solution $\bar{z}$ of RMP and so it is not a solution of P_7. The constraints $\bar{u}(b - E\bar{z}) \leqslant 0$ must be added to RMP to form a new RMP [25].

(2) (Production of integer exclusion cut)
The optimal value of dual is finite and $-\frac{1}{2}x^tHx - (b - E\bar{z})^tv + \alpha\beta - \xi \leqslant 0$. So with adding this constraint to RMP, the optimal value does not change. If the optimality condition is not satisfied, the algorithm continues by changing the current integer solution using the following cut:

$$\sum_{i \in P} z_i - \sum_{j \in Q} z_j \leqslant |P| - 1$$

where P is the set of indices of variables that equals to 1, i.e. $P = \{i : z_i^* = 1\}$. Similarly Q is the set of indices of variables that equals to 0, i.e. $Q = \{j : z_j^* = 0\}$. By adding this constraint to RMP, integer optimal solution of problem change and is produced new SP.

(3) (Production of optimality cut)
The optimal value of dual P_5 is bounded but

$$-\frac{1}{2}x^tHx - (b - E\bar{z})^tv + \alpha\beta - \xi > 0$$

So the constraint $-\frac{1}{2}x^tHx - (b - E\bar{z})^tv + \alpha\beta - \xi \leqslant 0$ be added to P_7 and a new RMP be made.

In case where the integer decision variables appear in the upper level objective function as $c_{13}x_3$, $c_{23}x_3$ terms, we have:

$$\begin{aligned}
&(\alpha + c_{11}x_1 + c_{12}x_2 + c_{13}x_3)(\beta + c_{21}x_1 + c_{22}x_2 + c_{23}x_3)\\
=&(\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) + c_{13}x_3(\beta + c_{21}x_1 + c_{22}x_2)\\
+&c_{23}x_3(\alpha + c_{11}x_1 + c_{12}x_2) + c_{13}c_{23}x_3^2\\
=&(\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2)\\
+&x_3[c_{13}(\beta + c_{21}x_1 + c_{22}x_2) + c_{23}(\alpha + c_{11}x_1 + c_{12}x_2)] + c_{13}c_{23}x_3^2
\end{aligned}$$

If $[c_{13}(\beta + c_{21}x_1 + c_{22}x_2) + c_{23}(\alpha + c_{11}x_1 + c_{12}x_2)] = 0$ then we have $(\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) + c_{13}c_{23}x_3^2$. Therefore we obtain the following form to objective function of dual P_6:

$$\min_{x_3}(c_{13}c_{23}x_3^2 + \max_{x_1,x_2}(\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2))$$

the variable ξ in the produced optimality cuts is replaced by $(\xi = \xi + c_{13}c_{23}x_3^2)\xi + c_{13}c_{23}x_3^2$.
Note that when we find dual variables may face with a dual gap then by writing the dual problem we find the value of the gap and add or deduce it to the given cut.

4. Examples

Consider the following problem:

$$\begin{aligned}
\min_{x_1,x_2} \quad & (20-x_1)(1+x_2) \\
\text{S.t.} \quad & \min_{x_2} \quad -30x_1 - 8x_2 \\
& \text{S.t.} \quad g_1 : 10x_1 + 3x_2 + 8z \leqslant 75 \\
& \qquad\quad g_2 : 5x_1 + 4z \leqslant 5 \\
& \qquad\quad g_3 : 5x_1 + x_2 + 2z \leqslant 85 \\
& \qquad\quad z \in \{0,1\},\ x_1, x_2 \geqslant 0
\end{aligned}$$

This problem is decomposed into the master and slaves problems:

RMP:

$$\begin{aligned}
\min \quad & F(z,\xi) = \xi \\
\text{S.t.} \quad & -M \leqslant \xi \leqslant M \\
& z \in \{0,1\} \\
& M = 235
\end{aligned}$$

$\mathbf{SP(\bar{z})}$:

$$\begin{aligned}
\min \quad & F_1(x_1,x_2) = (20-x_1)(1+x_2) \\
\text{S.t.} \quad & \min \quad F_2(x_1,x_2) = -30x_1 - 8x_2 \\
& \quad \text{S.t.} \quad 10x_1 + 3x_2 \leqslant 75 - 8\bar{z} \\
& \qquad\qquad 5x_1 \leqslant 5 - 4\bar{z} \\
& \qquad\qquad 5x_1 + x_2 \leqslant 85 - 2\bar{z} \\
& \qquad\qquad x_2, y_3 \geqslant 0
\end{aligned}$$

In the first, we fix the binary variable with $z = 0$ arbitrarily, so we have problem $SP(0)$:

SP(0):

$$\begin{aligned}
\min \quad & F_1(x_1,x_2) = (20-x_1)(1+x_2) \\
\text{S.t.} \quad & \min \quad F_2(x_1,x_2) = -30x_1 - 8x_2 \\
& \quad \text{S.t.} \quad 10x_1 + 3x_2 \leqslant 75 \\
& \qquad\qquad 5x_1 \leqslant 5 \\
& \qquad\qquad 5x_1 + x_2 \leqslant 85 \\
& \qquad\qquad x_2, y_3 \geqslant 0
\end{aligned}$$

The initial value of the lower and upper bound of the algorithm are $LB = -235$

and $UB = \infty$.

$$z = 1, \xi = -235 \Rightarrow F_3(z,\xi) = -235 \Rightarrow LB = -235$$

By using Kuhn-Tucker method and active set strategy, we obtain the following problem:

$$\begin{aligned}
\min \quad & F_1(x_1,x_2) = (20 - x_1)(1 + x_2) \\
\text{S.t.} \quad & g_1 : 10x_1 + 3x_2 \leqslant 75 \\
& g_2 : 5x_1 \leqslant 5 \\
& g_3 : 5x_1 + x_2 \leqslant 85 \\
& 10u_1 + 5u_2 + 5u_3 - u_4 = 30 \\
& 3u_1 + u_3 - u_5 = 8 \\
& u_1 - Mv_1 \leqslant 0,\ 75 - 10x_1 - 3x_2 - M(1 - v_1) \leqslant 0 \\
& u_2 - Mv_2 \leqslant 0,\ 5 - 5x_1 - M(1 - v_2) \leqslant 0 \\
& u_3 - Mv_3 \leqslant 0,\ 85 - 5x_1 - x_2 - M(1 - v_3) \leqslant 0 \\
& u_4 - Mv_4 \leqslant 0,\ x_1 - M(1 - v_4) \leqslant 0 \\
& u_5 - Mv_5 \leqslant 0,\ x_2 - M(1 - v_5) \leqslant 0 \\
& x_1, x_2 \geqslant 0,\ u_1,u_2,u_3,u_4,u_5 \geqslant 0,\ v_1,v_2,v_3,v_4,v_5 \in \{0,1\}
\end{aligned}$$

The solving of this problem shows that the inducible region of the linear multiplicative-linear bilevel problem is the line: $10x_1 + 3x_2 = 75$: So the first constraint of the linear multiplicative-linear bilevel problem is an active constraint. The associated linear multiplicative problem(LMP) is to the following form:

LMP:

$$\begin{aligned}
& \min F_1(x_1,x_2) = (20 - x_1)(1 + x_2) \\
\text{S.t.} \quad & 10x_1 + 3x_2 = 75 \Rightarrow w_1 \\
& 5x_1 \leqslant 5 \Rightarrow w_2 \\
& 5x_1 + x_2 \leqslant 85 \Rightarrow w_3 \\
& x_2, y_3 \geqslant 0
\end{aligned}$$

The optimal solution of LMP is equal to 430.67 (obtained with Lingo) at the point $(x_1, x_2, z) = (1, 21.66, 0)$. Therfore new upper bound is $UB_{new} = 430.67$.

$$LB = -235, UB = 430.67 \qquad UB - LB = 430.67 + 235 = 665.67 \not< 0 = \epsilon$$

The optimality condition is not satisfied and the current LMP produce the first cut. The optimal solution of P_5' is finite and the constraint $-\frac{1}{2}x^tHx - (b - E\bar{z})^tv + \alpha\beta - \bar{\xi} \leqslant 0$ is not satisfied, so optimality cut provide. Let (w_1, w_2, w_3) is the value of dual variables associated with LMP then we write LMP problem as following:

$$\begin{aligned} \max \quad & x_1x_2 - 75w_1 - 5w_2 - 85w_3 + 20 \\ \text{S.t.} \quad & -x_2 + 10w_1 + 5w_2 + 5w_3 = 1 \\ & -x_1 + 3w_1 + w_3 = -20 \\ & w_1 \text{ of any sign, } w_2, w_3 \geqslant 0 \end{aligned}$$

The optimal solution is equal to 452.33 at the point $(x_1, x_2, w_1, w_2, w_3) = (0, 0, -6.67, 13.53, 0)$ then we have:

$$w_1 = -6.67, w_2 = 13.53, w_3 = 0$$

In this problem the value of dual gap is equal to $a = 21.66$. By attention to dual values, we product
$-\frac{1}{2}x^tHx - (b - E\bar{z})^tv + \alpha\beta - \bar{\xi} \leqslant 0$ cut as following:

$$\begin{aligned} & (x_1x_2)\begin{pmatrix} 0 & .5 \\ .5 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (w_1, w_2, w_3)\begin{pmatrix} 75 - 8z \\ 5 - 4z \\ 85 - 2z \end{pmatrix} - a + \alpha\beta - \xi \leqslant 0 \\ & \Rightarrow x_1x_2 + 18.16z + 432.33 + 20 - 21.66 - \xi \leqslant 0 \\ & \Rightarrow x_1x_2 + 18.16z + 430.67 - \xi \leqslant 0 \end{aligned}$$

This cut is added to the new RMP is produced:

RMP:

$$\begin{aligned} \min \quad & F_3(\xi) = \xi \\ \text{S.t.} \quad & -M \leqslant \xi \leqslant M \\ & x_1x_2 + 18.16z + 430.67 - \xi \leqslant 0 \\ & z \in \{0, 1\} \\ & M = 235 \end{aligned}$$

The resolution of the RMP gives a new lower bound $LB_{new} = 430.67$. Comparison of the UB and the new LB satisfy the RMP optimality condition of the algorithm which it is $\epsilon = 0$. The optimal solution of the problem is the point $(x_1, x_2) = (1, 21.67)$, $z = 0$. In this example, the upper bound of slack variables is $M = 85$ and it has solution for values $M \geqslant 85$ but may be infeasible for values $M < 85$.
An complication in this class of problems is that first level objective function is nonlinear and it product the dual gap; therefore, by writing the dual problem we find the gap value and add or deduce it to the given cut.

5. Conclusions

In this paper, an algorithm for solving of the mixed integer linear multiplicative-linear bilevel problem was presented. The algorithm is based on Benders decomposition method and, can be considered as a reformulation algorithm. After decomposing the initial problem, the bilevel problem was solved using from KKT optimality conditions. The presented algorithm decomposes the initial problem to two subprolem that are named slave problems (SP) and the restricted master problem (RMP). The RMP is a relaxation of the initial problem, so it provides a lower bound for the algorithm in the case of minimization. The SP represents a restriction of the initial problem because they are resulted from the initial problem by fixing the integer variables. Thus, the solution of the SP in each iteration provides an upper bound in the case of minimization. The RMP results the optimal solution after the addition of cuts produced from the SP. The convergence condition of the algorithm is satisfied when the difference between the upper and lower bound of the algorithm is less than an arbitrary tolerance.

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