# An homogenization result in $W^{1,p} \times L^q$


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**Abstract**

An integral representation result is provided for the $\Gamma$-limit of integral functionals arising in homogenization problems for the study of coherent thermochemical equilibria in multiphase solids.



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## 1 Introduction

The target of this paper is the treatment of a single scale homogenization problem, formulated in terms of an integral energy, occurring in the description of elastic materials which exhibit an overall behavior depending not only on the strain but also on the chemical composition.

Homogenization theory deals with composites whose overall behavior is established taking into account their microstructure. Indeed such materials are characterized by the fact that they contain two or more several mixed constituents, that in a first approximation, can be thought to be periodically distributed, but even more general dependences can be considered. The size of the heterogeneities is very small compared with the dimension of the composite: the ratio between the microscopic and the macroscopic dimensions is the 'so called' homogenization parameter $\varepsilon$.

In detail we study the asymptotic behavior, as $\varepsilon \to 0^+$ of integral functionals of the form

$$\int_\Omega f_\varepsilon\left(x, \nabla u(x), v(x)\right) dx \tag{1.1}$$

where $f_\varepsilon$ is some oscillating integrand, $\Omega$ is an open bounded subset in $\mathbb{R}^N$ and $\nabla u$ represents the deformation gradient of some field $u$ belonging to some Sobolev space whose fields are $p$-th power summable, and $v$ is an $L^q$-function, (not necessarily scalar valued in our analysis), taking into account the chemical composition of the material.

This type of integrals find applications not only in the study of coherent thermochemical equilibria for multiphase solids as in [29, 28], but even in the 'directors' theory in Elasticity, (cf. [37] in the framework of thin structures), and, when $u$ is a field of Bounded Variation, the integrand can be intended as a $TV$ model (total variation model) for image decomposition (see [40], [43]).

For energies growing linearly without considering the chemical composition of the material, these kind of homogenization problems have been sucessively studied in [7], [23] and in [9] with an extra surface energy term.

To understand the asymptotic behavior of the (almost) minimizers of energies in the form of (1.1), we perform a $\Gamma-$ convergence analysis (see [11, 22] for a detailed description of this subject), showing that the $\Gamma$-limit still admits an integral representation. The presence of the two vector fields with different growths lead us to the crucial notion of quasiconvexity-convexity which requires an appropriate Lipschitz continuity property (see Proposition 2.11).


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Similar problems, when the integrands depend just on one field and exhibit a periodic behavior in the spatial variable, i.e., $f_\varepsilon(x,\xi,b) = f\left(\frac{x}{\varepsilon},\xi\right)$ or $f\left(\frac{x}{\varepsilon},b\right)$ have been studied by many authors with different sets of assumptions and techniques. In the first case for energies with superlinear growth, i.e., $\frac{1}{C}|\xi|^p - C \leq f(x,\xi) \leq C(|\xi|^p+1)$, $p>1$ we refer to pioneering papers [38] and [17] (where in the scalar case $f = f(y,\xi)$ is assumed to be convex with respect to $\xi$). The vectorial case is presented in the independent works of [10] and [41]. A wide literature has been produced since the present time with different methods, among the others we recall the papers [1] where the two-scale convergence method (see [42]) has been adopted, in the scalar setting, [18] with the approach of the unfolding method (see [19, 20]) and recently [30] where the unfolding method has been used to deal with linear differential constraints. The case when the function $f_\varepsilon$ is periodic in the first variable and it has just dependence on $b$ has been treated in [39], adopting the two-scale convergence method.

For what concerns the multiple scale case, for example, $f_\varepsilon(x,\xi) = f\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\xi,\right)$ we refer, in particular to [5, 12, 14, 36], (see also [3] in the realm of thin structures). In details, in [5], with very mild hypotheses a characterization as $\varepsilon \to 0^+$ of a family of integral functionals of the type $\int_\Omega f\left(x,\frac{x}{\varepsilon},\nabla u(x)\right)dx$ where $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ and $p>1$ is obtained, using $\Gamma-$convergence techniques combined with techniques of two-scale convergence. Moreover, we recall [4, 6] where the approach through Young measures has been adopted.

Besides we provide an integral representation theorem for the $\Gamma$-limit (up to a subsequence) of the functionals in (1.1) (see Theorem 3.2), generalizing the results of [11, 16]. To deal with the presence of the new vector field which is in the $L^q$ space, we use an approximation argument which allows to work with piecewise constant functions. We emphasize that we are mainly concerned with a single scale model, i.e., $f_\varepsilon(x,\xi,b) = f\left(\frac{x}{\varepsilon},\xi,b\right)$, leaving to a forthcoming paper the multiple scales case. The case $p=q$ has already been studied in [13], in the realm of $\mathcal{A}$-quasiconvexity, even if under the continuity assumption on $f$ on all the variables. In the present work, we consider any $p,q>1$ and we only require $f$ to be a Carathéodory integrand satisfying

(H$_1$) $f(\cdot,\xi,b)$ is $Q-$periodic, for all $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$, ($Q$ being the unit cube in $\mathbb{R}^N$);

(H$_2$) there exist $p,\ q>1$ and a positive constant $C$ such that

$$\frac{1}{C}(|\xi|^p+|b|^q) - C \leq f(x,\xi,b) \leq C(1+|\xi|^p+|b|^q),$$

for a.e. $x\in\Omega$ and for every $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$.

For $\varepsilon>0$, we define the family of functionals $F_\varepsilon : L^p(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$ by

$$F_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega f\left(\tfrac{x}{\varepsilon},\nabla u\left(x\right),v\left(x\right)\right)dx & \text{if } (u,v)\in W^{1,p}(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise.}\end{cases} \tag{1.2}$$

We are interested in studying the asymptotic behavior of $F_\varepsilon$ as $\varepsilon\to0^+$, using $\Gamma$-convergence, i.e., we want to show that the following functionals

$$\mathcal{F}^-_{\{\varepsilon\}}(u,v) := \inf\left\{\liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon,v_\varepsilon) : u_\varepsilon\to u \text{ in } L^p(\Omega;\mathbb{R}^d), v_\varepsilon\rightharpoonup v \text{ in } L^q(\Omega;\mathbb{R}^m)\right\}$$

$$\mathcal{F}^+_{\{\varepsilon\}}(u,v) := \inf\left\{\limsup_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon,v_\varepsilon) : u_\varepsilon\to u \text{ in } L^p(\Omega;\mathbb{R}^d), v_\varepsilon\rightharpoonup v \text{ in } L^q(\Omega;\mathbb{R}^m)\right\}$$

coincide, denoting the common value by $\mathcal{F}_{\{\varepsilon\}}$, the $\Gamma$-limit of $\{F_\varepsilon\}$, we will provide an integral representation for it. Indeed, cf. Theorem 1.1, we will show that it coincides with the functional $F_{\rm hom} : L^p(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$, such that

$$F_{\rm hom}(u,v) := \begin{cases} \displaystyle\int_\Omega f_{\rm hom}(\nabla u\left(x\right),v\left(x\right))\,dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases}$$

where the energy density $f_{\hom}$ is defined as

$$
\begin{aligned}
f_{\hom}(\xi,b):=\lim_{T\to\infty}\inf\Bigg\{&\tfrac{1}{T^N}\int_{(0,T)^N} f(y,\xi+\nabla\varphi(y),b+\eta(y))\,dy:\\
&\varphi\in W_0^{1,p}((0,T)^N;\mathbb{R}^d),\ \eta\in L^q((0,T)^N;\mathbb{R}^m):\int_{(0,T)^N}\eta(y)\,dy=0\Bigg\}.
\end{aligned}
\tag{1.3}
$$

Using the classical techniques of Γ-convergence (see [22]), integral representation theorems, together with the local Lipschitz continuity properties of integrands (see formula (2.5) below) we prove our main result.

**Theorem 1.1** *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and let $f:\Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\to\mathbb{R}$ be a Carathéodory function. Let $\{F_\varepsilon\}$ be the family of functionals defined in (1.2). Under the hypotheses $(H_1)$ and $(H_2)$, the sequence $\{F_\varepsilon\}$ Γ-converges to $F_{\hom}$, as $\varepsilon\to 0^+$, i.e.,*

$$\mathcal{F}_{\{\varepsilon\}}(u,v)=F_{\hom}(u,v),\ \forall\,(u,v)\in L^p(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m).$$

The proof is achieved as an application of a compactness result for Γ-limits of general families of integral functionals depending on two fields, one in $W^{1,p}$ through its gradient, and the other in $L^q$ (cf. Theorem 3.2). Moreover, we call the attention that the existence of the Γ-limit for the whole family in (1.2) deeply relies on a fundamental estimate suited to the present context (see Remark 3.6). Finally we apply an integral representation result proven in Theorem 3.1.

Moreover, in order to achieve Theorem 1.1 and to characterize the convexity properties of the limit energy density $f_{\hom}$ in (1.3), namely its quasiconvexity-convexity in the last two variables (see Definition 2.9) we prove the relaxation result below.

**Theorem 1.2** *Let $1<p<\infty$ and $1<q<\infty$ and assume that $f:\Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\to\mathbb{R}$ is a Carathéodory function that satisfies*

$$\frac{1}{C}\left(|\xi|^p+|b|^q\right)-C\le f(x,\xi,b)\le C\left(1+|\xi|^p+|b|^q\right)$$

*for a.e. $x\in\Omega$, for every $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$ and for some $C>0$.*

*Then for every $u\in W^{1,p}\left(A;\mathbb{R}^d\right)$, $v\in L^q(A;\mathbb{R}^m)$ and $A\in\mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u,v;A)=\int_A QCf(x,\nabla u(x),v(x))\,dx,$$

*where $\mathcal{F}(u,v;A)$ stands for the sequential lower semicontinuous envelope with respect to $W_w^{1,p}\times L_w^q$ convergence, namely*

$$\mathcal{F}(u,v;A)=\inf\left\{\liminf_{n\to+\infty}\int_A f(x,\nabla u_n(x),v_n(x))dx: u_n\rightharpoonup u \text{ in } W^{1,p}(A;\mathbb{R}^d), v_n\rightharpoonup v \text{ in } L^q(A;\mathbb{R}^m)\right\}, \tag{1.4}$$

*and $QCf$ stands for the quasiconvex-convex envelope of $f$ with respect to the last two variables (cf. (2.3)).*

**Remark 1.3** *For what concerns the case $p=1$, the same result can be achieved using the arguments of [13, Theorem 1.3], even if it is not known a pripori that the functional in (1.4) is sequentially weakly lower semicontinuous with respect to the $W_w^{1,1}\times L_w^q$ topology.*

*On the other hand, if we introduce the sequentially weakly lower semicontinuous envelope of the functional*

$$(u,v,A)\mapsto\int_A f(x,\nabla u(x),v(x))\,dx$$

*with respect to the $W_w^{1,1}\times L_w^q$ topology, call this functional $\mathcal{G}$, one can get that $\mathcal{F}$ and $\mathcal{G}$ coincide, so $\mathcal{F}$ is indeed the sequential lower semicontinuous envelope. To see this we use the sequential weak lower semicontinuity of*

$$(u,v,A)\mapsto\int_A QCf(x,\nabla u(x),v(x))\,dx,$$

*which follows as in the first part of the proof of Theorem 1.2, implying that $\mathcal{F}$ is sequentially weakly lower semicontinuous, and finally from the sequential weak lower semicontinuity of $\mathcal{G}$ one gets $\mathcal{G}\le\mathcal{F}$ and thus, by definition of $\mathcal{G}$, the identity.*

Theorem 1.2 provides also an extension of the relaxation theorem in [29] to the case where $f$ exhibits also dependence on $x$, (see also [44] for the homogeneous constrained case).

The paper is organized as follows. In section 2 we recall the notion of $\Gamma-$convergence and present standard results on this theory. A local Lipschitz property inherited by quasiconvex-convex functions which satisfies $(\mathrm{H}_2)$ is derived. In section 3 we provide an integral representation result for functionals depending on the strain and the chemical composition in the spirit of that obtained in the nonlinear elastic setting by Buttazzo and Dal Maso in [16] to local functionals defined in $W^{1,p}\left(\Omega;\mathbb{R}^d\right)\times L^q\left(\Omega;\mathbb{R}^m\right)$. This result is applied to obtain an integral representation for a general family of functionals (see Theorem 3.2 below). In section 4, Theorem 1.2 and Theorem 1.1 are proved as an application of Theorem 3.2 .

# 2 Preliminaries

This section is devoted to recall and prove concepts and results that will be exploited throughout the paper.

In the following $\Omega\subset\mathbb{R}^N$ is an open bounded set and we denote by $\mathcal{A}\left(\Omega\right)$ the family of all open subsets of $\Omega$. The unit cube in $\mathbb{R}^N$, $\left(-\frac{1}{2},\frac{1}{2}\right)^N$, is denoted by $Q$ and we set $Q\left(x_0,\varepsilon\right):=x_0+\varepsilon Q$ for $\varepsilon>0$. We write $B_\rho\left(x\right)$ for the open ball in $\mathbb{R}^N$ centered at $x$ with radius $\rho>0$.

The constant $C$ may vary from line to line.

## 2.1 Γ-convergence

First we remind De Giorgi's notion of Γ-convergence and some of its properties (see De Giorgi and Dal Maso [24] and De Giorgi and Franzoni [25]). For a more extended treatment of the subject we refer to the books [11] and [22].

Let $(X,d)$ be a metric space.

**Definition 2.1 (Γ-convergence for a sequence of functionals)** *Let $\{F_n\}$ be a sequence of functionals defined on $X$ with values in $\overline{\mathbb{R}}$. The functional $F:X\to\overline{\mathbb{R}}$ is said to be the $\Gamma-\liminf$ (resp. $\Gamma-\limsup$) of $\{F_n\}$ with respect to the metric $d$ if for every $u\in X$*

$$F(u)=\inf\left\{\liminf_{n\to\infty}F_n(u_n):u_n\in X,u_n\to u\ in\ X\right\}\ \ (resp.\ \limsup_{n\to\infty}).$$

*Thus we write*

$$F=\Gamma-\liminf_{n\to\infty}F_n\ \ (resp.\ F=\Gamma-\limsup_{n\to\infty}F_n).$$

*Moreover, the functional $F$ is said to be the $\Gamma-$limit of $\{F_n\}$ if*

$$F=\Gamma-\liminf_{n\to\infty}F_n=\Gamma-\limsup_{n\to\infty}F_n,$$

*and we may write*

$$F=\Gamma-\lim_{n\to\infty}F_n.$$

For every $\varepsilon>0$, let $F_\varepsilon$ be a functional over $X$ with values in $\overline{\mathbb{R}}$, $F_\varepsilon:X\to\overline{\mathbb{R}}$.

**Definition 2.2 (Γ-convergence for a family of functionals)** *A functional $F:X\to\overline{\mathbb{R}}$ is said to be the Γ-liminf (resp. Γ-limsup or Γ-limit) of $\{F_\varepsilon\}$ with respect to the metric $d$, as $\varepsilon\to0^+$, if for every sequence $\varepsilon_n\to0^+$*

$$F=\Gamma-\liminf_{n\to\infty}F_{\varepsilon_n}\,(resp.\ F=\Gamma-\limsup_{n\to\infty}F_{\varepsilon_n}\ \ or\ F=\Gamma-\lim_{n\to\infty}F_{\varepsilon_n}),$$

*and we write*

$$F=\Gamma-\liminf_{\varepsilon\to0^+}F_\varepsilon\,(resp.\ F=\Gamma-\limsup_{\varepsilon\to0^+}F_\varepsilon\ \ or\ F=\Gamma-\lim_{\varepsilon\to0^+}F_\varepsilon).$$

Next we state the Urysohn property for Γ-convergence in a metric space.

**Proposition 2.3** *Given $F : X \to \overline{\mathbb{R}}$ and $\varepsilon_n \to 0^+$, $F = \Gamma - \lim_{n\to\infty} F_{\varepsilon_n}$ if and only if for every subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$ there exists a further subsequence $\left\{\varepsilon_{n_{j_k}}\right\} \equiv \{\varepsilon_k\}$ such that $\{F_{\varepsilon_k}\}$ $\Gamma-$converges to $F$.*

In addition, if the metric space is also separable the following compactness property holds.

**Proposition 2.4** *Each sequence $\varepsilon_n \to 0^+$ has a subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$ such that $\Gamma - \lim_{j\to\infty} F_{\varepsilon_j}$ exists.*

**Proposition 2.5** *If $F = \Gamma - \liminf_{\varepsilon\to 0^+} F_\varepsilon$ (or $\Gamma - \limsup_{\varepsilon\to 0^+} F_\varepsilon$) then $F$ is lower semicontinuous (with respect to the metric $d$). Clearly, if $F = \Gamma - \lim_{\varepsilon\to 0^+} F_\varepsilon$ then $F$ is lower semicontinuous.*

**Definition 2.6** *A family of functionals $\{F_\varepsilon\}$ is said to be equi-coercive if for every real number $\lambda$ there exists a compact set $K_\lambda$ in $X$ such that for each sequence $\varepsilon_n \to 0^+$,*

$$\{u \in X : F_{\varepsilon_n}(u) \leq \lambda\} \subset K_\lambda \text{ for every } n \in \mathbb{N}.$$

The next result states that $\Gamma$-convergence is a variational convergence, in fact under suitable compactness conditions, there is convergence of minimizers (or almost minimizers) of a family of equi-coercive functionals to the minimum of the limiting functional.

**Theorem 2.7 (Fundamental Theorem of $\Gamma-$convergence)** *If $\{F_\varepsilon\}$ is a family of equi-coercive functionals on $X$ and if*

$$F = \Gamma - \lim_{\varepsilon\to 0^+} F_\varepsilon,$$

*then the functional $F$ has a minimum on $X$ and*

$$\min_{u\in X} F(u) = \lim_{\varepsilon\to 0^+} \inf_{u\in X} F_\varepsilon(u).$$

*Moreover, given $\varepsilon_n \to 0^+$ and $\{u_n\}$ a converging sequence such that*

$$\lim_{n\to\infty} F_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \inf_{u\in X} F_{\varepsilon_n}(u), \tag{2.1}$$

*then its limit is a minimum point for $F$ on $X$.*

If (2.1) holds, then $\{u_n\}$ is said to be a sequence of almost-minimizers for $F$.

Now we recall the notion of $\overline{\Gamma}-$convergence for sequences of functionals on a suitable rich family of sets. Let $\mathcal{A}_0(\Omega)$ be the family of all open subsets of $\Omega$ compactly included in $\Omega$ and $\mathcal{E}(\Omega)$ any class of subsets of $\Omega$ containing $\mathcal{A}_0(\Omega)$.

**Definition 2.8** *We say that $\{F_n\}$ $\overline{\Gamma}-$converges to $F$ in $X$ if $F$ is the inner regular envelope of both $\Gamma - \liminf_{n\to\infty} F_n$ and $\Gamma - \limsup_{n\to\infty} F_n$, this means*

$$\begin{aligned} F(u; A) &= \sup\left\{\Gamma - \liminf_{n\to\infty} F_n(u; B) : B \in \mathcal{E}(\Omega),\ B \subset\subset A\right\} \\ &= \sup\left\{\Gamma - \limsup_{n\to\infty} F_n(u; B) : B \in \mathcal{E}(\Omega),\ B \subset\subset A\right\} \end{aligned}$$

*for any $A \in \mathcal{A}(\Omega)$.*

## 2.2 Quasiconvexity-convexity and Lipschitz continuity

Following [29, 37], see also [28] and [27] we recall the definition of quasiconvexity-convexity.

**Definition 2.9** *A Borel measurable function* $h : \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ *is said to be quasiconvex-convex if there exists a bounded open set* $D$ *of* $\mathbb{R}^N$ *such that*

$$h(\xi, b) \leq \frac{1}{|D|} \int_D h(\xi + \nabla\varphi(x), b + \eta(x))\, dx, \tag{2.2}$$

*for every* $(\xi, b) \in \mathbb{R}^{d\times N}\times\mathbb{R}^m$*, for every* $\eta \in L^\infty(D;\mathbb{R}^m)$*, with* $\int_D \eta(x)\,dx = 0$ *and for every* $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^d\right)$*.*

If $h : \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is any given Borel measurable function bounded from below, it can be defined the quasiconvex-convex envelope of $h$, that is the largest quasiconvex-convex function below $h$:

$$QCh(\xi, b) := \sup\{g(\xi, b) :\ g \leq h,\ g \text{ quasiconvex-convex}\}.$$

Moreover, by Theorem 4.16 in [37]

$$QCh(\xi, b) = \inf\left\{\frac{1}{|D|}\int_D h(\xi + \nabla\varphi(x), b + \eta(x))\, dx :\ \varphi \in W_0^{1,\infty}(D;\mathbb{R}^d), \eta \in L^\infty(D;\mathbb{R}^m), \int_D \eta(x)dx = 0\right\}. \tag{2.3}$$

**Remark 2.10** *i) It can be easily proved that, if* $h$ *is quasiconvex-convex, then, both condition (2.2) and (2.3) hold for any bounded open set* $D \subset \mathbb{R}^N$*.*

*ii) It can be also showed that if* $h$ *satisfies a growth condition of the type* $(H_2)$ *then in (2.2) and (2.3) the spaces* $L^\infty$ *and* $W_0^{1,\infty}$ *can be replaced by* $L^q$ *and* $W_0^{1,p}$*, respectively.*

*iii) In the remainder of the paper when we will say that a function* $f$*, possibly defined in* $\Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$*, is quasiconvex-convex, this property has to be understood with respect to the last two variables.*

*iv) Any function quasiconvex-convex is separately convex.*

Next we state and prove the local Lipschitz property inherited by a separately convex function $f$ which satisfies a $p - q$ growth condition. We follow along the lines the proof of Proposition 2.32 in [21].

**Proposition 2.11** *Let* $f : \mathbb{R}^{d\times N} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ *be a separately convex function verifying the growth condition*

$$|f(\xi, b)| \leq c(1 + |b|^q + |\xi|^p),\ \forall\ (\xi, b) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m \tag{2.4}$$

*for some* $p,\ q > 1$*.*

*Then, denoting by* $p'$ *and* $q'$*, the conjugate exponent of* $p$ *and* $q$*, respectively, there exists a constant* $\gamma > 0$ *such that*

$$|f(\xi, b) - f(\xi', b')| \leq \gamma\left(1 + |b|^{q/p'} + |\xi|^{p-1} + |\xi'|^{p-1}\right)|\xi - \xi'| + \gamma\left(1 + |b|^{q-1} + |b'|^{q-1} + |\xi'|^{p/q'}\right)|b - b'| \tag{2.5}$$

*for every* $b, b' \in \mathbb{R}^m$ *and for every* $\xi, \xi' \in \mathbb{R}^{d\times N}$*.*

**Remark 2.12** *By Remark 2.10 iv) this result applies, in particular, to quasiconvex-convex functions satisfying the growth condition (2.4).*

**Proof.** For any $(\xi, b),\ (\xi', b') \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$ we have

$$|f(\xi, b) - f(\xi', b')| \leq |f(\xi, b) - f(\xi', b)| + |f(\xi', b) - f(\xi', b')|.$$

Therefore to achieve the Lipschitz condition stated in the theorem, it is enough to estimate each of the two terms appearing in the right-hand of the previous inequality.

We recall that given any convex function $g : \mathbb{R} \to \mathbb{R}$, it results for every $\lambda > \mu > 0$ and for every $t \in \mathbb{R}$, that

$$\frac{g(t \pm \mu) - g(t)}{\mu} \leq \frac{g(t \pm \lambda) - g(t)}{\lambda}.$$

We will apply these inequalities to $f$, for a convenient choice of $\lambda$ and $\mu$, when all but one of the components of $(\xi, b)$ are fixed. Let $\hat{\xi_1} := (\xi_2, \dots, \xi_{d\times N})$ and define for every $b \in \mathbb{R}^m$ and $t \in \mathbb{R}$

$$g(t) := f((t, \hat{\xi_1}), b).$$

Choose $\lambda := 1 + |\xi| + |\xi'| + |b|^{q/p}$ and $\mu := \xi'_1 - \xi_1$ (where without loss of generality it has been assumed that $\xi'_1 > \xi_1$). In order to evaluate $|g(\xi_1) - g(\xi'_1)|$ we observe that

$$\begin{aligned} g(\xi'_1) - g(\xi_1) = g(\xi_1 + (\xi'_1 - \xi_1)) - g(\xi_1) &\leq (\xi'_1 - \xi_1)\frac{g(\xi_1 + \lambda) - g(\xi_1)}{\lambda} \\ &\leq |\xi'_1 - \xi_1| \frac{c(1 + |b|^q + |(\xi_1 + \lambda, \hat{\xi_1})|^p) + c(1 + |b|^q + |\xi|^p)}{\lambda} \\ &\leq C(1 + |b|^{q/p'} + |\xi|^{p-1} + |\xi'|^{\,p-1})|\xi_1 - \xi'_1|, \end{aligned}$$

where we have used the $p - q$ growth condition (2.4).

Arguing in the same way, one deduce that

$$g(\xi_1) - g(\xi'_1) = g(\xi'_1 - (\xi'_1 - \xi_1)) - g(\xi'_1) \leq C(1 + |b|^{q/p'} + |\xi|^{p-1} + |\xi'|^{p-1})|\xi_1 - \xi'_1|,$$

hence

$$|g(\xi_1) - g(\xi'_1)| \leq C(1 + |b|^{q/p'} + |\xi|^{p-1} + |\xi'|^{\,p-1})|\xi_1 - \xi'_1|. \tag{2.6}$$

Consequently, since

$$\begin{aligned} f(\xi, b) - f(\xi', b) =& f((\xi_1, \hat{\xi_1}), b) - f((\xi'_1, \hat{\xi_1}), b) + \\ &+ \sum_{i=1}^{d\times N-2} [f(\xi'_1, \dots, \xi'_i, \xi_{i+1}, \xi_{i+2}, \dots, \xi_{d\times N}, b) - f(\xi'_1, \dots, \xi'_i, \xi'_{i+1}, \xi_{i+2}, \dots, \xi_{d\times N}, b)] + \\ &+ f(\xi'_1, \xi'_2, \dots, \xi'_{d\times N-1}, \xi_{d\times N}, b) - f(\xi', b). \end{aligned} \tag{2.7}$$

Applying to each term, in the sum above, the estimate analogous to (2.6) one obtains

$$|f(\xi, b) - f(\xi', b)| \leq C\left(1 + |b|^{q/p'} + |\xi|^{p-1} + |\xi'|^{p-1}\right)|\xi - \xi'|. \tag{2.8}$$

Analogously, let $\hat{b_1} := (b_2, \dots, b_m)$ and define the convex function $h : \mathbb{R} \longrightarrow \mathbb{R}$ by $h(t) := f(\xi', (t, \hat{b_1}))$. Clearly, choosing $\lambda := 1 + |b| + |b'| + |\xi'|^{p/q}$ and $\mu := b'_1 - b_1$, (assuming $b'_1 > b_1$) and arguing as above it results that

$$|h(b_1) - h(b'_1)| \leq C(1 + |b|^{q-1} + \left|b'\right|^{q-1} + |\xi|^{p/q'})|b - b'|.$$

Finally, by splitting the difference $f(\xi', b) - f(\xi', b')$ in $m$ terms as in (2.7) one gets

$$|f(\xi, b) - f(\xi', b)| \leq C\left(1 + |\xi'|^{q-1} + \left|b'\right|^{q-1} + |\xi'|^{p/q'}\right)|b - b'|. \tag{2.9}$$

Putting together (2.8) and (2.9) and choosing suitably the constant $\gamma$ we conclude the proof. ■

# 3 General results

In this section we provide sufficient conditions for which a functional defined in $W^{1,p}(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$ admits an integral representation. Next we apply this result to represent the $\Gamma$-limit of certain sequence of functionals.

## 3.1 Integral representation theorem

In this subsection we prove an integral representation theorem for local functionals defined on the product of Sobolev spaces and the space of $L^q$ functions and on open sets, by following the proof of a classical integral representation result proved by Buttazzo and Dal Maso (see [16] and the monograph of Buttazzo [15]) dealing with functionals defined on Sobolev spaces and open sets.

**Theorem 3.1** *Let $p \geq 1$, $q > 1$ and $F : W^{1,p}\left(\Omega;\mathbb{R}^d\right) \times L^q\left(\Omega;\mathbb{R}^m\right) \times \mathcal{A}\left(\Omega\right) \to \mathbb{R}$ satisfying*

*i) $F$ is local on $\mathcal{A}\left(\Omega\right)$, i.e.*

$$F\left(u, v; A\right) = F\left(\overline{u}, \overline{v}; A\right)$$

*whenever $A \in \mathcal{A}\left(\Omega\right)$, and $u = \overline{u}$, $v = \overline{v}$ a.e. on $A$;*

*ii) $F\left(u, v; \cdot\right)$ is the restriction to $\mathcal{A}\left(\Omega\right)$ of a Radon measure;*

*iii) there exists $C > 0$ such that*

$$\left|F\left(u, v; A\right)\right| \leq C \int_A \left(1 + \left|\nabla u\left(x\right)\right|^p + \left|v\left(x\right)\right|^q\right) dx$$

*for any $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v \in L^q\left(\Omega;\mathbb{R}^m\right)$ and $A \in \mathcal{A}\left(\Omega\right)$;*

*iv) $F$ is translation invariant in $u$, i.e., for every $A \in \mathcal{A}\left(\Omega\right)$, $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v \in L^q\left(\Omega;\mathbb{R}^m\right)$, $c \in \mathbb{R}^d$,*

$$F\left(u + c, v; A\right) = F\left(u, v; A\right);$$

*v) for every $A \in \mathcal{A}\left(\Omega\right)$, $F\left(\cdot, \cdot; A\right)$ is sequentially weak lower semicontinuous in $W^{1,p}\left(\Omega;\mathbb{R}^d\right) \times L^q\left(\Omega;\mathbb{R}^m\right)$.*

*Then there exists a Carathéodory function $g : \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ such that*

*a) $\left|g\left(x, \xi, b\right)\right| \leq C\left(1 + \left|\xi\right|^p + \left|b\right|^q\right)$ for a.e. $x \in \Omega$, for any $\left(\xi, b\right) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$;*

*b) for every $A \in \mathcal{A}\left(\Omega\right)$, $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ and $v \in L^q\left(\Omega;\mathbb{R}^m\right)$ the following integral representation holds*

$$F\left(u, v; A\right) = \int_A g\left(x, \nabla u\left(x\right), v\left(x\right)\right) dx.$$

*Moreover, if*

$$F\left(u_\xi, v_b; B_\rho\left(y\right)\right) = F\left(u_\xi, v_b; B_\rho\left(z\right)\right) \tag{3.1}$$

*for every $y$, $z \in \Omega$, for $\rho > 0$ such that $B_\rho\left(y\right) \cup B_\rho\left(z\right) \subset \Omega$, and for every $\left(\xi, b\right) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$ where $u_\xi\left(x\right) := \xi x$ and $v_b \equiv b$, then $g$ is independent of $x$ and it is quasiconvex-convex.*

**Proof.** The proof follows the same argument as Theorem 4.3.2 in [15]. We start by proving the integral representation for piecewise affine functions $u$ in $W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ and piecewise constant functions $v$. Then we will use a density argument to get the full result.

For every $\xi \in \mathbb{R}^{d\times N}$, and for every $b \in \mathbb{R}^m$, we will denote by $u_\xi$ the affine map $u_\xi(x) = \xi x$ and by $v_b$ the constant map $v_b \equiv b$.

By hypothesis *iii*), we can assume, without loss of generality, that $F \geq 0$. Using hypothesis *ii*) and *iii*), for every $\left(\xi, b\right) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$ we have that $F\left(u_\xi, v_b, \cdot\right)$ is absolutely continuous with respect to the Lebesgue measure.

For every $x \in \Omega$, $\left(\xi, b\right) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$ set

$$g\left(x, \xi, b\right) := \limsup_{\rho\to 0^+} \frac{F\left(u_\xi, v_b; B_\rho\left(x\right)\right)}{\left|B_\rho\left(x\right)\right|}. \tag{3.2}$$

By Besicovitch derivation theorem $g\left(\cdot, \xi, b\right) \in L^1\left(\Omega\right)$ and

$$F\left(u_\xi, v_b; A\right) = \int_A g\left(x, \xi, b\right) dx. \tag{3.3}$$

Moreover, from hypothesis $iii)$ it follows that $g$ satisfies the growth condition $a)$.

Let $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ be a piecewise affine function and $v$ a piecewise constant function. Precisely, let $\{\Omega_i\}_{i\in I}$ be a finite family of open pairwise disjoint subsets of $\Omega$ such that, for some $b_i \in \mathbb{R}^m$,

$$u|_{\Omega_i} \text{ is affine}, \qquad v = b_i \text{ on } \Omega_i, \qquad \text{for each } i \in I \tag{3.4}$$

and $\left|\Omega \backslash \bigcup_{i\in I} \Omega_i\right| = 0$.

From (3.3) and hypotheses $i) - iii)$ it follows that

$$F\left(u,v;A\right) = \int_A g\left(x, \nabla u\left(x\right), v\left(x\right)\right) dx$$

for every $u$ and $v$ verifying $(3.4)$.

We claim that $g\left(x,\cdot,\cdot\right)$ is separately convex for every $x \in \Omega$, i.e.,

$$\xi_i \longmapsto g\left(x, \xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots \xi_{d\times N}, b\right) \tag{3.5}$$

is convex for every $i \in \{1,\ldots,d\times N\}$ and

$$b \longmapsto g\left(x,\xi,\cdot\right) \tag{3.6}$$

is convex.

We leave the proof of the claim to the end and proceed with the rest of the argument.

By Proposition 2.11, $g$ satisfies the Lipschitz condition (2.5) which ensures $g$ is a Carathéodory function.

By Lebesgue dominated convergence theorem

$$\left(u,v\right) \mapsto \int_A g\left(x, \nabla u\left(x\right), v\left(x\right)\right) dx \tag{3.7}$$

is strongly continuous in $W^{1,p}\left(\Omega;\mathbb{R}^d\right) \times L^q\left(\Omega;\mathbb{R}^m\right)$.

We will now prove the integral representation for general functions $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ and $v \in L^q\left(\Omega;\mathbb{R}^m\right)$.

Let $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $A \in \mathcal{A}\left(\Omega\right)$ with $A \subset\subset \Omega$ and $\widehat{u} \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ be with compact support in $\Omega$ and such that $u = \widehat{u}$ on $A$. We may find a sequence $\{u_n\}$ of piecewise affine functions converging to $\widehat{u}$ strongly in $W^{1,p}\left(\Omega;\mathbb{R}^d\right)$.

Let $v \in L^q\left(\Omega;\mathbb{R}^m\right)$. Using a density argument, we obtain that, for every $n \in \mathbb{N}$ there exists $\widehat{v}_n \in C_c^\infty\left(\Omega;\mathbb{R}^m\right)$ such that $\|\widehat{v}_n - v\|_{L^q} < \frac{1}{n}$.

Let

$$K_n := \operatorname{supp} \widehat{v}_n$$

which is included in an open subset $A_n$ of $\Omega$, and let $\eta > 0$. For $\delta > 0$ let $\left\{Q_i^\delta\right\}$ be a family of pairwise disjoint open cubes with side less than $\delta$ and such that $K_n \subset \cup_{i=1}^{M_\delta} \overline{Q_i^\delta} \subset A_n$ and let

$$m_i^{\delta,n} := \inf_{Q_i^\delta} \widehat{v}_n = \min_{\overline{Q_i^\delta}} \widehat{v}_n, \qquad s_n^\delta := \sum_{i=1}^{M_\delta} m_i^{\delta,n} \chi_{Q_i^\delta}.$$

For sufficiently small $\delta$, it is possible to get

$$\left\|s_n^\delta - \widehat{v}_n\right\|_{L^\infty} < \eta. \tag{3.8}$$

In fact, since $\widehat{v}_n$ is uniformly continuous in $\overline{\Omega}$ then

$$\forall\, n \in \mathbb{N}, \quad \forall\, \eta > 0, \quad \exists\, \delta_n > 0 : |x - x'| < \delta_n \Longrightarrow |\widehat{v}_n\left(x\right) - \widehat{v}_n\left(x'\right)| < \eta.$$

In particular, in each cube $Q_i^\delta$

$$\left\|s_n^\delta - \widehat{v}_n\right\|_{L^\infty} = \left\|\inf_{Q_i^\delta} \widehat{v}_n - \widehat{v}_n\right\|_{L^\infty} < \eta.$$

On the other hand, if $x \notin \cup_{i=1}^{M_\delta} Q_i^\delta$ then $x \notin K_n$ and thus $\widehat{v}_n = s_n^\delta = 0$. Hence it follows $(3.8)$. Observe that

$$\left\|v - s_n^\delta\right\|_{L^q} \leq \|v - \widehat{v}_n\|_{L^q} + \left\|\widehat{v}_n - s_n^\delta\right\|_{L^q} < \frac{1}{n} + \left(\int_\Omega \left|\widehat{v}_n(x) - s_n^\delta(x)\right|^q dx\right)^{\frac{1}{q}} < \frac{1}{n} + |\Omega|^{\frac{1}{q}} \eta.$$

Choosing $\eta < \frac{1}{n}$ and letting $n \to \infty$ we conclude that $s_n^\delta \to v$ in $L^q(\Omega; \mathbb{R}^m)$.

Hence

$$\begin{aligned} F(u, v; A) = F(\widehat{u}, v; A) &\leq \liminf_{n\to\infty} F\left(u_n, s_n^\delta; A\right) = \liminf_{n\to\infty} \int_A g\left(x, \nabla u_n(x), s_n^\delta(x)\right) dx \\ &= \int_A g(x, \nabla \widehat{u}(x), v(x))\, dx = \int_A g(x, \nabla u(x), v(x))\, dx \end{aligned}$$

where we have used the fact that $F(\cdot, \cdot; A)$ is sequentially weak lower semicontinuous and the strong continuity of $(3.7)$ in $W^{1,p}\left(\Omega; \mathbb{R}^d\right) \times L^q(\Omega; \mathbb{R}^m)$. Hence

$$F(u, v; A) \leq \int_A g(x, \nabla u(x), v(x))\, dx$$

for every $u \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$ and $v \in L^q(\Omega; \mathbb{R}^m)$.

To prove the reverse inequality, let us fix $u \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$, $v \in L^q(\Omega; \mathbb{R}^m)$ and denote by $H : W^{1,p}\left(\Omega; \mathbb{R}^d\right) \times L^q(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \to \mathbb{R}$ the functional defined by

$$H(\overline{u}, \overline{v}; A) := F(u + \overline{u}, v + \overline{v}; A), \;\forall\, (\overline{u}, \overline{v}) \in W^{1,p}\left(\Omega; \mathbb{R}^d\right) \times L^q(\Omega; \mathbb{R}^m),\; A \in \mathcal{A}(\Omega).$$

Since $H$ satisfies the conditions of the theorem then there exists a Carathéodory function $h$ satisfying the $p - q$ growth condition $a)$ and such that

$$H(\overline{u}, \overline{v}; A) = \int_A h(x, \nabla \overline{u}(x), \overline{v}(x))\, dx$$

for every $\overline{u} \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$ piecewise affine and $\overline{v}$ piecewise constant.

Moreover, we have proved that

$$H(\overline{u}, \overline{v}; A) \leq \int_A h(x, \nabla \overline{u}(x), \overline{v}(x))\, dx$$

for $\overline{u} \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$, $\overline{v} \in L^q(\Omega; \mathbb{R}^m)$ and $A \subset\subset \Omega$.

Fix $A \in \mathcal{A}(\Omega)$ such that $A \subset\subset \Omega$ and let, as before, $\widehat{u} \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$ be with compact support in $\Omega$ and such that $u = \widehat{u}$ on $A$, $\{u_n\}$ a sequence of piecewise affine functions converging to $\widehat{u}$ strongly in $W^{1,p}\left(\Omega; \mathbb{R}^d\right)$, and $v_n \in C_c^\infty(\Omega; \mathbb{R}^m)$ converging strongly to $v$ in $L^q(\Omega; \mathbb{R}^m)$.

We obtain

$$\begin{aligned} \int_A h(x, 0, 0)\, dx = H(0, 0; A) = F(u, v; A) &\leq \int_A g(x, \nabla u(x), v(x))\, dx \\ &= \int_A g(x, \nabla \widehat{u}(x), v(x))\, dx = \lim_{n\to\infty} \int_A g(x, \nabla u_n(x), v_n(x))\, dx \\ &= \lim_{n\to\infty} F(u_n, v_n; A) = \lim_{n\to\infty} H(u_n - u, v_n - v; A) \\ &\leq \lim_{n\to\infty} \int_A h(x, \nabla u_n(x) - \nabla u(x), v_n(x) - v(x))\, dx \\ &= \lim_{n\to\infty} \int_A h(x, \nabla u_n(x) - \nabla \widehat{u}(x), v_n(x) - v(x))\, dx \\ &= \int_A h(x, 0, 0)\, dx, \end{aligned}$$

where we have used in the last identity the strong continuity of

$$(u,v) \mapsto \int_A h(x, \nabla u(x), v(x))\, dx$$

in $W^{1,p}(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$, which follows from Lebesgue dominated convergence theorem.

Hence

$$F(u, v; A) = \int_A g(x, \nabla u(x), v(x))\, dx \tag{3.9}$$

for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $v \in L^q(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$. By virtue of (3.9) on open sets $A$ well contained in $\Omega$ and by the inner regularity of the integral and of $F$ (recall that $F$ is a measure as assumed in ii)), the equality $F(u, v; A) = \int_A g(x, \nabla u(x), v(x))dx$ holds for every $A \in \mathcal{A}(\Omega)$, $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$.

To finish the proof it remains to prove that $g(x, \cdot, \cdot)$ is separately convex.

The convexity described in (3.5) follows from Zig-Zag Lemma 4.3.5 in [15] (see also Lemma 20.2 in [22]). To prove (3.6), we argue as in [2], Theorem 5.1. Let $\xi \in \mathbb{R}^N$ and define

$$v_b := tb_1 + (1-t)\, b_2$$

for $t \in (0,1)$ and $b_1,\ b_2 \in \mathbb{R}^m$. To prove the convexity of $g$ it suffices to prove

$$F(u_\xi, v_b; B_\rho(x)) \leq tF(u_\xi, b_1; B_\rho(x)) + (1-t)\, F(u_\xi, b_2; B_\rho(x))$$

for every fixed $x \in \Omega$ and for every $\rho > 0$.

Let $x \in \Omega$, $A := Q\left(x, \sqrt[N]{t}\rho\right)$ and define $v_n(y) := b_1\chi(ny) + b_2(1-\chi(ny))$, where $\chi$ denotes the characteristic function of $A$ defined in the cube $Q(x, \rho)$ and extended by periodicity to $\mathbb{R}^N$.

By Riemann-Lebesgue lemma it follows that $v_n \rightharpoonup tb_1 + (1-t)b_2$ in the weak topology of $L^q(B_\rho(x);\mathbb{R}^m)$.

Let us consider the open set $A_n := \{y \in B_\rho(x) : \chi(ny) = 1\}$. Since $v_n$ are piecewise constants and $F(\cdot, \cdot; B_\rho(x))$ is sequentially weak lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$ we obtain

$$\begin{aligned}
F(u_\xi, v_b; B_\rho(x)) &\leq \liminf_{n\to\infty} F(u_\xi, v_n; B_\rho(x)) \\
&= \liminf_{n\to\infty} \left( \int_{B_\rho(x)\cap A_n} g(y, \xi, b_1)\, dy + \int_{B_\rho(x)\setminus A_n} g(y, \xi, b_2)\, dy \right) \\
&= t \int_{B_\rho(x)} g(y, \xi, b_1)\, dy + (1-t) \int_{B_\rho(x)} g(y, \xi, b_2)\, dy \\
&= t\, F(u_\xi, b_1; B_\rho(x)) + (1-t)\, F(u_\xi, b_2; B_\rho(x)) \,.
\end{aligned}$$

So we conclude that $g$ is separately convex.

By (3.1) and (3.2) one has

$$g(y, \xi, b) = \limsup_{\rho\to 0^+} \frac{F(u_\xi, v_b; B_\rho(y))}{\rho^N} = \limsup_{\rho\to 0^+} \frac{F(u_\xi, v_b; B_\rho(z))}{\rho^N} = g(z, \xi, b)\,.$$

Thus given $(\xi, b) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$ we have that $g(y, \xi, b) = g(z, \xi, b)$ for any $y, z \in \Omega$. Hence $g$ is independent of $x$. By Theorem 4.4 in [29] we conclude that $g$ is quasiconvex-convex. ∎

## 3.2 Compactness

This subsection is devoted to prove that general families of integral functionals, essentially under hypotheses $(H_1)$ and $(H_2)$ (for $p, q > 1$) admit a subsequence $\Gamma(L^p \times L^q_w)$-converging to a functional which is still a measure and that can admit an integral formulation.

In this subsection $p, q > 1$.

First we will establish a compactness result for general families of functionals $H_\varepsilon : L^p(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0,\infty]$ of the form

$$H_\varepsilon(u,v;A) := \begin{cases} \displaystyle\int_A f_\varepsilon(x,\nabla u(x), v(x))\,dx & \text{if } (u,v) \in W^{1,p}(A;\mathbb{R}^d) \times L^q(A;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.10}$$

where $f_\varepsilon : \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is a family of Carathéodory functions satisfying uniform $p-q$ growth and $p-q$ coercivity conditions as in $(H_2)$, namely

$$\frac{1}{C}\left(|\xi|^p + |b|^q\right) - C \le f_\varepsilon(x,\xi,b) \le C(1+|\xi|^p+|b|^q) \tag{3.11}$$

for some $C>0$, for a.e. $x\in\Omega$ and for every $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$.

This compactness result will ensure the existence of $\Gamma-$convergent subsequences of $H_\varepsilon$, whose $\Gamma-$limit admits an integral representation in $W^{1,p}\left(\Omega;\mathbb{R}^d\right)\times L^q\left(\Omega;\mathbb{R}^m\right)$.

Let $\mathcal{H}^-_{\{\varepsilon_j\}}$ and $\mathcal{H}^+_{\{\varepsilon_j\}}$ be defined in $L^p\left(\Omega;\mathbb{R}^d\right)\times L^q\left(\Omega;\mathbb{R}^m\right)\times\mathcal{A}(\Omega)$ by

$$\mathcal{H}^-_{\{\varepsilon_j\}}(u,v;A) := \inf\left\{\liminf_{j\to\infty} H_{\varepsilon_j}(u_j,v_j;A) : u_j \to u \text{ in } L^p(A;\mathbb{R}^d), v_j \rightharpoonup v \text{ in } L^q(A;\mathbb{R}^m)\right\},$$
$$\mathcal{H}^+_{\{\varepsilon_j\}}(u,v;A) := \inf\left\{\limsup_{j\to\infty} H_{\varepsilon_j}(u_j,v_j;A) : u_j \to u \text{ in } L^p(A;\mathbb{R}^d), v_j \rightharpoonup v \text{ in } L^q(A;\mathbb{R}^m)\right\}.$$

If $\mathcal{H}^+_{\{\varepsilon_j\}}(u,v;A) = \mathcal{H}^-_{\{\varepsilon_j\}}(u,v;A)$ for each $A\in\mathcal{A}(\Omega)$, for every $u\in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ and $v\in L^q\left(\Omega;\mathbb{R}^m\right)$ then we denote $\mathcal{H}_{\{\varepsilon_j\}}(u,v;A) := \Gamma - \lim_{j\to\infty} H_{\varepsilon_j}(u,v;A)$.

**Theorem 3.2** *Let $f_\varepsilon : \Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\to\mathbb{R}$ be a family of Carathéodory functions satisfying* (3.11)*. Let $H_\varepsilon$ be the functional defined in* (3.10)*. For every sequence $\{\varepsilon_n\}$ converging to zero there exists a subsequence $\left\{\varepsilon_{n_j}\right\}\equiv\{\varepsilon_j\}$ such that $\mathcal{H}_{\{\varepsilon_j\}}$ exists for all $u\in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v\in L^q\left(\Omega;\mathbb{R}^m\right)$ and $A\in\mathcal{A}(\Omega)$.*

*Moreover, there exists a Carathéodory function $g^{\{\varepsilon_j\}} : \Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\to\mathbb{R}$ such that*

$$\mathcal{H}_{\{\varepsilon_j\}}(u,v;A) = \int_A g^{\{\varepsilon_j\}}(x,\nabla u(x), v(x))\,dx$$

*for every $u\in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v\in L^q\left(\Omega;\mathbb{R}^m\right)$, $A\in\mathcal{A}(\Omega)$ and*

$$\left|g^{\{\varepsilon_j\}}(x,\xi,b)\right| \le C\left(1+|\xi|^p+|b|^q\right)$$

*for a.e. $x\in\Omega$, and for every $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$.*

Let $\mathcal{C}$ be a countable collection of open subsets of $\Omega$ such that for any $\delta>0$ and any $A\in\mathcal{A}(\Omega)$ there exists a finite union $C_A$ of disjoint elements of $\mathcal{C}$ satisfying

$$\begin{cases} \overline{C_A}\subset A, \\ \mathcal{L}^N(A)\le\mathcal{L}^N(C_A)+\delta. \end{cases}$$

We may take $\mathcal{C}$ as the set of open cubes with faces parallel to the axes, centered at $x\in\Omega\cap\mathbb{Q}^N$ and with rational edge length. We denote by $\mathcal{R}$ the countable collection of all finite unions of elements of $\mathcal{C}$, i.e.,

$$\mathcal{R} := \left\{\bigcup_{i=1}^k C_i : k\in\mathbb{N},\ C_i\in\mathcal{C}\right\}.$$

We start by proving that the $\Gamma-$limit exists for any element $C\in\mathcal{R}$.

**Lemma 3.3** *For every sequence $\{\varepsilon_n\}$ converging to zero there exists a subsequence $\{\varepsilon_{n_j}\} \equiv \{\varepsilon_j\}$ (depending on $\mathcal{R}$) such that*

$$\mathcal{H}_{\{\varepsilon_j\}}(u, v; C) \tag{3.12}$$

*exists for all $u \in L^p(\Omega; \mathbb{R}^d)$, $v \in L^q(\Omega; \mathbb{R}^m)$ and $C \in \mathcal{R} \cup \{\Omega\}$.*

**Proof.** Observing that the dual of $L^p(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$ is a separable metric space, by virtue of Kuratowski's compactness theorem (see Theorem 8.5 and Corollary 8.12 in [22]) and via a diagonal argument, we may say that there exists a subsequence $\{\varepsilon_j\}$, depending on $\mathcal{R}$ such that the $\Gamma$-limit of $H_{\varepsilon_j}$ exists for every $C \in \mathcal{R}\cup\{\Omega\}$, and $(u, v) \in L^p(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$, and, moreover, this $\Gamma$-limit is $+\infty$ in $(L^p(\Omega;\mathbb{R}^d) \setminus W^{1,p}(\Omega;\mathbb{R}^d)) \times L^q(\Omega;\mathbb{R}^m)$. ■

In order to conclude the proof of Theorem 3.2, we prove that the $\Gamma$-liminf is the trace of a Radon measure. To this end we will invoke the following result (see [33]) which is based on De Giorgi-Letta's criterion (see [26]).

**Lemma 3.4 (Fonseca-Malý)** *Let $X$ be a locally compact Hausdorff space, let $\Pi : \mathcal{A}(X) \to [0, \infty]$ be a set function and $\mu$ be a finite Radon measure on $X$ satisfying*

*i)* $\Pi(A) \leq \Pi(B) + \Pi\left(A \backslash \overline{C}\right)$ *for all $A$, $B$, $C \in \mathcal{A}(X)$ such that $C \subset\subset B \subset\subset A$;*

*ii) given $A \in \mathcal{A}(X)$, for all $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{A}(X)$ such that $A_\varepsilon \subset\subset A$ and $\Pi\left(A \backslash \overline{A_\varepsilon}\right) \leq \varepsilon$;*

*iii)* $\Pi(X) \geq \mu(X)$;

*iv)* $\Pi(A) \leq \mu\left(\overline{A}\right)$ *for all $A \in \mathcal{A}(X)$.*

*Then, $\Pi = \mu\lfloor_{\mathcal{A}(X)}$.*

We are now in position to prove that the $\Gamma - \liminf$ is the trace of a Radon measure.

**Lemma 3.5** *For each $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$, for every $A \in \mathcal{A}(\Omega)$, let $\{\varepsilon_j\}$ be the sequence given by Lemma 3.3. Then there exists a further subsequence $\{\varepsilon_{j_k}\} \equiv \{\varepsilon_k\}$ such that $\mathcal{H}^-_{\{\varepsilon_k\}}(u, v, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a finite Radon measure.*

**Proof.** The proof develops following by now standard techniques (see for instance [8]). We will see that we are in conditions to apply Lemma 3.4 with $\Pi(\cdot) := \mathcal{H}^-_{\{\varepsilon_k\}}(u, v, \cdot)$ for some sequence $\{\varepsilon_k\}$ to be chosen.

Let $A$, $B$, $C \in \mathcal{A}(X)$ such that $C \subset\subset B \subset\subset A$, fix $\eta > 0$ and find $\{u_j\} \subset L^p(\Omega;\mathbb{R}^d)$ and $\{v_j\} \subset L^q(\Omega;\mathbb{R}^m)$ such that $u_j \to u$ in $L^p(A \setminus \overline{C};\mathbb{R}^d)$, $v_j \rightharpoonup v$ in $L^q(A \setminus \overline{C};\mathbb{R}^m)$ and

$$\liminf_{j\to\infty} \int_{A\setminus\overline{C}} f_{\varepsilon_j}(x, \nabla u_j(x), v_j(x))\, dx \leq \mathcal{H}^-_{\{\varepsilon_j\}}(u, v; A \setminus C) + \eta. \tag{3.13}$$

Moreover, up to a subsequence (not relabeled), we may assume that

$$\lim_{j\to\infty} \int_{A\setminus\overline{C}} f_{\varepsilon_j}(x, \nabla u_j(x), v_j(x))\, dx = \liminf_{j\to\infty} \int_{A\setminus\overline{C}} f_{\varepsilon_j}(x, \nabla u_j(x), v_j(x))\, dx. \tag{3.14}$$

Let $B_0 \in \mathcal{R}$ be such that $C \subset\subset B_0 \subset\subset B$, in particular $\mathcal{L}^N(\partial B_0) = 0$. Then, by Lemma 3.3, $\mathcal{H}^-_{\{\varepsilon_j\}}(u, v; B_0)$ is a $\Gamma$-limit, and thus there exists a sequence $\{u'_j\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ and $\{v'_j\} \subset L^q(\Omega;\mathbb{R}^m)$ such that $u'_j \to u$ in $L^p(B_0, \mathbb{R}^d)$, $v'_j \rightharpoonup v$ in $L^q(B_0;\mathbb{R}^m)$ and

$$\lim_{j\to\infty} \int_{B_0} f_{\varepsilon_j}(x, \nabla u'_j(x), v'_j(x))\, dx = \mathcal{H}^-_{\{\varepsilon_j\}}(u, v; B_0). \tag{3.15}$$

For every $\overline{u} \in L^p(\Omega;\mathbb{R}^d)$ and $\overline{v} \in L^q(\Omega;\mathbb{R}^m)$ consider the functional

$$G(\overline{u}, \overline{v}; A) := \int_A (1 + |\nabla\overline{u}(x)|^p + |\overline{v}(x)|^q)\, dx.$$

By virtue of the coercivity condition (3.11), up to a subsequence, there exists a nonnegative Radon measure $\nu$ such that $\nu_{j_k} := G(u_{j_k}, v_{j_k};\cdot) + G(u'_{j_k}, v'_{j_k};\cdot)$ restricted to $B_0\backslash\overline{C}$ converges weakly star in the sense of measures to $\nu$.

We claim that

$$\mathcal{H}^-_{\{\varepsilon_k\}}(u,v;A) \leq \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;B) + \mathcal{H}^-_{\{\varepsilon_k\}}\left(u,v;A\backslash\overline{C}\right)$$

for all $A,B,C \in \mathcal{A}(\Omega)$ such that $C \subset\subset B \subset\subset A$, for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and for every $v \in L^q(\Omega;\mathbb{R}^m)$.

For every $t > 0$, let $B_t := \{x \in B_0 : \mathrm{dist}(x, \partial B_0) > t\}$. For $0 < \delta < \eta' < \eta$ such that $\nu(\partial B_{\eta'}) = 0$, define $L_\delta := B_{\eta'-2\delta} \setminus \overline{B_{\eta+\delta}}$ and take a smooth cut-off function $\varphi_\delta \in C_0^\infty(B_{\eta-\delta};[0,1])$ such that $\varphi_\delta(x) = 1$ on $B_\eta$. As the thickness of the strip is of order $\delta$, we have an upper bound of the type $\|\nabla\varphi_\delta\|_{L^\infty(B_{\eta-\delta})} \leq \frac{C}{\delta}$.

Define

$$\overline{u}_k := u'_k\varphi_\delta + (1-\varphi_\delta)u_k, \qquad \overline{v}_k := v'_k\varphi_\delta + (1-\varphi_\delta)v_k.$$

Clearly $\{\overline{u}_k\}$ and $\{\overline{v}_k\}$ converge strongly to $u$ in $L^p(A;\mathbb{R}^d)$ and weakly to $v$ in $L^q(A;\mathbb{R}^m)$, respectively.

By (3.11) it follows that

$$\begin{aligned}\int_A f_{\varepsilon_k}(x, \nabla\overline{u}_k(x), \overline{v}_k(x))\,dx &\leq \int_{B_\eta} f_{\varepsilon_k}(x, \nabla u'_k(x), v'_k(x))\,dx + \int_{A\backslash\overline{B_{\eta-\delta}}} f_{\varepsilon_k}(x, \nabla u_k(x), v_k(x))\,dx \\ &\quad + C(G(u'_k, v'_k; L_\delta) + G(u_k, v_k; L_\delta)) + \frac{C}{\delta^p}\int_{L_\delta} |u'_k(x) - u_k(x)|^p dx \\ &\leq \int_{B_0} f_{\varepsilon_k}(x, \nabla u'_k(x), v'_k(x))\,dx + \int_{A\backslash\overline{C}} f_{\varepsilon_k}(x, \nabla u_k(x), v_k(x))\,dx \\ &\quad + C(G(u'_k, v'_k; L_\delta) + G(u_k, v_k; L_\delta)) + \frac{C}{\delta^p}\int_{L_\delta} |u'_k(x) - u_k(x)|^p dx.\end{aligned}$$

Passing to the limit on $k$ and using (3.13), (3.14) and (3.15), we have

$$\begin{aligned}\mathcal{H}^-_{\{\varepsilon_k\}}(u,v;A) &\leq \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;B_0) + \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;A\setminus\overline{C}) + \eta + C\nu(\overline{L_\delta}) \\ &\leq \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;B) + \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;A\setminus\overline{C}) + \eta + C\nu(\overline{L_\delta}),\end{aligned}$$

where it has been used the fact that the $\Gamma$-liminf of a sequence is below the $\liminf$ on any subsequence. Letting $\delta \to 0^+$ we obtain

$$\mathcal{H}^-_{\{\varepsilon_k\}}(u,v;A) \leq \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;B) + \mathcal{H}^-_{\{\varepsilon_k\}}(u,v;A\setminus\overline{C}) + \eta + \nu(B_{\eta'}\backslash\overline{B}_\eta).$$

Letting $\eta \to 0^+$ and since $\nu(\partial B_{\eta'}) = 0$ we have proven the subadditivity of $\mathcal{H}^-_{\{\varepsilon_k\}}(u,v;\cdot)$.

To establish condition $ii)$ in Lemma 3.4 let $A \in \mathcal{A}(\Omega)$, $\varepsilon > 0$ and consider $A_\varepsilon \in \mathcal{A}(\Omega)$ such that $\overline{A_\varepsilon} \subset A$ and

$$\int_{A\backslash\overline{A_\varepsilon}} (1 + |\nabla u(x)|^p + |v(x)|^q)\,dx < \frac{\varepsilon}{C}, \tag{3.16}$$

where $C$ is the constant given by condition (3.11).

Due to the growth conditions (3.11) and (3.16)

$$\begin{aligned}\mathcal{H}^-_{\{\varepsilon_k\}}\left(u,v;A\backslash\overline{A_\varepsilon}\right) &\leq \liminf_{k\to\infty}\int_{A\backslash\overline{A_\varepsilon}} f_{\varepsilon_k}(x, \nabla u(x), v(x))\,dx \\ &\leq C\int_{A\backslash\overline{A_\varepsilon}} (1 + |\nabla u(x)|^p + |v(x)|^q)\,dx < \varepsilon.\end{aligned}$$

Hence condition $ii)$ holds.

Up to a subsequence, there exists $\{\varepsilon_k\}$ such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^d)$, $v_k \rightharpoonup v$ in $L^q(\Omega;\mathbb{R}^m)$ and $\mathcal{H}^-_{\{\varepsilon_k\}}(u,v;\Omega) = \lim_{k\to\infty}\int_\Omega f_{\varepsilon_k}(x, \nabla u_k(x), v_k(x))\,dx$. Let $\mu_k := f_{\varepsilon_k}(x, \nabla u_k, v_k)\mathcal{L}^N\lfloor\Omega$ and let $\mu$ be defined, up to a subsequence, as the limit of $\{\mu_k\}$ in the sense of measures.

By definition, it follows that

$$\mathcal{H}^{-}_{\{\varepsilon_k\}}(u,v;A) \leq \liminf_{k\to\infty} \int_A f_{\varepsilon_k}(x, \nabla u_k(x), v_k(x))\,dx \leq \mu(\overline{A})$$

and we attained *iv*).

Finally, to establish *iii*), take $\Omega' \subset\subset \Omega$. Since $\{\mu_k\}$ converges weakly star in the sense of measures to $\mu$ then

$$\mu(\Omega') \leq \lim_{k\to\infty} \int_\Omega f_{\varepsilon_k}(x, \nabla u_k(x), v_k(x))\,dx = \mathcal{H}^{-}_{\{\varepsilon_k\}}(u,v;\Omega).$$

Therefore

$$\mu(\Omega') \leq \mathcal{H}^{-}_{\{\varepsilon_k\}}(u,v;\Omega)$$

for all $\Omega' \subset\subset \Omega$. Hence

$$\mu(\Omega) \leq \mathcal{H}^{-}_{\{\varepsilon_k\}}(u,v;\Omega).$$

As a consequence of Lemma 3.4 we conclude that

$$\mathcal{H}^{-}_{\{\varepsilon_k\}}(u,v;A) = \mu(A)$$

for all $A \in \mathcal{A}(\Omega)$. ■

**Remark 3.6** *Following the argument of Proposition 12.2 in [12] and assuming* (3.11) *we may conclude that* $H_\varepsilon$ *satisfies the* $L^p \times L^q-$ *fundamental estimate. Precisely, for every* $U,\ U',\ V \in \mathcal{A}(\Omega)$ *with* $U' \subset\subset U$ *and* $\sigma > 0$ *there exist* $M_\sigma > 0$ *and* $\varepsilon_\sigma > 0$ *such that for all* $u,\ \overline{u} \in L^p(\Omega;\mathbb{R}^d)$, $v,\ \overline{v} \in L^q(\Omega;\mathbb{R}^m)$ *and* $\varepsilon < \varepsilon_\sigma$ *there exists a cut-off function* $\varphi \in C_0^\infty(U;[0,1])$ *such that* $\varphi \equiv 1$ *on* $U'$ *and there exists* $r > 0$ *such that* $U'_r := \{x \in U : \operatorname{dist}(x, U') < r\}$ *and*

$$\begin{aligned} H_\varepsilon\left(\varphi u + (1-\varphi)\overline{u}, \chi_{U'_r} v + \left(1-\chi_{U'_r}\right)\overline{v}; U' \cup V\right) &\leq (1+\sigma)\left(H_\varepsilon(u,v;U) + H_\varepsilon(\overline{u},\overline{v};V)\right) \\ &\quad + M_\sigma \int_{(U\cap V)\setminus U'} |u(x) - \overline{u}(x)|^p\,dx + \sigma \end{aligned}$$

*where* $\chi_{U'_r}$ *stands for the characteristic function of* $U'_r$. *By Proposition 18.3 in [22] we conclude that for every* $A, B, C \in \mathcal{A}(\Omega)$ *such that* $C \subset\subset B \subset\subset A$

$$\mathcal{H}^{+}_{\{\varepsilon\}}(u,v;A) \leq \mathcal{H}^{+}_{\{\varepsilon\}}(u,v;B) + \mathcal{H}^{+}_{\{\varepsilon\}}(u,v;A\setminus\overline{C}). \tag{3.17}$$

**Proof of Theorem 3.2.** Since the dual of $W^{1,p}(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$ is separable, by virtue of the coercivity condition (3.11), we may apply Theorem 16.9 in [22], which ensures that every sequence of increasing functionals $\{H_{\varepsilon_n}\}$ admits a subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$, $\overline{\Gamma}$-converging to a functional $H$, namely the inner regular envelope of $\mathcal{H}^{-}_{\{\varepsilon_j\}}$ and $\mathcal{H}^{+}_{\{\varepsilon_j\}}$ coincide with $H$ for every $A \in \mathcal{A}(\Omega)$. On the other hand, by virtue of Lemma 3.5, we have that $\mathcal{H}^{-}_{\{\varepsilon_j\}}$ is a measure hence coinciding with its inner regular envelope. Moreover, arguing as in the proof of Proposition 18.6 in [22], by virtue of (3.17) and the growth condition (3.11) we may conclude that $H$ coincides also with $\mathcal{H}^{+}_{\{\varepsilon_j\}}$, thus concluding the existence of the $\Gamma$- limit.

To prove that $\mathcal{H}_{\{\varepsilon_j\}}$ admits an integral representation we will verify that the hypotheses of Theorem 3.1 hold.

Hypotheses *i*) and *v*) are consequence of the definition of the $\Gamma-$limit. Hypothesis *iii*) comes from (3.11) and *iv*) is easily attained. Condition *ii*) follows from Lemma 3.5. ■

Next we prove, using the same techniques as in [8], that $\mathcal{H}^{-}_{\{\varepsilon\}}$ is independent of the boundary data for $v$ constant. This result will be useful in order to achieve Theorem 1.1.

**Lemma 3.7** *Let* $\mathcal{H}^{*}_{\varepsilon} : W^{1,p}(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0,\infty)$ *be defined by*

$$\begin{aligned} \mathcal{H}^{*}_{\{\varepsilon\}}(u,v;A) := \inf\Bigg\{ &\liminf_{\varepsilon\to 0^+} H_\varepsilon(u_\varepsilon, v_\varepsilon; A) : u_\varepsilon \to u \text{ in } L^p(A;\mathbb{R}^d),\ v_\varepsilon \rightharpoonup v \text{ in } L^q(A;\mathbb{R}^m) \\ &u_\varepsilon = u \text{ on a neighborhood of } \partial A,\ \int_A v_\varepsilon(x)\,dx = \int_A v(x)\,dx \Bigg\}. \end{aligned}$$

*Then, under the growth condition* $(3.11)$ *,*

$$\mathcal{H}^{-}_{\{\varepsilon\}}(u,v_b;A)=\mathcal{H}^{*}_{\{\varepsilon\}}(u,v_b;A)$$

*for every* $A\in\mathcal{A}(\Omega)$*,* $u\in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$ *,* $b\in\mathbb{R}^m$ *where* $v_b\equiv b$*.*

Contrary to the case where there is no dependence on $v$, we emphasize that in general one cannot expect to have $\mathcal{H}^{-}_{\{\varepsilon\}}(u,v;A)=\mathcal{H}^{*}_{\{\varepsilon\}}(u,v;A)$ for every $v$. However, the achievement of the lemma will be enough to prove Theorem 1.1, since in the proof it will be sufficient to apply the fundamental theorem of $\Gamma$ - convergence just on constant functions $v$.

**Proof.** Clearly $\mathcal{H}^{-}_{\{\varepsilon\}}(u,v;A)\leq\mathcal{H}^{*}_{\{\varepsilon\}}(u,v;A)$ for every $u\in W^{1,p}\left(A;\mathbb{R}^d\right)$, $v\in L^q\left(A;\mathbb{R}^m\right)$ and $A\in\mathcal{A}(\Omega)$. To prove the reverse inequality, let

$$G_{p,q}(u,v,r;A):=\int_A\left(1+|\nabla u(x)|^p+(|v(x)|+r)^q\right)dx$$

be defined for every $u\in W^{1,p}\left(A;\mathbb{R}^d\right)$, $v\in L^q\left(A;\mathbb{R}^m\right)$, $r\in\mathbb{R}^+$ and $A\in\mathcal{A}(\Omega)$. Given $\rho>0$ consider $u_\varepsilon\in W^{1,p}\left(A;\mathbb{R}^d\right)$, $v_\varepsilon\in L^q\left(A;\mathbb{R}^m\right)$ such that $u_\varepsilon\to u$ in $L^p\left(A;\mathbb{R}^d\right)$, $v_\varepsilon\rightharpoonup v$ in $L^q\left(A;\mathbb{R}^m\right)$ and

$$\mathcal{H}^{-}_{\{\varepsilon\}}(u,v_b;A)+\rho>\liminf_{\varepsilon\to0^+}H_\varepsilon(u_\varepsilon,v_\varepsilon;A).$$

Due to the coercivity of $H_\varepsilon$, we may extract subsequences $\{u_{\varepsilon_k}\}$ and $\{v_{\varepsilon_k}\}$ such that

$$\liminf_{\varepsilon\to0^+}H_\varepsilon(u_\varepsilon,v_\varepsilon;A)=\lim_{k\to\infty}H_{\varepsilon_k}(u_{\varepsilon_k},v_{\varepsilon_k};A)$$

and the sequence of measures $\nu_k:=G_{p,q}(u_{\varepsilon_k},v_{\varepsilon_k},0;\cdot)+G_{p,q}(u,v_b,1;\cdot)$ converges weakly star in the sense of measures to some Radon measure $\nu$.

For every $t>0$, let $A_t:=\{x\in A:\operatorname{dist}(x,\partial A)>t\}$, fix $\eta>0$ and for every $0<2\delta<\eta'<\eta$ such that $\nu(\partial A_{\eta'})=0$ we define $L_\delta:=A_{\eta-2\delta}\backslash\overline{A}_{\eta+\delta}$. Consider a smooth cut-off function $\varphi_\delta\in C_0^\infty(A_{\eta-\delta};[0,1])$ such that $\varphi_\delta\equiv1$ on $A_\eta$ and $\|\nabla\varphi_\delta\|_{L^\infty(A)}\leq\frac{C}{\delta}$.

Define

$$\overline{u}_{\varepsilon_k}:=u_{\varepsilon_k}\varphi_\delta+u(1-\varphi_\delta),\qquad\overline{v}_{\varepsilon_k}:=v_{\varepsilon_k}\chi_{A_\eta}+c_{\varepsilon_k}\left(1-\chi_{A_\eta}\right)$$

where

$$c_{\varepsilon_k}:=\frac{\int_A\left(v_b(x)-v_{\varepsilon_k}(x)\chi_{A_\eta}(x)\right)dx}{\int_A\left(1-\chi_{A_\eta}(x)\right)dx}.$$

Clearly, $\overline{u}_{\varepsilon_k}\to u$ in $L^p\left(A;\mathbb{R}^d\right)$ and $\overline{u}_{\varepsilon_k}=u$ on a neighborhood of $\partial A$. Moreover, $c_{\varepsilon_k}\to b$, $\overline{v}_{\varepsilon_k}\rightharpoonup v_b$ in $L^q\left(A;\mathbb{R}^m\right)$ and

$$\int_A\overline{v}_{\varepsilon_k}(x)\,dx=\int_A v_b(x)\,dx.$$

Thus

$$\begin{aligned}H_{\varepsilon_k}(\overline{u}_{\varepsilon_k},\overline{v}_{\varepsilon_k};A)&\leq H_{\varepsilon_k}(\overline{u}_{\varepsilon_k},\overline{v}_{\varepsilon_k};A_\eta)+H_{\varepsilon_k}\left(\overline{u}_{\varepsilon_k},\overline{v}_{\varepsilon_k};A\backslash\overline{A}_{\eta-\delta}\right)+H_{\varepsilon_k}(\overline{u}_{\varepsilon_k},\overline{v}_{\varepsilon_k},L_\delta)\\&\leq H_{\varepsilon_k}(u_{\varepsilon_k},v_{\varepsilon_k};A_\eta)+H_{\varepsilon_k}\left(u,c_{\varepsilon_k};A\backslash\overline{A}_{\eta-\delta}\right)+C\int_{L_\delta}\left(1+|\nabla\overline{u}_{\varepsilon_k}(x)|^p+|\overline{v}_{\varepsilon_k}(x)|^q\right)dx\\&\leq H_{\varepsilon_k}(u_{\varepsilon_k},v_{\varepsilon_k};A_\eta)+C\int_{A\backslash\overline{A}_{\eta-\delta}}\left(1+|\nabla u(x)|^p+(|v_b(x)|+1)^q\right)dx\\&+C\int_{L_\delta}\left(1+|\nabla\overline{u}_{\varepsilon_k}(x)|^p+|\overline{v}_{\varepsilon_k}(x)|^q\right)dx.\end{aligned}$$

Since

$$\begin{aligned}\int_{L_\delta}|\nabla u_{\varepsilon_k}(x)|^p\,dx&\leq C\int_{L_\delta}|\nabla u(x)|^p+|\nabla u_{\varepsilon_k}(x)|^p+|\nabla\varphi_\delta(x)\otimes(u_{\varepsilon_k}(x)-u(x))|^p\,dx\\&\leq C\int_{L_\delta}|\nabla u(x)|^p+|\nabla u_{\varepsilon_k}(x)|^p+\frac{1}{\delta^p}|u_{\varepsilon_k}(x)-u(x)|^p\,dx\end{aligned}$$

and

$$\int_{L_\delta} |\overline{v}_{\varepsilon_k}(x)|^q \, dx = \int_{L_\delta \cap A_\eta} |v_{\varepsilon_k}(x)|^q \, dx + \int_{L_\delta \setminus A_\eta} |c_{\varepsilon_k}|^q \, dx \leq \int_{L_\delta} |v_{\varepsilon_k}(x)|^q \, dx + \int_{L_\delta} (|v_b(x)| + 1)^q \, dx$$

we have

$$H_{\varepsilon_k}(\overline{u}_{\varepsilon_k}, \overline{v}_{\varepsilon_k}; A) \leq H_{\varepsilon_k}(u_{\varepsilon_k}, v_{\varepsilon_k}; A_\eta) + C\nu_k\left(A \setminus \overline{A}_{\eta-\delta}\right) + C\nu_k(L_\delta) + \frac{C}{\delta^p} \int_{L_\delta} |u_{\varepsilon_k}(x) - u(x)|^p \, dx.$$

Letting $k \to \infty$ and $\delta \to 0^+$ in this order one obtains

$$\liminf_{k \to \infty} H_{\varepsilon_k}(\overline{u}_{\varepsilon_k}, \overline{v}_{\varepsilon_k}; A) \leq \mathcal{H}^-_{\{\varepsilon\}}(u, v_b; A) + \rho + C\nu\left(A \setminus \overline{A}_\eta\right) + C\nu(\partial A_{\eta'}).$$

Since $\nu(\partial A_{\eta'}) = 0$ letting $\eta \to 0^+$ one obtains

$$\mathcal{H}^*_{\{\varepsilon\}}(u, v_b; A) \leq \liminf_{k \to \infty} H_{\varepsilon_k}(\overline{u}_{\varepsilon_k}, \overline{v}_{\varepsilon_k}; A) \leq \mathcal{H}^-_{\{\varepsilon\}}(u, v_b; A) + \rho.$$

Letting $\rho$ go to zero we attain the claim. ■

# 4 Applications

In this section we apply the integral representation results and the compactness theorem for the Γ-convergence of a family of general integral functionals obtained in the previous section to provide an explicit integral representation result for the Γ-limit of (1.2).

## 4.1 Relaxation in $W^{1,p} \times L^q_w$

Let $f$ be a Carathéodory function as in the statement of Theorem 1.2 and define $F : W^{1,p}\left(\Omega; \mathbb{R}^d\right) \times L^q(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \to \mathbb{R}$ by

$$F(u, v; A) := \int_A f(x, \nabla u(x), v(x)) \, dx.$$

Considering the relaxed functional defined as in (1.4), our goal is to find an integral representation for $\mathcal{F}$. The proof is based on blow-up techniques developed in [34]. We refer also to [32]. We also emphasize that the relaxation theorem below holds for $p \geq 1$ and $q > 1$. Moreover, the presence of two fields will require in the proof below the use of the decomposition lemma (see [35] and [13]) in two times, first for the gradients and then for the unconstrained fields.

**Proof of Theorem 1.2.** We start showing that, for every $u \in W^{1,p}\left(A; \mathbb{R}^d\right)$, $v \in L^q(A; \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}(u, v; A) \geq \int_A QCf(x, \nabla u(x), v(x)) \, dx.$$

Let $u_n \rightharpoonup u$ in $W^{1,p}(A; \mathbb{R}^d)$, $v_n \rightharpoonup v$ in $L^q(A; \mathbb{R}^m)$, and assume, without loss of generality, that

$$\liminf_{n \to \infty} \int_A f(x, \nabla u_n(x), v_n(x)) \, dx = \lim_{n \to \infty} \int_A f(x, \nabla u_n(x), v_n(x)) \, dx < \infty.$$

By the growth condition on $f$, up to a subsequence, there exists a nonnegative Radon measure $\mu$ such that

$$f(x, \nabla u_n(x), v_n(x)) \mathcal{L}^N \lfloor A \overset{\star}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly star in the sense of measures.

We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq QCf(x_0, \nabla u(x_0), v(x_0)) \tag{4.1}$$

for a.e. $x_0 \in A$.

If (4.1) holds then the desired inequality follows immediately. Indeed, by Proposition 1.203 $i)$ in [31] we have

$$\liminf_{n\to\infty}\int_A f\left(x,\nabla u_n\left(x\right),v_n\left(x\right)\right)dx \geq \mu\left(A\right) \geq \int_A \frac{d\mu}{d\mathcal{L}^N}(x)\,dx \geq \int_A QCf\left(x,\nabla u\left(x\right),v\left(x\right)\right)dx.$$

To show (4.1) we apply Lusin's theorem (see Theorem 1.94 in [31]) to obtain a compact set $K_j\subset A$ with $|A\backslash K_j|\leq\frac{1}{j}$ such that $f|_{K_j}:K_j\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\to\mathbb{R}$ is continuous. Let $K_j^*\subset A$ be the set of Lebesgue points of $\chi_{K_j}$ and set $\omega:=\bigcup_{j=1}^{\infty}\left(K_j\cap K_j^*\right)$. Then

$$|A\backslash\omega|\leq|A\backslash K_j|\leq\frac{1}{j}\to 0 \text{ as } j\to\infty.$$

Fix $x_0\in\omega$ a Lebesgue point of $u$ such that

$$\begin{aligned}
&\frac{d\mu}{d\mathcal{L}^N}\left(x_0\right)=\lim_{\varepsilon\to0^+}\frac{\mu\left(Q\left(x_0,\varepsilon\right)\right)}{\varepsilon^N}<\infty,\\
&\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N+1}}\int_{Q(x_0,\varepsilon)}\left|u\left(x\right)-u\left(x_0\right)-\nabla u\left(x_0\right)\left(x-x_0\right)\right|dx=0,\\
&\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}\left|v\left(x\right)-v\left(x_0\right)\right|^q dx=0.
\end{aligned}\tag{4.2}$$

Choosing $\varepsilon_k\to0^+$ such that $\mu\left(\partial Q\left(x_0,\varepsilon_k\right)\right)=0$ and applying Proposition 1.203 $iii)$ in [31] one has

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}\left(x_0\right)&=\lim_{k\to\infty}\frac{\mu\left(Q\left(x_0,\varepsilon_k\right)\right)}{\varepsilon_k^N}\\
&=\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\varepsilon_k^N}\int_{Q(x_0,\varepsilon_k)}f\left(x,\nabla u_n\left(x\right),v_n\left(x\right)\right)dx\\
&=\lim_{k\to\infty}\lim_{n\to\infty}\int_Q f\left(x_0+\varepsilon_k y,\nabla w_{n,k}\left(y\right),v_{n,k}\left(y\right)\right)dy
\end{aligned}$$

where

$$w_{n,k}\left(y\right):=\frac{u_n\left(x_0+\varepsilon_k y\right)-u\left(x_0\right)}{\varepsilon_k},\qquad v_{n,k}\left(y\right):=v_n\left(x_0+\varepsilon_k y\right).$$

Clearly $w_{n,k}\in W^{1,p}\left(Q;\mathbb{R}^d\right)$ and, by $(4.2)$, $\lim_{k\to\infty}\lim_{n\to\infty}\|w_{n,k}-w_0\|_{L^1(Q;\mathbb{R}^d)}=0$ where $w_0\left(y\right):=\nabla u\left(x_0\right)y$. Let $\{\varphi_R\}$ be a countable dense set of functions in $L^{q'}\left(Q;\mathbb{R}^m\right)$. Then by $(4.2)_3$

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_Q\left(v_{n,k}\left(y\right)-v\left(x_0\right)\right)\varphi_R\left(y\right)dy=0.$$

By a standard diagonalization argument, we may extract subsequences $w_k:=w_{n_k,k}$ and $\widetilde{v}_k:=v_{n_k,k}$ such that $\{w_k\}$ converges to $w_0$ in $L^1\left(Q;\mathbb{R}^d\right)$, $\sup\limits_{k\in\mathbb{N}}\int_Q|\nabla w_k\left(y\right)|^p dy<\infty$, $\{\widetilde{v}_k\}$ converges weakly to $v\left(x_0\right)$ in $L^q\left(Q;\mathbb{R}^m\right)$ and

$$\frac{d\mu}{d\mathcal{L}^N}\left(x_0\right)\geq\lim_{k\to\infty}\int_Q f\left(x_0+\varepsilon_k y,\nabla w_k\left(y\right),\widetilde{v}_k\left(y\right)\right)dy.$$

Notice that if $p=1$ the sequence $\{\nabla w_k\}$ is already $p-$equi-integrable. If $p>1$ by the decomposition lemma (see Lemma 1.2 in [35]), and up to a subsequence, we may find $\{\overline{w}_k\}\subset W^{1,p}\left(Q;\mathbb{R}^d\right)$ such that $\{|\nabla\overline{w}_k|^p\}$ is equi-integrable, $\overline{w}_k=w_0$ on $\partial Q$, $\overline{w}_k\rightharpoonup w_0$ in $W^{1,p}\left(Q;\mathbb{R}^d\right)$ and

$$|\{y\in Q:w_k\left(y\right)\neq\overline{w}_k\left(y\right)\ \text{ or }\nabla w_k\left(y\right)\neq\nabla\overline{w}_k\left(y\right)\}|\to0.$$

Then, applying the decomposition lemma to $\{\widetilde{v}_k\}$ in $L^q$ (see Proposition 2.3 in [13]) we may find, up to a subsequence, $\{\overline{v}_k\}\subset L^q\left(Q;\mathbb{R}^m\right)$ $q-$equi-integrable in $Q$ such that

$$|\{y\in Q:\widetilde{v}_k\left(y\right)\neq\overline{v}_k\left(y\right)\}|\to0\text{ as }k\to\infty,\qquad\int_Q\overline{v}_k\left(y\right)dy=v\left(x_0\right),\quad\text{for every }k\in\mathbb{N}$$

and $\overline{v}_k \rightharpoonup v(x_0)$ in $L^q(Q;\mathbb{R}^m)$.

Hence

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{k\to\infty} \int_{\{\overline{v}_k = v_k \text{ and } \overline{w}_k = w_k\}} f(x_0 + \varepsilon_k y, \nabla \overline{w}_k(y), \overline{v}_k(y))\, dy$$

where we have used the fact that $f \geq 0$. Since $x_0 \in \omega$ there exists $j_0 \in \mathbb{N}$ such that $x_0 \in K_{j_0} \cap K^*_{j_0}$ and using the continuity of $f$ there exists $0 < \rho_j < 1$ such that

$$f(x_0, \xi, b) \leq f(x, \xi, b) + \frac{1}{j}$$

for all $(x, \xi, b) \in K_{j_0} \times \overline{B_j^{d\times N}(0)} \times \overline{B_j^m(0)}$ with $|x - x_0|, \ |u(x) - u(x_0)| \leq \rho_j$.

Set

$$E_{k,j} := \{y \in Q : w_k(y) = \overline{w}_k(y), \ |\varepsilon_k \overline{w}_k(y)| \leq \rho_j, \ |\nabla \overline{w}_k(y)| \leq j, \ \widetilde{v}_k(y) = \overline{v}_k(y), \ |\overline{v}_k(y)| \leq j\}.$$

The sequence $\{\overline{w}_k\}$ is bounded in $W^{1,p}(Q;\mathbb{R}^d)$, $\{\overline{v}_k\}$ is bounded in $L^q(Q;\mathbb{R}^m)$ and $\lim_{j\to\infty}\lim_{k\to\infty}|Q\backslash E_{k,j}| = 0$.

Thus

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{j\to\infty}\liminf_{k\to\infty}\int_{E_{k,j}} f(x_0 + \varepsilon_k y, \nabla \overline{w}_k(y), \overline{v}_k(y))\, dy \\
&= \liminf_{j\to\infty}\liminf_{k\to\infty}\frac{1}{\varepsilon_k^N}\int_{D_{k,j}} f\left(x, \nabla \overline{w}_k\left(\frac{x-x_0}{\varepsilon_k}\right), \overline{v}_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx
\end{aligned}$$

where $D_{k,j} := x_0 + \varepsilon_k E_{k,j}$.

Hence

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{j\to\infty}\liminf_{k\to\infty}\frac{1}{\varepsilon_k^N}\int_{D_{k,j}\cap K_{j_0}} f\left(x, \nabla \overline{w}_k\left(\frac{x-x_0}{\varepsilon_k}\right), \overline{v}_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx \\
&\geq \liminf_{j\to\infty}\liminf_{k\to\infty}\frac{1}{\varepsilon_k^N}\int_{D_{k,j}\cap K_{j_0}} f\left(x_0, \nabla \overline{w}_k\left(\frac{x-x_0}{\varepsilon_k}\right), \overline{v}_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) - \frac{1}{j}\, dx.
\end{aligned}$$

Using the fact that $|\nabla \overline{w}_k| \leq j$ and $|\overline{v}_k| \leq j$ in $E_{k,j}$ and, by the growth conditions on $f$ , we have that

$$\begin{aligned}
&\frac{1}{\varepsilon_k^N}\int_{D_{k,j}\backslash K_{j_0}} f\left(x_0, \nabla \overline{w}_k\left(\frac{x-x_0}{\varepsilon_k}\right), \overline{v}_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx \\
&\leq C a(x_0, u(x_0))(1 + j^p + j^q)\frac{|Q(x_0,\varepsilon_k)\backslash K_{j_0}|}{\varepsilon_k^N} \to 0
\end{aligned}$$

as $k \to \infty$, because $x_0$ is a Lebesgue point of $\chi_{K_{j_0}}$.

Consequently

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{j\to\infty}\liminf_{k\to\infty}\frac{1}{\varepsilon_k^N}\int_{D_{k,j}} f\left(x_0, \nabla \overline{w}_k\left(\frac{x-x_0}{\varepsilon_k}\right), \overline{v}_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx \\
&= \liminf_{j\to\infty}\liminf_{k\to\infty}\int_{E_{k,j}} f(x_0, \nabla \overline{w}_k(y), \overline{v}_k(y))\, dy \\
&= \liminf_{k\to\infty}\int_Q f(x_0, \nabla \overline{w}_k(y), \overline{v}_k(y))\, dy,
\end{aligned}$$

where we have used the growth conditions on $f$, the equi-integrability of $\{|\nabla \overline{w}_k|^p\}$ and $\{|\overline{v}_k|^q\}$ and the fact that $|Q\backslash E_{k,j}| \to 0$.

Since $\overline{w}_k = w_0$ on $\partial Q$, $\displaystyle\int_Q \overline{v}_k(x)\, dx = v(x_0)$ and using (2.3) it follows that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq QCf(x_0, \nabla u(x_0), v(x_0)).$$

To prove the reverse inequality, that is

$$\mathcal{F}(u,v;A) \leq \int_A QCf(x,\nabla u(x), v(x))\,dx,$$

we assume without loss of generality that $f \geq 0$. Arguing as in the proof of Theorem 3.2 it is easily seen that (1.4) fullfills all the assumptions of Theorem 3.1 thus

$$\mathcal{F}(u,v;A) = \int_A g(x,\nabla u(x), v(x))\,dx$$

for some Carathéodory function $g$, for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and every $v \in L^q(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

By Scorza-Dragoni theorem (see Theorem 6.35 in [31]) since $f$ is Carathéodory, for each $j \in \mathbb{N}$, there exists a compact set $K_j \subset A$, with $|A \backslash K_j| < \frac{1}{j}$ such that the restriction of $f$ to $K_j \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$ is continuous. Let $K_j^* \subset A$ be the set of Lebesgue points of $\chi_{K_j}$ and set $\omega := \cup_{j=1}^{\infty}(K_j \cap K_j^*)$. Then

$$|A\backslash\omega| \leq |A\backslash K_j| < \frac{1}{j} \to 0 \qquad \text{as} \qquad j \to \infty.$$

Moreover, since for a.e. $x_0 \in A$

$$g(x_0,\xi_0,b_0) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u_{\xi_0}, v_{b_0}, Q(x_0,\varepsilon))}{\varepsilon^N} \tag{4.3}$$

where $u_{\xi_0}(x) := \xi_0 x$ and $v_0 \equiv b_0$, it is enough to prove that

$$g(x_0,\xi_0,b_0) \leqslant QCf(x_0,\xi_0,b_0)$$

for any $x_0 \in \omega$ satisfying (4.3), any $\xi_0 \in \mathbb{R}^{d\times N}$ and any $b_0 \in \mathbb{R}^m$.

Let $(x_0,\xi_0,b_0)$ be such triple. Fix $\delta > 0$ and let $w \in W_0^{1,\infty}(Q;\mathbb{R}^d)$ and $\eta \in L^\infty(Q;\mathbb{R}^m)$ with $\int_Q \eta(x)\,dx = 0$ be such that

$$\int_Q f(x_0,\xi_0 + \nabla w(x), b_0 + \eta(x))\,dx \leqslant QCf(x_0,\xi_0,b_0) + \delta.$$

Still denoting by $w$ and $\eta$ the extension of these functions to $\mathbb{R}^N$ by $Q-$periodicity, let

$$w_{n,\varepsilon}(x) := \frac{\varepsilon}{n} w\left(n\frac{x-x_0}{\varepsilon}\right) \qquad \text{and} \qquad \eta_{n,\varepsilon}(x) := \eta\left(n\frac{x-x_0}{\varepsilon}\right).$$

Clearly, up to a subsequence, $w_{n,\varepsilon} \rightharpoonup 0$ in $W^{1,p}(Q(x_0,\varepsilon);\mathbb{R}^d)$ as $n \to \infty$ and by Riemann-Lebesgue lemma (see Lemma 2.85 in [31]) $\eta_{n,\varepsilon} \rightharpoonup 0$ in $L^q(Q(x_0,\varepsilon);\mathbb{R}^m)$ as $n \to \infty$.

Therefore, by (4.3) and the definition of $\mathcal{F}$,

$$g(x_0,\xi_0,b_0) \leqslant \liminf_{\varepsilon\to 0^+} \liminf_{n\to\infty} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f(x,\xi_0 + \nabla w_{n,\varepsilon}(x), b_0 + \eta_{n,\varepsilon}(x))\,dx.$$

Let $L := 1 + |\xi_0| + \|\nabla w\|_{L^\infty} + |b_0| + \|\eta\|_{L^\infty}$. Since $x_0 \in \omega$, there exists $j_0 \in \mathbb{N}$ such that $x_0 \in K_{j_0} \cap K_{j_0}^*$ and by the uniform continuity of $f$ on $K_{j_0} \times \overline{B_L^{d\times N}(0)} \times \overline{B_L^m(0)}$, one has the existence of $\rho > 0$ such that if $(x,\xi,b)$, $(\overline{x},\overline{\xi},\overline{b}) \in K_{j_0} \times B_L^{d\times N}(0) \times B_L^m(0)$ such that if $|(x,\xi,b) - (\overline{x},\overline{\xi},\overline{b})| < \rho$ then $|f(x,\xi,b) - f(\overline{x},\overline{\xi},\overline{b})| < \delta$. Therefore for $\varepsilon$ sufficiently small ($\varepsilon < \rho$), and applying the growth condition assumed on $f$,

$$
\begin{aligned}
g(x_0,\xi_0,b_0) &\leqslant \liminf_{\varepsilon\to 0^+}\ \liminf_{n\to\infty}\left(\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap K_{j_0}} f\left(x,\xi_0+\nabla w_{n,\varepsilon}(x), b_0+\eta_{n,\varepsilon}(x)\right)dx\right.\\
&\qquad\left.+\frac{C}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\setminus K_{j_0}} 1+\left(|\xi_0|+\|\nabla w\|_{L^\infty}\right)^p+\left(|b_0|+\|\eta\|_{L^\infty}\right)^q dx\right)\\
&\leqslant \liminf_{\varepsilon\to 0^+}\ \liminf_{n\to\infty}\left(\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap K_{j_0}} f\left(x_0,\xi_0+\nabla w_{n,\varepsilon}(x), b_0+\eta_{n,\varepsilon}(x)\right)dx+\delta\right.\\
&\qquad\left.+C\frac{|Q(x_0,\varepsilon)\setminus K_{j_0}|}{\varepsilon^N}\right)\\
&\leqslant \liminf_{\varepsilon\to 0^+}\ \liminf_{n\to\infty}\left(\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(x_0,\xi_0+\nabla w_{n,\varepsilon}(x), b_0+\eta_{n,\varepsilon}(x)\right)dx+\delta\right.\\
&\qquad\left.+C\frac{|Q(x_0,\varepsilon)\setminus K_{j_0}|}{\varepsilon^N}\right)\\
&=\int_Q f\left(x_0,\xi_0+\nabla w(z), b_0+\eta(z)\right)dz+\delta+\lim_{\varepsilon\to 0^+} C\frac{|Q(x_0,\varepsilon)\setminus K_{j_0}|}{\varepsilon^N}\\
&\leqslant QCf(x_0,\xi_0,b_0)+2\delta,
\end{aligned}
$$

where we have used the periodicity of $f$ and the fact that $x_0$ is a Lebesgue point of $\chi_{K_{j_0}}$ to get $\frac{|Q(x_0,\varepsilon)\setminus K_{j_0}|}{\varepsilon^N}\to 0$ as $\varepsilon\to 0^+$.

Letting $\delta\to 0^+$ we obtain the desired inequality. ■

## 4.2 Homogenization

In this section we prove Theorem 1.1.

Let $F_\varepsilon : L^p\left(\Omega;\mathbb{R}^d\right)\times L^q\left(\Omega;\mathbb{R}^m\right)\times\mathcal{A}(\Omega)\to\overline{\mathbb{R}}$ be given by

$$
F_\varepsilon(u,v;A):=\begin{cases}\displaystyle\int_A f\left(\tfrac{x}{\varepsilon},\nabla u(x), v(x)\right)dx & \text{if } (u,v)\in W^{1,p}\left(A;\mathbb{R}^d\right)\times L^q\left(A;\mathbb{R}^m\right),\\ +\infty & \text{otherwise.}\end{cases}\tag{4.4}
$$

Our goal is to show that the $\Gamma-$limit of $\{F_\varepsilon\}$ admits an integral representation. Precisely,

$$
\mathcal{F}_{\{\varepsilon\}}(u,v;A)=\int_A f_{\text{hom}}\left(\nabla u(x), v(x)\right)dx \tag{4.5}
$$

for all $u\in W^{1,p}\left(A;\mathbb{R}^d\right)$, $v\in L^q\left(A;\mathbb{R}^m\right)$ and $A\in\mathcal{A}(\Omega)$, where $\mathcal{F}_{\{\varepsilon\}}$ is the $\Gamma-$limit of $\{F_\varepsilon\}$ and $f_{\text{hom}}$ is given by $(1.3)$.

We start by showing that the limit in (1.3) is well defined. The proof is an adaptation of Proposition 14.4 in [12] and we present it here for convenience of the reader, since it contains more and accurate details.

**Proposition 4.1** *Let $f:\mathbb{R}^N\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\to\mathbb{R}$ be a Borel function satisfying $(H_1)$ and $(H_2)$ such that $\sup_{x\in\mathbb{R}^N} f(x,\xi,b)<\infty$ for every $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$. Then $f_{\text{hom}}$ is well defined and satisfies $(H_2)$.*

**Proof.** Let $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$ be fixed and for $t>0$ define

$$
\begin{aligned}
g_t:=\tfrac{1}{t^N}\inf\Bigg\{&\int_{(0,t)^N} f(x,\xi+\nabla\varphi(x), b+\eta(x))\,dx :\\
&\varphi\in W_0^{1,p}((0,t)^N;\mathbb{R}^d),\ \eta\in L^q((0,t)^N;\mathbb{R}^m),\ \int_{(0,t)^N}\eta(x)\,dx=0\Bigg\}.
\end{aligned}
$$

Let $\varphi_t \in W_0^{1,p}\left((0,t)^N;\mathbb{R}^d\right)$, $\eta_t \in L^q\left((0,t)^N;\mathbb{R}^m\right)$ be such that

$$g_t + \frac{1}{t} \geq \frac{1}{t^N} \int_{(0,t)^N} f\left(x, \xi + \nabla\varphi_t(x), b + \eta_t(x)\right) dx.$$

Let $s > t$ and $I := \left\{i = (i_1, \ldots, i_N) \in \mathbb{N}_0^N : 0 < ([t]+1)(i_j+1) \leq s\right\}$ where we denote by $[t]$ the integer part of $t$.

Let $Q_s := \cup_{i\in I} i([t]+1) + (0,[t]+1]^N$ and define on $Q_s$ the maps $\varphi_s$ and $\eta_s$ as the extension by $([t]+1)-$periodicity of $\varphi_t$ and $\eta_t$, respectively. Then extend by zero these functions to $(0,s)^N$ still denoting them by $\varphi_s$ and $\eta_s$, respectively. More precisely, on $(0,s)^N$ define

$$\varphi_s(x) := \begin{cases} \varphi_t(x - i([t]+1)) & \text{if } x - i([t]+1) \in (0,[t]+1]^N, \ i \in I, \\ 0 & \text{elsewhere,} \end{cases}$$

$$\eta_s(x) := \begin{cases} \eta_t(x - i([t]+1)) & \text{if } x - i([t]+1) \in (0,[t]+1]^N, \ i \in I, \\ 0 & \text{elsewhere.} \end{cases}$$

Notice that $\varphi_s \in W_0^{1,p}((0,s)^N;\mathbb{R}^d)$, $\eta_s \in L^q((0,s)^N;\mathbb{R}^m)$ and $\int_{(0,s)^N} \eta_s(x)\, dx = 0$.

Let $R_s := (0,s)^N \backslash Q_s$, then

$$|R_s| \leq s^N - \left(\frac{s}{t+1} - 1\right)^N t^N.$$

Moreover, denoting by $\sharp I$ the number of elements of $I$,

$$\sharp I = \left[\frac{s}{[t]+1}\right]^N \leq \left(\frac{s}{[t]+1} + 1\right)^N \leq \left(\frac{s}{t} + 1\right)^N. \tag{4.6}$$

Using the periodicity of $f$, (4.6) and the growth conditions $(H_2)$ we have

$$\begin{aligned} g_s &\leq \frac{1}{s^N} \int_{(0,s)^N} f\left(x, \xi + \nabla\varphi_s(x), b + \eta_s(x)\right) dx \\ &= \frac{1}{s^N} \left( \sum_{i\in I} \int_{i([t]+1)+(0,[t]+1]^N} f\left(x, \xi + \nabla\varphi_s(x), b + \eta_s(x)\right) dx + \int_{R_s} f(x,\xi,b)\, dx \right) \\ &\leq \frac{1}{s^N} \left( \left(\frac{s}{t}+1\right)^N \int_{(0,t)^N} f\left(x, \xi + \nabla\varphi_t(x), b + \eta_t(x)\right) dx + \left(\frac{s}{t}+1\right)^N \int_{(t,[t]+1]^N} f(x,\xi,b)\, dx + C|R_s| \right) \\ &\leq \frac{t^N}{s^N} \left(\frac{s}{t}+1\right)^N \left(g_t + \frac{1}{t}\right) + C\left(\frac{1}{t} + \frac{1}{s}\right)^N + C\left(1 - \left(\frac{t}{t+1} - \frac{t}{s}\right)^N\right). \end{aligned}$$

Taking the upper limit on $s$ and then the lower limit on $t$ we get

$$\limsup_{s\to\infty} g_s \leq \limsup_{t\to\infty} g_t$$

and thus the desired result.

It is easy to see that $f_{\text{hom}}$ satisfies $(H_2)$. Indeed, by taking $\varphi \equiv 0$ and $\eta \equiv 0$ one has

$$f_{\text{hom}}(\xi, b) \leq \limsup_{T\to\infty} \frac{1}{T^N} \int_{(0,T)^N} f(x,\xi,b)\, dx \leq C\left(1 + |\xi|^p + |b|^q\right). \tag{4.7}$$

On the other hand, since $|\cdot|^p$, $|\cdot|^q$ are convex and using Jensen's inequality

$$\begin{aligned}\frac{1}{T^N}\int_{(0,T)^N} f\left(x,\xi+\nabla\varphi\left(x\right),b+\eta\left(x\right)\right)dx &\geq \frac{1}{T^N}\int_{(0,T)^N}\left(\frac{1}{C}\left(\left|\xi+\nabla\varphi\left(x\right)\right|^p+\left|b+\eta\left(x\right)\right|^q\right)-C\right)dx\\ &\geq \frac{1}{C}\left|\frac{1}{T^N}\int_{(0,T)^N}\left(\xi+\nabla\varphi\left(x\right)\right)dx\right|^p\\ &+\frac{1}{C}\left|\frac{1}{T^N}\int_{(0,T)^N}\left(b+\eta\left(x\right)\right)dx\right|^q-C,\end{aligned}$$

where we have used the coercivity of $f$. By taking the infimum over all $\varphi\in W_0^{1,p}\left((0,T)^N;\mathbb{R}^d\right)$ and over all $\eta\in L^q\left((0,T)^N;\mathbb{R}^m\right)$ such that $\int_{(0,T)^N}\eta\left(x\right)dx=0$, one obtains

$$f_{\text{hom}}\left(\xi,b\right)\geq\frac{1}{C}\left(|\xi|^p+|b|^q\right)-C. \tag{4.8}$$

From (4.7) and (4.8) one concludes that $f_{\text{hom}}$ satisfies $(H_2)$. ∎

**Lemma 4.2** *Let $y,\ z\in\Omega$, and $\rho>0$ such that $B_\rho\left(y\right)\cup B_\rho\left(z\right)\subset\Omega$. Then, for any sequence $\{\varepsilon\}$ there is a subsequence $\{\varepsilon_j\}$ such that, under assumptions $(H_1)$ and $(H_2)$,*

$$\mathcal{F}^-_{\{\varepsilon_j\}}\left(u_\xi,v_b;B_\rho\left(y\right)\right)=\mathcal{F}^-_{\{\varepsilon_j\}}\left(u_\xi,v_b;B_\rho\left(z\right)\right) \tag{4.9}$$

*holds, where $u_\xi\left(x\right):=\xi x$ and $v_b\equiv b$ with $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$.*

**Proof.** Fix $\rho>0$, $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$. By Proposition 11.7 in [12] there exist $\{u_k\}\subset W_0^{1,p}\left(B_\rho\left(y\right);\mathbb{R}^d\right)$, $\{v_k\}\subset L^q\left(B_\rho\left(y\right);\mathbb{R}^m\right)$ such that $u_k\to0$ in $L^p\left(B_\rho\left(y\right);\mathbb{R}^d\right)$ and $v_k\rightharpoonup0$ in $L^q\left(B_\rho\left(y\right);\mathbb{R}^m\right)$ and

$$\lim_{k\to\infty}F_{\varepsilon_{j_k}}\left(u_\xi+u_k,v_b+v_k;B_\rho\left(y\right)\right)=\mathcal{F}^-_{\{\varepsilon_j\}}\left(u_\xi,v_b;B_\rho\left(y\right)\right).$$

Following the argument of Proposition 14.3 in [12], we extend $u_k$ and $v_k$ by 0 outside $B_\rho\left(y\right)$. Let $r>1$, let $\tau_k\in\mathbb{R}^N$ be given by

$$\left(\tau_k\right)_i:=\varepsilon_{j_k}\left[\frac{z_i-y_i}{\varepsilon_{j_k}}\right]$$

and let

$$\overline{u}_k\left(x\right):=u_k\left(x-\tau_k\right),\qquad \overline{v}_k\left(x\right):=v_k\left(x-\tau_k\right).$$

Note that $\tau_k\to z-y$ and $\tau_k$ is a period for $x\longmapsto f\left(\frac{x}{\varepsilon_{j_k}},\xi,b\right)$ for all $(\xi,b)\in\mathbb{R}^{d\times N}\times\mathbb{R}^m$. Thus

$$\begin{aligned}F_{\varepsilon_{j_k}}\left(u_\xi+\overline{u}_k,v_b+\overline{v}_k,\tau_k+B_\rho\left(y\right)\right)&=\int_{\tau_k+B_\rho(y)}f\left(\frac{x}{\varepsilon_{j_k}},\xi+\nabla\overline{u}_k\left(x\right),b+\overline{v}_k\left(x\right)\right)dx\\ &=\int_{\tau_k+B_\rho(y)}f\left(\frac{x}{\varepsilon_{j_k}},\xi+\nabla u_k\left(x-\tau_k\right),b+v_k\left(x-\tau_k\right)\right)dx\\ &=\int_{B_\rho(y)}f\left(\frac{t+\tau_k}{\varepsilon_{j_k}},\xi+\nabla u_k\left(t\right),b+v_k\left(t\right)\right)dt\\ &=\int_{B_\rho(y)}f\left(\frac{t}{\varepsilon_{j_k}},\xi+\nabla u_k\left(t\right),b+v_k\left(t\right)\right)dt\\ &=F_{\varepsilon_{j_k}}\left(u_\xi+u_k,v_b+v_k;B_\rho\left(y\right)\right)\end{aligned}$$

where we have used the fact that $\frac{t+\tau_k}{\varepsilon_{j_k}}=\frac{t}{\varepsilon_{j_k}}+\left[\frac{z_i-y_i}{\varepsilon_{j_k}}\right]$ and the periodicity of $f\left(\cdot,\xi,b\right)$.

Moreover, $\overline{u}_k \to 0$ in $L^p\left(B_{\rho r}(z), \mathbb{R}^d\right)$ and $\overline{v}_k \rightharpoonup 0$ in $L^q\left(B_{\rho r}(z); \mathbb{R}^m\right)$. In fact,

$$\int_{B_{\rho r}(z)} |\overline{u}_k(x)|^p\,dx = \int_{B_{\rho r}(z)} |u_k(x-\tau_k)|^p\,dx = \int_{B_\rho(z)+\tau_k} |u_k(t)|^p\ dt \le \int_{B_\rho(y)} |u_k(t)|^p\,dt \to 0.$$

And, for any measurable set $E \subset B_{\rho r}(z)$,

$$\begin{aligned}\int_{B_{\rho r}(z)} \overline{v}_k(x)\,\chi_E(x)\,dx &= \int_{\mathbb{R}^N} \overline{v}_k(x)\,\chi_E(x)\,dx = \int_{\mathbb{R}^N} v_k(x-\tau_k)\,\chi_E(x)\,dx \\ &= \int_{\mathbb{R}^N} v_k(t)\,\chi_{\tau_k+E}(t)\,dt \to \int_{\mathbb{R}^N} 0\chi_{z-y+E}(t)\,dt = 0.\end{aligned}$$

Since $\chi_E$ are dense in $L^{q'}$ we obtain the weak convergence in $L^q$. Hence, assuming that without loss of generality $f \ge 0$ and using the growth condition $(H_2)$

$$\begin{aligned}\mathcal{F}^-_{\{\varepsilon_{j_k}\}}(\xi x, b; B_\rho(z)) &\le \mathcal{F}^-_{\{\varepsilon_{j_k}\}}(u_\xi, v_b; B_{\rho r}(z)) \\ &\le \liminf_{k\to\infty} F_{\varepsilon_{j_k}}(u_\xi + \overline{u}_k, v_b + \overline{v}_k; B_{\rho r}(z)) \\ &\le \liminf_{k\to\infty} F_{\varepsilon_{j_k}}(u_\xi + u_k, v_b + v_k; B_\rho(y)) \\ &+ C\,|B_{\rho r}(y) \setminus B_\rho(y)|\,(1 + |\xi|^p + |b|^q) \\ &= \mathcal{F}^-_{\{\varepsilon_{j_k}\}}(u_\xi, v_b; B_\rho(y)) + C\,|B_{\rho r}(y) \setminus B_\rho(y)|\,(1 + |\xi|^p + |b|^q).\end{aligned}$$

Letting $r \to 1$ then $|B_{\rho r}(y) \setminus B_\rho(y)| \to 0$. Thus we obtain (4.9). ■

**Proof of Theorem 1.1.** To prove that the $\Gamma-$limit expressed in the theorem exists, we will prove that for any sequence $\{\varepsilon_n\} \searrow 0$ there is a subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$ for which the $\Gamma-$limit is the functional $F_{\text{hom}}$. Therefore, since the $\Gamma-$limit for the subsequence $\{\varepsilon_j\}$ is characterized, we get the existence of the $\Gamma-$limit for the sequence $\{\varepsilon_n\}$ and we achieve the result. Let then $\varepsilon_n \searrow 0$ and apply Theorem 3.2 to get, for some subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$,

$$\mathcal{F}_{\{\varepsilon_j\}}(u, v; A) = \int_A g^{\{\varepsilon_j\}}(x, \nabla u(x), v(x))\,dx$$

for some Carathéodory function $g^{\{\varepsilon_j\}} : \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ and for every $u \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$ and $v \in L^q(\Omega; \mathbb{R}^m)$. Moreover, by Lemma 3.5, Lemma 4.2 and by Theorem 3.1, $g^{\{\varepsilon_j\}}$ is independent of $x$ and it is quasiconvex-convex.

We claim that

$$g^{\{\varepsilon_j\}} = f_{\text{hom}}.$$

By (2.3) and Remark 2.10 *ii)*

$$\begin{aligned}g^{\{\varepsilon_j\}}(\xi, b) &= \min\left\{\int_Q g^{\{\varepsilon_j\}}(\xi + \nabla\varphi(x), b + \eta(x))\,dx : \varphi \in W_0^{1,p}\left(Q; \mathbb{R}^d\right),\ \eta \in L^q(Q; \mathbb{R}^m),\ \int_Q \eta(x)\,dx = 0\right\} \\ &= \min\left\{\mathcal{F}_{\{\varepsilon_j\}}(u, v, Q) : u = u_\xi + \varphi,\ v = v_b + \eta,\ \varphi \in W_0^{1,p}\left(Q; \mathbb{R}^d\right),\ \eta \in L^q(Q; \mathbb{R}^m),\right. \\ &\qquad\left. \int_Q \eta(x)\,dx = 0\right\},\end{aligned}$$

where $u_\xi(x) := \xi x$ and $v_b \equiv b$, for every $(\xi, b) \in \mathbb{R}^{d\times N} \times \mathbb{R}^m$.

Thus by the fundamental theorem of $\Gamma-$convergence (see Theorem 2.7) we have

$$g^{\{\varepsilon_j\}}(\xi,b) = \lim_{j\to\infty} \inf \left\{ F_{\varepsilon_j}(u,v,Q) : u = u_\xi + \varphi,\ v = v_b + \eta,\ \varphi \in W_0^{1,p}\left(Q;\mathbb{R}^d\right),\ \eta \in L^q\left(Q;\mathbb{R}^m\right), \int_Q \eta(x)\,dx = 0 \right\}$$

$$= \lim_{j\to\infty} \inf \left\{ \int_Q f\left(\frac{y}{\varepsilon_j}, \nabla u(y), v(y)\right) dy : u = u_\xi + \varphi,\ v = v_b + \eta,\ \varphi \in W_0^{1,p}\left(Q;\mathbb{R}^d\right), \eta \in L^q\left(Q;\mathbb{R}^m\right), \int_Q \eta(x)\,dx = 0 \right\},$$

where we have used Lemma 3.7.

Changing variables one obtains the desired identity. ■

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