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Optimal Forms of Shallow Shells With Circular Boundary

Part 1: Maximum Fundamental Frequency

Thin, shallow, elastic shells with given circular boundary are considered. We seek the axisymmetric shell form which maximizes the fundamental frequency of vibration. The boundary conditions, material, surface area, and uniform thickness of the shell are specified. We employ a bimodal formulation and use an iterative procedure based on the optimality condition to obtain optimal forms. Results are presented for clamped and simply supported boundary conditions. For the clamped case, the optimal forms have zero slope at the boundary. The maximum fundamental frequency is significantly higher than that for the corresponding spherical shell if the boundary is clamped, but only slightly higher if it is simply supported.

Introduction

One basic problem in structural optimization is that of maximizing the fundamental frequency of vibration. This may be desirable to avoid resonance if the structure is subjected to dynamic forces or excitation of its supports. Many frequency optimization results have been obtained for bars, beams, and plates, in which the material is distributed in an optimal manner (e.g., see [1-3]).

Recently, shallow arches were optimized with respect to vibrations. In [4], the form of the arch was varied for given cross section, length, and span, while both the form and material distribution were varied in [5]. For the case of clamped ends, the optimal arch forms have zero slope at the ends and have a significantly higher fundamental vibration frequency than the corresponding circular arch. If the ends are simply supported, however, the circular form is almost optimal.

In this study, we consider thin, shallow, elastic shells with a given circular boundary. The shells are either clamped or simply supported at the boundary. The material, surface area, and uniform thickness of the shell are specified. Hence, the total volume and mass of the shell are given. Our objective in Part 1 is the determination of the axisymmetric form of the shell that has maximum fundamental vibration frequency. (In Parts 2 and 3, we will optimize buckling load and enclosed volume, respectively [6, 7].)

Marguerre's equations [8], including a transverse inertia term, are linearized to yield the governing equations for small vibrations. The optimality condition is derived by the calculus of variations. We allow for the possibility that the fundamental frequency is a double eigenvalue (i.e., the solution is bimodal). An iterative procedure is utilized, based on the optimality condition, and a shooting method is used to obtain numerical results. Optimal forms are determined for several cases, and the maximum fundamental frequencies are compared to those for the corresponding spherical shells.

Shell Equations

Consider a thin, shallow, elastic shell with a circular boundary of radius a. The shell has constant thickness h, density ρ, Young's modulus E, and Poisson's ratio ν. We define polar coordinates r and θ in the base plane, and denote time by T. The middle surface of the shell is assumed to be axisymmetric and is denoted $Z(r)$, with $Z(a)=0$. The slope $Z'(a)$ at the edge is not specified.

The area S of the middle surface is given by

$$S=2\pi\int_0^a [(Z')^2+1]^{1/2} r\,dr. \tag{1}$$

For shallow shells, we can write (1) approximately as

$$S=\pi a^2+\pi\int_0^a (Z')^2 r\,dr. \tag{2}$$

We define the nondimensional surface area parameter λ by

$$\lambda^4=4(S-\pi a^2)/(\pi c^2) \tag{3}$$

where

$$c=h/\sqrt{12(1-\nu^2)}. \tag{4}$$

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For shallow spherical shells with central height H, we have $Z(r)=(a^2-r^2)H/a^2$ [9], and (3) yields $\lambda^4=4(H/c)^2$, which is the usual definition of the geometric parameter λ for such shells.

We let $W(r,\theta,T)$ be the downward deflection of the shell and $F(r,\theta,T)$ be a stress function [9]. In the equations of motion, we include a downward axisymmetric distributed load $q(r)$ for reference in Parts 2 and 3, and we neglect inplane inertia. Marguerre's theory leads to the following equations [8, 9]:

$$D\nabla^2\nabla^2 W=\frac{1}{r}\left(F_r+\frac{1}{r}F_{\theta\theta}\right)(W_{rr}-Z'')+\frac{1}{r^2}F_{rr}W_{\theta\theta}+\frac{1}{r}(W_r-Z')F_{rr}-\frac{2}{r^2}\left(F_{r\theta}-\frac{1}{r}F_\theta\right)\left(W_{r\theta}-\frac{1}{r}W_\theta\right)-\rho h\,W_{TT}+q, \tag{5}$$

$$\frac{1}{Eh}\nabla^2\nabla^2 F=\frac{1}{r}\left(W_r+\frac{1}{r}W_{\theta\theta}\right)(Z''-W_{rr})+\frac{1}{r}Z'W_{rr}+\frac{1}{r^2}\left(W_{r\theta}-\frac{1}{r}W_\theta\right)^2, \tag{6}$$

where

$$D=\frac{Eh^3}{12(1-\nu^2)},\ \nabla^2(\ )=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\ ), \tag{7}$$

and subscripts r,θ, and T denote partial derivatives.

We introduce the nondimensional quantities

$$x=r/a,\ z=Z/c,\ w=W/c,\ f=F/D,\quad t=T\sqrt{D/(\rho h a^4)},\ p=qa^4/(4\lambda^4 Dc) \tag{8}$$

and also define

$$g(x)=z'(x). \tag{9}$$

In the optimization procedure, $g(x)$ will be the design function. For a spherical shell, $g(x)=-\lambda^2 x$. In nondimensional terms, equation (2) becomes

$$\int_0^1 g^2\,x\,dx=\lambda^4/4, \tag{10}$$

while (5) and (6) become

$$\nabla^2\nabla^2 w=\frac{1}{x}\left(f_x+\frac{1}{x}f_{\theta\theta}\right)(w_{xx}-g')+\frac{1}{x^2}f_{xx}w_{\theta\theta}+\frac{1}{x}(w_x-g)f_{xx}-\frac{2}{x^2}\left(f_{x\theta}-\frac{1}{x}f_\theta\right)\left(w_{x\theta}-\frac{1}{x}w_\theta\right)-w_{tt}+4\lambda^4 p, \tag{11}$$

$$\nabla^2\nabla^2 f=\frac{1}{x}\left(w_x+\frac{1}{x}w_{\theta\theta}\right)(g'-w_{xx})+\frac{1}{x}g\,w_{xx}+\frac{1}{x^2}\left(w_{x\theta}-\frac{1}{x}w_\theta\right)^2, \tag{12}$$

where

$$\nabla^2(\ )=\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)(\ ) \tag{13}$$

and subscripts x,θ, and t denote partial derivatives.

To examine small, natural vibrations of the shell, we set $p=0$ and linearize (11) and (12) in w and f. We then write

$$w(x,\theta,t)=\sum_{n=0}^{\infty}\beta_n(x)\cos n\theta\cos\omega_n t, \tag{14}$$

$$f(x,\theta,t)=\sum_{n=0}^{\infty}\psi_n(x)\cos n\theta\cos\omega_n t, \tag{15}$$

where the ω_n are nondimensional vibration frequencies. This leads to the equations

$$L_nL_n\beta_n=-\frac{1}{x}\left(\psi_n'-\frac{n^2}{x}\psi_n\right)g'-\frac{1}{x}g\psi_n''+\omega_n^2\beta_n, \tag{16}$$

$$L_nL_n\psi_n=\frac{1}{x}\left(\beta_n'-\frac{n^2}{x}\beta_n\right)g'+\frac{1}{x}g\,\beta_n'', \tag{17}$$

where $n=0,1,2,\ \ldots$, and

$$L_n(\ )=\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{n^2}{x^2}\right)(\ ). \tag{18}$$

For a given value of n, equations (16) and (17), along with appropriate boundary conditions, lead to an infinite number of eigenvalues ω_n. We will let ω_n denote the lowest of these eigenvalues for that n. As an example, ω_0 will denote the lowest frequency associated with axisymmetric vibrations, and $\beta_0(x)$ and $\psi_0(x)$ will be the corresponding vibration mode and stress function, respectively.

If the edge of the shell is simply supported (with no radial and tangential displacements), the boundary conditions at $x=1$ are [10]

$$\beta_n=0, \tag{19}$$

$$\beta_n''+\nu\beta_n'=0, \tag{20}$$

$$\psi_n''-\nu\psi_n'+\nu n^2\psi_n=0, \tag{21}$$

$$\psi_n'''-[1-\nu+(2+\nu)n^2]\psi_n'+3n^2\psi_n-g\beta_n'=0. \tag{22}$$

If the edge is clamped (with no radial and tangential displacements), the boundary conditions are (19), (21), (22), and

$$\beta_n'=0. \tag{23}$$

At $x=0$, finiteness requirements and symmetry properties lead to the following conditions (see Appendix of [7]):

$$\beta_0'=\beta_0'''=\psi_0'=\psi_0'''=0, \tag{24}$$

$$\beta_1=\beta_1''=\psi_1=\psi_1''=0, \tag{25}$$

$$\beta_2=\beta_2'=\beta_2'''=\psi_2=\psi_2'=\psi_2'''=0, \tag{26}$$

$$\beta_n=\beta_n'=\ \ldots\ =\beta_n^{(n-1)}=\psi_n=\psi_n'=\ \ldots\ =\psi_n^{(n-1)}=0\quad\text{for}\quad n\geq 3. \tag{27}$$

Optimization Equations

Based on (16)–(27), we can write ω_n^2 as the following Rayleigh quotient:

$$\omega_n^2=\Gamma_n\Big/\int_0^1\beta_n^2 x\,dx \tag{28}$$

where

$$\Gamma_n=\int_0^1\left\{(L_n\beta_n)^2-(L_n\psi_n)^2-\frac{2}{x}\left(\psi_n'-\frac{n^2}{x}\psi_n\right)g\beta_n'+\frac{2n^2}{x}\left(\frac{1}{x}\psi_n\right)'g\,\beta_n\right\}x\,dx+(1+\nu)\left[(\psi_n')^2-n^2\left(\frac{1}{x}\psi_n^2\right)'\right]_{x=1}+(\nu-1)\left[\left(\beta_n'\right)^2\right]_{x=1} \tag{29}$$

(see [9]). It is convenient to normalize the vibration mode $\beta_n(x)$ by

$$\int_0^1\beta_n^2 x\,dx=1. \tag{30}$$

Our objective is to determine the design function $g(x)$ which maximizes the fundamental vibration frequency for a given value of the surface area parameter λ. In some cases the optimal design is associated with a double frequency, say

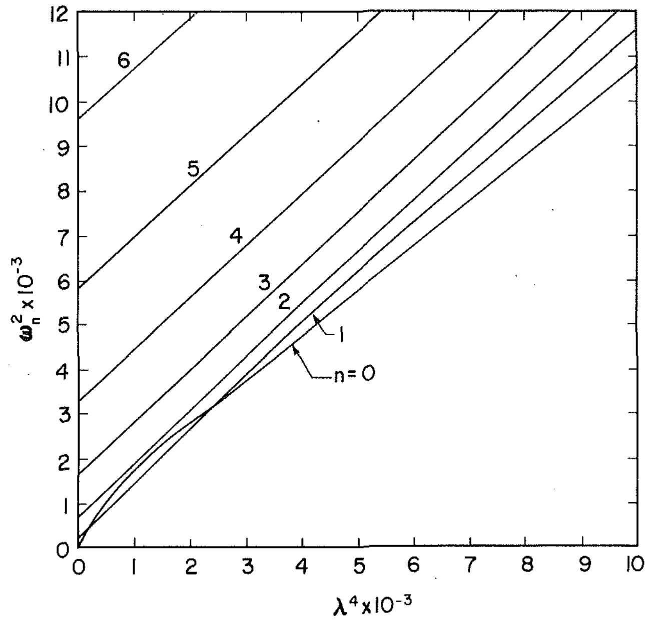

Fig. 1 Vibration frequencies for simply supported spherical shells

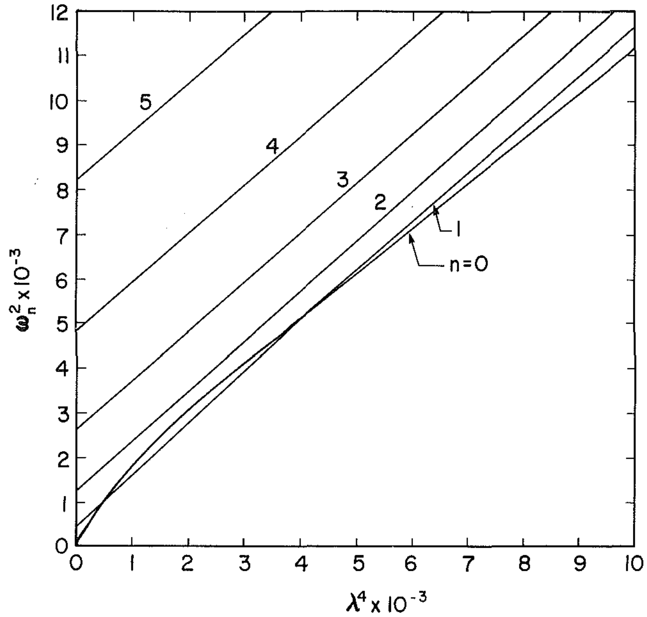

Fig. 2 Vibration frequencies for clamped spherical shells

$\omega_n = \omega_m$, and involves two independent modes, $\beta_n(x)$ and $\beta_m(x)$. Hence we allow for this possibility and consider a bimodal formulation [5].

The objective function is given by (28) and the constraints by $\omega_n = \omega_m$, equation (10), and (30) for $\beta_n(x)$ and $\beta_m(x)$. With the use of Lagrange multipliers ϵ, γ, μ_n, and μ_m, we construct the following augmented functional:

$$\Gamma_n^* = \Gamma_n - \epsilon(\Gamma_n - \Gamma_m) - \gamma\left[\int_0^1 g^2\, x\, dx - (\lambda^4/4)\right]$$

$$- \mu_n\left(\int_0^1 \beta_n^2\, x\, dx - 1\right) - \mu_m\left(\int_0^1 \beta_m^2 x\, dx - 1\right). \quad (31)$$

If the fundamental frequency ω_n^2 is unimodal, then $\epsilon = 0$ and $\mu_m = 0$.

Stationarity of Γ_n^* with respect to $\beta_n(x)$ and $\psi_n(x)$ leads to (16) and (17), respectively, with the appropriate boundary conditions, and to $\mu_n = 0$. A similar result occurs for $\beta_m(x)$ and $\psi_m(x)$. Stationarity of Γ_n^* with respect to the design function $g(x)$ leads to the optimality condition

$$\gamma g(x) = (1-\epsilon)s_n(x) + \epsilon s_m(x) \quad (32)$$

where

$$s_n(x) = \frac{n^2}{x}\left(\frac{\psi_n}{x}\right)'\beta_n - \frac{1}{x}\left(\psi_n{}' - \frac{n^2}{x}\psi_n\right)\beta_n{}' \quad (33)$$

and $s_m(x)$ is defined similarly. If the edge of the shell is clamped, we note that (19), (23), and (32) imply that $g(1) = 0$, i.e., the optimal form has zero slope at the clamped edge.

Solution Procedure

The solution procedure involves successive iterations. A spherical shell is usually taken as the initial form, and modifications are based on the optimality condition (32), which provides a formula for the design function $g(x)$. The procedure is concluded when the increase in fundamental frequency for successive designs becomes insignificant.

We specify Poisson's ratio ν, the surface area parameter λ, and the boundary conditions. For the design function $g_i(x)$ at the ith iteration, we solve (16) and (17) with various values of n to determine the fundamental frequency ω_n and the corresponding functions $\beta_n(x)$ and $\psi_n(x)$. The numerical technique used here is described in the Appendix of Part 3 [7]. The next step depends on whether ω_n is distinct or a double eigenvalue.

For the unimodal case (distinct ω_n), $\epsilon = 0$ in (32). If we write $\hat{g}(x) = s_n(x)/\gamma$, then γ^2 is determined by substituting $\hat{g}(x)$ into (10), the sign of γ is chosen so that the shell form corresponding to $\hat{g}(x)$ lies above the base plane, and the design function to be used in the next iteration is given by the recurrence relation

$$g_{i+1}(x) = b[g_i(x) + k\hat{g}(x)]. \quad (34)$$

In (34), k is a parameter which we choose to facilitate convergence of the procedure (e.g., $k = 0.1$), and b is chosen such that $g_{i+1}(x)$ satisfies (10).

For the bimodal case, with $\omega_n = \omega_m$, we denote $g(x) = \hat{g}(x)$ in (32). Substitution of $\hat{g}(x)$ into (10) yields one equation in γ and ϵ. A second equation is obtained by substituting $\hat{g}(x)$ into the bimodal condition $\Gamma_n = \Gamma_m$, along with the functions $\beta_n(x)$, $\psi_n(x)$, $\beta_m(x)$, and $\psi_m(x)$ associated with $g_i(x)$; this is based on the need to increase the two equal frequencies simultaneously. The two equations can be solved for γ and ϵ, with $0 < \epsilon < 1$. We then use those values in $\hat{g}(x)$ and apply (34) to determine the design function for the next iteration.

Results

In all the numerical results, we use $\nu = 1/3$. For comparison purposes, the frequencies of a spherical shell were determined first, i.e., we solved (16) and (17) with $g(x) = -\lambda^2 x$. The resulting values of ω_n^2 for small n are plotted versus λ^4 in Figs. 1 and 2 for simply supported and clamped edge conditions, respectively. (We recall that ω_n denotes the lowest frequency for that value of n.) The frequency ω_0, associated with axisymmetric vibration, is the fundamental one for sufficiently small and sufficiently large values of λ, while ω_1, associated with a mode having a single nodal diameter, is the fundamental frequency in an intermediate range. For $n \geq 1$, the relation between ω_n^2 and λ^4 is almost linear, and is approximately given by

$$\omega_n^2 = \omega_{np}^2 + 1.1\lambda^4, \quad n \geq 1, \quad (35)$$

where ω_{np} is the frequency for the corresponding plate[1]. We

[1] Using another shell theory, Soedel [11] obtained a formula similar to (35) but with the coefficient 1.1 replaced by unity, independent of the value of ν.

Table 1 Optimal forms $\bar{z}(x)$

x	Spherical	$S;\lambda=\sqrt{10}$	$S;\lambda=7$	$C;\lambda=4$	$C;\lambda=10$
0.0	1.00	1.09	1.02	1.13	1.03
0.1	0.99	1.08	1.01	1.11	1.02
0.2	0.96	1.04	0.99	1.07	0.97
0.3	0.91	0.97	0.94	0.99	0.89
0.4	0.84	0.88	0.86	0.88	0.77
0.5	0.75	0.76	0.77	0.74	0.59
0.6	0.64	0.63	0.65	0.58	0.42
0.7	0.51	0.48	0.50	0.39	0.42
0.8	0.36	0.32	0.34	0.21	0.34
0.9	0.19	0.16	0.17	0.07	0.14
1.0	0.00	0.00	0.00	0.00	0.00
Frequency ratio		1.03	1.03	1.21	1.42

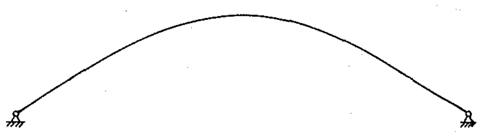

Fig. 3 Optimal form for maximum fundamental frequency: simply supported edge, $\lambda=\sqrt{10}$

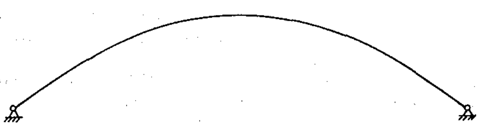

Fig. 4 Optimal form for maximum fundamental frequency: simply supported edge, $\lambda=7$

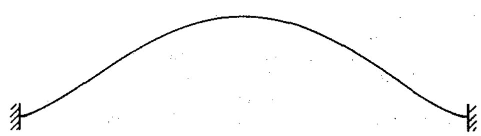

Fig. 5 Optimal form for maximum fundamental frequency: clamped edge, $\lambda=4$

Fig. 6 Optimal form for maximum fundamental frequency: clamped edge, $\lambda=10$

note that the dimensional frequency, say Ω_n, is related to the nondimensional frequency ω_n by $\Omega_n=\omega_n\sqrt{D/(\rho h a^4)}$.

Optimal forms were determined for simply supported shells with $\lambda=\sqrt{10}$ and $\lambda=7$, and for clamped shells with $\lambda=4$ and $\lambda=7$. They are listed in Table 1 (where S denotes simply supported and C denotes clamped) and depicted in Figs. 3–6, using the nondimensional function

$$\bar{z}(x)=2z(x)/(\lambda^2). \qquad (36)$$

The spherical form, $\bar{z}(x)=1-x^2$, is also listed in Table 1, as well as the ratio between the fundamental frequency for the optimal form and that for the spherical cap.

For the simply supported shell with $\lambda=\sqrt{10}$, the spherical shell has $\omega_0=16.2$ and $\omega_1=17.9$, while the optimal form has $\omega_0=16.7$ and $\omega_1=18.0$. A bimodal solution did not occur in this case. For $\lambda=7$, we have $\omega_0=55.9$ and $\omega_1=56.5$ for the spherical shell. After a number of iterations, ω_0 and ω_1 coalesced and then were raised further using the bimodal formulation, finally reaching the value $\omega_0=\omega_1=57.8$. Relative to the spherical cap, the increase in fundamental frequency for these two cases is only 3 percent. The optimal forms, shown in Figs. 3 and 4, are convex functions near the edge.

For both cases with a clamped edge, the optimal solutions are bimodal. When $\lambda=4$, the spherical shell has $\omega_0=24.8$ and $\omega_1=27.5$, while the optimal form has $\omega_0=\omega_1=30.1$. When $\lambda=10$, the spherical shell has $\omega_0=106$ and $\omega_1=108$, while the optimal form has $\omega_0=\omega_1=150$. Here the improvement in fundamental frequency is significant. Both optimal forms, shown in Figs. 5 and 6, have zero slope at the edge, as mentioned earlier. Also, the optimal form for the case $\lambda=10$ has a plateau around $x=0.65$. Perspectives of these optimal shells are depicted in Figs. 7 and 8.

Concluding Remarks

We have formulated the problem of maximizing the fundamental vibration frequency of a shallow shell when its form is variable. The shell has a given circular boundary and a given value of the surface area parameter λ. Numerical solutions have been obtained for two cases with a simply supported boundary and two cases with a clamped boundary. For a spherical cap, the fundamental vibration mode (i.e., the mode associated with the lowest frequency) is either axisymmetric ($n=0$) or has one nodal diameter ($n=1$). In the four examples analyzed here, the optimal form either has an

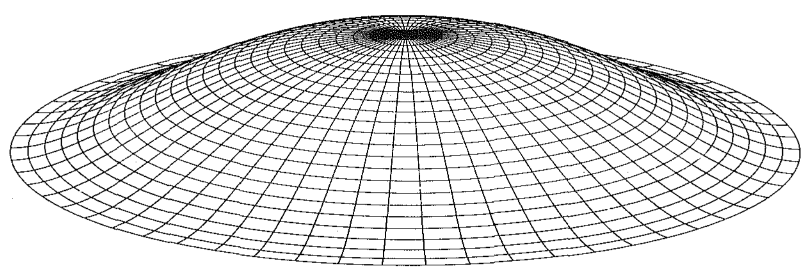

Fig. 7 Geometry of optimal shell for clamped edge, $\lambda=4$

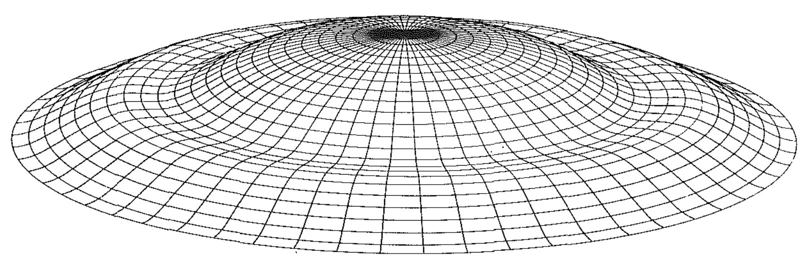

Fig. 8 Geometry of optimal shell for clamped edge, $\lambda=10$

axisymmetric fundamental vibration mode or a bimodal solution involving the $n = 0$ and $n = 1$ modes (i.e., $\omega_0 = \omega_1$).

The slope of the shell at the boundary is not specified, even when the boundary is clamped. If it were, the optimal form would have a discontinuous slope at the edge. All the optimal forms obtained in this part are convex in regions adjacent to the edge (see Figs. 3-6). For a clamped edge, the slope at the edge is zero, and the fundamental frequency is significantly higher than that for the corresponding spherical cap.

It is interesting to compare the optimal shell forms obtained here with the optimal arch forms determined in [4]. For maximum fundamental frequency, the optimal form of a simply supported arch is sinusoidal, which is almost identical to the form in Fig. 3, while the optimal clamped arch is somewhat similar in form to Fig. 5. For the arch, the optimal fundamental frequency corresponds to a symmetric mode for sufficiently small values of the length parameter, and an antisymmetric mode for sufficiently large values; a bimodal solution only occurs at the transition value. Here, for shallow shells, it appears that a bimodal solution will govern for most values of the geometric parameter λ.

The iterative solution procedure was effective in modifying a design to improve the objective function. The direction of the modification was based on the optimality condition (32). Typically, 10-15 iterations were sufficient to obtain satisfactory convergence of the fundamental frequency. At some steps, the parameter k in (34) has to be altered to avoid overshooting the maximum point. Since the governing set of equations is nonlinear, we cannot guarantee that the solution procedure leads to the global optimum. In a few cases, however, we chose different initial designs and arrived at the same final design.

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