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A Study of a Bi-Phasic Flow Problem in Porous Media

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Abstract

In this paper, we develop a simplified formulation of the hydrocarbon system used for the petroleum reservoirs simulation.This system is a simplified model which describes a two-phase flow (oil and gas) with a mass transfer in a porous medium, that leads to the fluid compressibility.this kind of flow is modelled by a system of parabolic degenerated non linear convection-diffusion equations.Under certrain hypothesis, such as validity of Darcy's law, incompressibility of the porous medium, compressibility of the fluids, mass transfer between the oil and the gas and negligible gravity, the global pressure is formulated.This formulation allows the establishment of theoretical results on the existence and uniquness of the solution.

Mathematics Subject Classification: 76D07, 76M25

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1 Introduction

The mathematical study of the multiphasic flows in porous medium has attracted attention to many researchers in oil industry for exploitation of oil or gas field, in water resource management and many environment problems.
The hydrocarbon system is a simplified model which describes a two-phase

flow (oil and gas) with a mass transfer in a porous medium, leading to the fluid compressibility.The global pressure formulation of Guy Chavent [3] will be introduced, this kind of flow (hydrocarbon system) is modeled by a system of parabolic degenerated non linear convection-diffusion equations. This formulation allows the establishment of theoretical results of the existence and the uniqueness of the solution.Based on different studies, for example, in [1] the authors studied the case of two phase incompressible fluids without mass transfer,[3] was devoted to analyze the one phase flow and [8] the black-oil model.Furthermore a study of different numerical schemes have been considered by many authors; among others we can refer the reader to [4] who purposed a finite element scheme, [5] who used a finite difference method while [6] a higher order Godunov scheme and the authors in [9] have proposed a finite volume method.All these schemes seem to suffer from a lack of numerical stability.It will be fruitfull if one can investigate numerical algorithms based on stable methods, see for example [10] and |11].
Here, we are concerned by compressible fluids and our aim is to seek conditions to prove the existence and uniqueness of the developed model's solution..

2 Mathematical model

Let us consider a bounded connected open domain Ω of $\mathbb{R}^d$ with $d = 2$ or 3, describing the porous medium (the reservoir), with a Lipchitz boundary Γ, and let t be the time variable $t \in [0, T[$, $T \prec \infty$.
We consider compressible flows, with constant dynamic viscosities and where the gravity effect is neglected.Under these hypotheses, the Darcy's law combined with mass's conservation equations for each of the component leads to the following system of partial differential equations of parabolic convection-diffusion type

$$\phi(x)\frac{\partial}{\partial t}(\rho_o\omega_o^h S_o) + div(\rho_o\omega_o^h U_o) = 0 \tag{1}$$

$$\phi(x)\frac{\partial}{\partial t}(\rho_g S_g + \rho_o\omega_o^l S_o) + div(\rho_g U_g + \rho_o\omega_o^l U_o) = 0 \tag{2}$$

$$U_i = -K(x)\frac{k_{ri}}{\mu_i}\nabla P_i \tag{3}$$

where S_i, U_i, P_i, ρ_i, μ_i, k_{ri} represent, the saturation, the velocity,the pressure, the density, the viscosity and the relative permeability, of the phase $i = o, g$, respectively, the parameters ϕ and K are the porosity and the absolute permeability of the medium and ω_o^c, $c = h, l$ is the massic fraction of component

c, denoted by h for the heavy component and by l for the light component in the oil phase.
We suppose that it is a saturated regime and is expressed by

$$S_o + S_g = 1 \tag{4}$$

The capillary pressure is given by

$$P_g - P_o = P_c(S_o) = p_c(S_o)p_{cM} \tag{5}$$

where

$$p_{cM} = \sup |P_c(S_o)| \quad \text{ and } \; 0 \leq p_c(S_o) \leq 1 \tag{6}$$

We define the mobility of each phase by

$$\lambda_i = \frac{k_{ri}}{\mu_i}, \qquad i = o, g \tag{7}$$

and the total mobility λ by

$$\lambda = \lambda_o + \lambda_g \tag{8}$$

For simplicity convenience, we set

$$\rho_o^h = \rho_o \omega_o^h, \; \rho = \rho_g + \rho_o, \; b = \rho_g \lambda_g + \rho_o \lambda_o, \text{ and } \; d = \rho_g - \rho_o \tag{9}$$

2.1 Reduced Saturation

Let us define by $S_{i,m}$, the residual saturation of the fluid, $i = o, g,$; we write

$$S_{i,m} \leq S_i, \qquad i = o, g \tag{10}$$

and by $S_{i,M}$,the maximum saturation of the fluid, $i = o, g$, such that

$$S_{g,M} = 1 - S_{o,m} \quad \text{ and } \quad S_{o,M} = 1 - S_{g,m} \tag{11}$$

$$S_{i,m} \leq S_i \leq S_{i,M}, \qquad i = o, g \tag{12}$$

This leads to the reduced saturation S set as

$$S = \frac{S_o - S_{o,m}}{1 - S_{g,m} - S_{o,m}} \tag{13}$$

$$0 \leq S \leq 1 \tag{14}$$

2.2 Global pressure

If $S = 0$, equation (13) disappears.This is one of the main reasons for which the terminology of the"global pressure" was introduced and written as

$$P = \frac{1}{2}\left(P_g + P_o\right) + \gamma\left(S\right) \tag{15}$$

with

$$\gamma\left(S\right) = \frac{1}{2}\int_{S_{o,m}}^{S} \left(\frac{\lambda_g - \lambda_o}{\lambda}\right) \; p_c'\left(\xi\right) \; p_{cM} d\xi \tag{16}$$

The total velocity is given by the sum of the velocities of the two components

$$U = U_g + U_o \tag{17}$$

and we write

$$\gamma(S) = \frac{1}{2}\int_{0}^{S} \alpha\left(\xi\right) d\xi \tag{18}$$

where

$$\alpha\left(S\right) = \left(\frac{\lambda_g\left(S\right) - \lambda_o\left(S\right)}{\lambda\left(S\right)}\right) \; p_c'\left(S\right) \; p_{cM} \tag{19}$$

is the capillary diffusion.

2.3 Boundary and initial conditions

The system must be completed with boundary and initial conditions.We suppose that the reservoir's boundary is not permeable, we write

$$U.\eta = 0, \qquad \text{on } \Gamma \times (0,T) \tag{20}$$

$$\alpha(S)\nabla S = 0, \qquad \text{on } \Gamma \times (0,T) \tag{21}$$

The parameter η denotes the normal vector.

The initial conditions for the saturation and the pressure are given by

$$S\left(x,0\right) = S^0\left(x\right) \qquad \text{in } \Omega \tag{22}$$

$$P\left(x,0\right) = P^0\left(x\right) \qquad \text{in } \Omega \tag{23}$$

Therefore, we write system $(1-3)$ as

$$\Phi(x)\frac{\partial}{\partial t}\left(\rho_o^h S\right) - div\left(K(x)\rho_o^h\lambda_o(S)\nabla P\right) + div\left(K(x)\rho_o^h\alpha(S)\nabla S\right) = f_1 \quad (24)$$

$$\Phi(x)\frac{\partial}{\partial t}(\rho S) - div(K(x)b(S,P)\nabla P) + div\left(K(x)d\left(P\right)\alpha(S)\nabla S\right) = f_2 \quad (25)$$

$$\nabla P.\eta = 0,\ \alpha(S)\nabla S = 0, \qquad \text{on } \Gamma\times(0,T) \quad (26)$$

$$S\left(x,0\right) = S^0\left(x\right),\ P\left(x,0\right) = P^0\left(x\right) \qquad \text{in } \Omega \quad (27)$$

Where

$$f_1 = -\phi(x)S_{o,m}\frac{\partial}{\partial t}\left(\rho_o^h\right) \text{ and } f_2 = -\phi(x)\frac{\partial}{\partial t}\left(\rho S_{o,m}+\rho_g\right)$$

3 Existence and uniqueness of the solution

Let Ω be a connected open set in $\mathbb{R}^d$ ($d=2$ or 3), with a Lipschiz boundary Γ, to ensure the existence and the uniqueness of the weak solution, for our case, we start by setting the following hypotheses:

1. $K\left(x\right)\in L^\infty\left(\Omega\right)$, such that

$$K_- \leq K\left(x\right) \leq K_+ \ \ a.e.\text{in } \Omega$$

2. $\phi(x)\in L^\infty\left(\Omega\right)$, such that:

$$0 \prec \phi_- \leq \phi\left(x\right) \leq \phi_+ \ \ \text{a.e.in } \Omega$$

3. $\rho_o^h(P)\in L^\infty\left(\Omega,(0,T)\right)\cap H^1\left(\Omega,(0,T)\right)$, such that:

$$\rho_{o-}^h \leq \rho_o^h\left(P\right) \leq \rho_{o+}^h \ \ \text{a.e.in } \Omega\times(0,T)$$

4. $\alpha(S)\in L^\infty\left(\Omega,(0,T)\right)$, such that:

$$\alpha_- \leq \alpha\left(S\right) \leq \alpha_+ \ \ \text{a.e.in } \Omega\times(0,T)$$

5. $\lambda_o(S)\in L^\infty\left(\Omega,(0,T)\right)$, such that:

$$\lambda_{o-} \leq \lambda_o\left(S\right) \leq \lambda_{o+} \ \ \text{a. e.in } \Omega\times(0,T)$$

6. $d(P) \in L^{\infty}(\Omega, (0,T))$, such that:

$$d_{-} \leq d(P) \leq d_{+} \quad \text{a.e.in } \Omega \times (0,T)$$

7. $b(S,P) \in L^{\infty}(\Omega, (0,T))$, such that:

$$b_{-} \leq b(S,P) \leq b_{+} \quad \text{a.e.in } \Omega \times (0,T)$$

8. $S^{o}, P^{o} \in L^{\infty}(\Omega, (0,T))$, such that:

$$0 \leq S(x,t) \leq 1 \qquad \text{a.e.in } \Omega \times (0,T)$$

9. $\rho_g(P) \in H^1(\Omega, (0,T))$.

Note that a.e stands for almost everywhere.Then we introduce the following functional spaces

$$H(div, \Omega) = \left\{ v \in \left(L^2(\Omega, (0,T))\right)^d, div(v) \in L^2(\Omega, (0,T)), d = 2,3 \right\} \tag{28}$$

$$V(\Omega) = \{ v \in H(div, \Omega), v.\eta = 0 \quad \text{on } \Gamma \} \tag{29}$$

$$W(\Omega) = \{ v \in V(\Omega), v(x,T) = 0 \quad \text{in } \Omega \}$$

The weak formulation of problem $(24-27)$ is written as

$$\left(\Phi(x)\rho_o^h S, \frac{\partial v}{\partial t} \right)_{\Omega} - \left(K(x)\rho_o^h \lambda_o(S) \nabla P, \nabla v \right)_{\Omega} +$$

$$\left(K(x)\rho_o^h \alpha(S) \nabla S, \nabla v \right)_{\Omega} = (f_1, v) \tag{30}$$

$$\left(\Phi(x)\rho S, \frac{\partial v}{\partial t} \right)_{\Omega} - (K(x)b(S,P)\nabla P, \nabla v)_{\Omega} +$$

$$(K(x)d(P)\alpha(S)\nabla S, \nabla v)_{\Omega} = (f_2, v) \tag{31}$$

where $(.,.)_{\Omega}$ is the inner product defined on $W(\Omega)$.

Proposition 3.1

Under assumption (1.−9.), problem $(24-27)$ has a unique solution $(S,P)\in (V(\Omega))^2$.

Proof

To prove this result, we adapt results in [1] and [3] to our model so that system $(24-25)$ can be written in a simplified form as:

$$\Phi(x)b(S,P)\frac{\partial}{\partial t}\left(\rho_o^h S\right)-b(S,P)\nabla.\left(K(x)\rho_o^h\lambda_o(S)\nabla P\right)+$$

$$+\,b(S,P)\nabla.\left(K(x)\rho_o^h\alpha(S)\nabla S\right)=b(S,P).f_1 \tag{32}$$

$$\Phi(x)\rho_o^h\lambda_o(S)\frac{\partial}{\partial t}\left(\rho S\right)-\rho_o^h\lambda_o(S)\nabla.(K(x)b(S,P)\nabla P)+$$

$$+\rho_o^h\lambda_o(S)\nabla.\left(K(x)d\left(P\right)\alpha(S)\nabla S\right)=\rho_o^h\lambda_o(S).f_2 \tag{33}$$

If we multiply equations (33) by v_1 and (34) by v_2 and integrate over Ω we get

$$\left(\Phi(x)\frac{\partial}{\partial t}\left(\rho_o^h S\right),v_1\right)-\left(K(x)\rho_o^h\lambda_o(S)\nabla P,\nabla v_1\right)+$$

$$+\left(K(x)\rho_o^h\alpha(S)\nabla S,\nabla v_1\right)=\left(f_1,v_1\right) \tag{34}$$

$$\left(\Phi(x)\frac{\partial}{\partial t}\left(\rho S\right),v_2\right)-\left(K(x)b(S,P)\nabla P,\nabla v_2\right)+$$

$$+\left(K(x)d\left(P\right)\alpha(S)\nabla S,\nabla v_2\right)=\left(f_2,v_2\right) \tag{35}$$

where $\nabla v_1=b(S,P)$ and $\nabla v_2=\rho_o^h\lambda_o(S)$. Therefore

$$\int_\Omega \Phi(x)\left(\frac{\partial}{\partial t}\left(\rho_o^h S\right).v_1-\frac{\partial}{\partial t}\left(\rho S\right).v_2\right)+$$

$$+\int_\Omega K\left(x\right)\rho_o^h\alpha(S)\left(b(S,P)-d\left(P\right)\lambda_o(S)\right)\nabla S=\int_\Omega f_1v_1-f_2v_2 \tag{36}$$

We deduce that

$$\Phi(x)\left(\frac{\partial}{\partial t}\left(\rho_o^h S\right).v_1-\frac{\partial}{\partial t}\left(\rho S\right).v_2\right)+$$

$$+K\left(x\right)\rho_o^h\alpha(S)\left(b(S,P)-d\left(P\right)\lambda_o(S)\right)\nabla S=f_1v_1-f_2v_2 \text{ a.e in } \Omega \qquad (37)$$

in the distribution's sense.
Now, we introduce a new unknown $V=K\left(x\right)\rho_o^h\alpha(S)\left(b(S,P)-d\left(P\right)\lambda_o(S)\right)\nabla S$ to get the following two equations:

$$\Phi(x)\left(\frac{\partial}{\partial t}\left(\rho_o^h S\right).v_1-\frac{\partial}{\partial t}\left(\rho S\right).v_2\right)+V=f_1v_1-f_2v_2 \qquad (*)$$

and

$$V=K\left(x\right)\rho_o^h\alpha(S)\left(b(S,P)-d\left(P\right)\lambda_o(S)\right)\nabla S \qquad (**)$$

We first assume that the unknown $V\in L^2\left(\Omega,(0,T)\right)$ is known and we try to find $S\in V\left(\Omega,(0,T)\right)$ from equation $(**)$. We multiply equation $(**)$ by a test fonction $v\in V\left(\Omega,(0,T)\right)$ and we integrate over Ω to get

$$\begin{array}{rcl} a\left(s,v\right) & = & (V,v) \\ \nabla S & = & 0 \qquad \text{on } \Gamma \end{array}$$

where

$$a\left(s,v\right)=\left(K\left(x\right)\rho_o^h\alpha(S)\left(b(S,P)-d\left(P\right)\lambda_o(S)\right)\nabla S,v\right)$$

This problem has a unique solution s in $V\left(\Omega\right)$ So,if we define $A\in \pounds\left(V,V'\right)$ as

$$(AS,v)=a\left(s,v\right) \qquad (38)$$

we can write

$$AS=V \quad \text{a.e in } (0,T) \qquad (39)$$

A is an isomorphism from V into V' (according to the above conditions $(1,3-7)$), thus equation $(**)$ has a unique solution $S\left(x,t\right)$ for $t\in(0,T)$ (from Lax-Milgram theorem [7]).
Let us now study equation $(*)$: $\Phi(x)\left(\frac{\partial}{\partial t}S_1\right)+V=f_1v_1-f_2v_2$ where $S_1=\left(\rho_o^h S\right).v_1-\frac{\partial}{\partial t}\left(\rho S\right).v_2$; from (39) and (40) we get

$$S=A^{-1}V$$

where A is an isomorphism.Note that at $t=0$: $V_0 = AS_0$.If we multiply equation $(*)$ by a test fonction $u \in H^1(\Omega)$ and integrate over Ω we get

$$\int_\Omega \frac{\partial}{\partial t}\left(A^{-1}V\right).u + \left(\frac{1}{\phi(x)}V, u\right) = (f_1 v_1 - f_2 v_2.v)$$

$$\int_\Omega \frac{\partial}{\partial t}A^{-1}V.u + \left(\frac{1}{\phi(x)}V, u\right) = (f_1 v_1 - f_2 v_2.v)$$

If we define the norm $((.,.))$ on Ω by $((u,v)) = \int_\Omega A^{-1}(u).v$, we have the relation

$$((V,u)) + \left(\frac{1}{\phi(x)}V, u\right) = 0, \quad V(0) = V_0$$

This problem has a unique solution V in $L^2(\Omega) \cap L^\infty(\Omega)$.

4 Conclusion

In this paper, we have introduced a simplified formulation of the Hydrocarbon system where the unknowns are the reduced saturation of one of the fluids and the global pressure.This formulation transforms a coupled degenerate non linear parabolic system to a familly of elliptic equations. Hence we prove a theorecal result for the existence and the uniquness of the solution of the resulting system.This is proved by decoupling the equations and using twice Lax-Milgram's theorem [7], first to determine the saturation S assuming that V $\left(V = K(x)\rho_o^h\alpha(S)\left(b(S,P) - d(P)\lambda_o(S)\right)\nabla S\right)$ is known from equation $(*)$, and then obtain V from equation $(**)$.

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