Identification of Certain Polynomial Nonlinear Structures by Adaptive Selectively-Sensitive Excitation

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This paper presents a method for identification of certain polynomial nonlinear dynamic systems by adaptive vibrational excitation. The identification is based on the concept of selective sensitivity and is implemented by an adaptive multihypothesis estimation algorithm. The central problem addressed by this method is reduction of the dimensionality of the space in which the model identification is performed. The method of selective sensitivity allows one to design an excitation which causes the response to be selectively sensitive to a small set of model parameters and insensitive to all the remaining model parameters. The identification of the entire system thus becomes a sequence of low-dimensional estimation problems. The dynamical system is modelled as containing both a linear and a nonlinear part. The estimation procedure presumes precise knowledge of the linear model and knowledge of the structure, though not the parameter values, of the nonlinear part of the model. The theory is developed for three different polynomial forms of the nonlinear model: quadratic, cubic and hybrid polynomial nonlinearities. The estimation procedure is illustrated through simulated identification of quadratic nonlinearities in the small-angle vibrations of a uniform elastic beam.

1 Introduction

In complaining about undue adherence to linear theories, Erwin Schrödinger once wrote to Max Born about the plethora of models in which "[a]ll is linear, linear—linear in the *n*th power I would say, if that was not a contradiction." [1, p. 381]. While much modern engineering is rooted in linear models, today's scientific engineer is intensely aware of the nonlinearities which abound in nature as well as in technology. Identification of mechanical damage in civil structures, for example, is just one of many fields in which the multiplicity of nonlinear phenomena is recognized [2].

Two classes of identification problems can be defined [3]: (*1*) the structure of the model (for linear mechanical systems this refers to the number of degrees of freedom and the type of damping), and (*2*) values of the model parameters (stiffnesses, inertias and so on).

The identification of linear elastic structures has traditionally exploited modal properties which are characteristic of such systems. The complexities inherent in modal analysis of a nonlinear system occur for several reasons [4]. Nonlinear systems do not obey the superposition principle of inputs and outputs, so the technique of orthogonal decomposition of inputs and outputs fails in nonlinear applications. Second, the reciprocity theorem of linear systems does not hold, so the system response is intrinsically and significantly dependent on the location of the excitation. Finally, the frequency response function of a nonlinear system depends on the form of the input signal, so global modal properties cannot be established which are independent of the input. The analytical rectification of these difficulties has led to the development and application of mathematical tools which are foreign to linear-system analysis, such as limit cycles and attractors [5] and Hilbert transforms [6].

While the traditional tools for identification of linear systems are in need of substantial revision when directed to nonlinear applications, the difficulties which characterize linear identification remain in force. In particular, a central problem is the "ill-conditioning" which often arises in high-dimensional inverse problems such as linear-system identification [7]. In practical terms, ill-conditioning appears as large variation of the identified quantities resulting from small variations of the measurements. This is a widespread property of large systems, for which the "curse of dimensionality" has long been recognized [8, p. 197]. Nonlinearities cannot be expected to diminish the intensity of these phenomena.

Various approaches have been studied for identifying nonlinear systems. Generalized transform methods based on Volterra integrals have been widely used [9, 10], estimation methods motivated by ARMA representations have been studied [11], and orthogonal decompositions of measurements have been exploited [12]. The latter approach is especially interesting since the orthogonal decomposition separately estimates each model parameter for the discrete-time nonlinear systems which were studied.

The present paper also concentrates on the question of separately estimating distinct model parameters. Our approach, however, stresses the design of the input to achieve independent

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estimation of a sequence of model parameters. The estimation is based on the idea of selective sensitivity [13–15] which enables one to design an excitation which causes the response to be selectivity sensitive to a small set of model parameters and insensitive to the remaining model parameters. The identification of the entire system thus becomes a sequence of low-dimensional estimation problems, which alleviates the ill-conditioning of the estimation.

While selective sensitivity motivates the adaptive identification to be presented, on a more fundamental level the present analysis of nonlinear systems is grounded on the idea of substructure parameterization [16, 17]. Even if the dynamics of a system are nonlinear, one can often model the behavior in such a way that the mathematical operators which define the dynamics are linear in the unknown model parameters. This of course does not remove the nonlinear dependence of the output on these parameters, but it does assure an important linearity in the sensitivity of the model to uncertainty in the unknown model parameters. This linear sensitivity is central to our analysis, as will become evident in section 3.

From among the wide selection of commonly studied nonlinear model structures, we concentrate on polynomial nonlinearities. The motivation is two-fold. First, such models are important and have been the subject of investigation [18]. Second, the continuity of polynomial functions, as opposed to discontinuous nonlinearities, makes the polynomial a suitable starting point for investigating a new method of analysis. We will consider specific forms of quadratic, cubic, and combined quadratic-cubic polynomial operators in sections 4, 5, and 6. In section 7 we will discuss an adaptive multihypothesis algorithm which is based on the selectively-sensitive excitations derived in earlier sections. In section 8 we present a simulated application to detection of quadratic stiffness nonlinearities in a uniform vibrating beam.

2 System Formulation

Consider the finite-dimensional nonlinear system whose state vector is N_D-dimensional and denoted $x(t)$. The dynamics of the system are represented by:

$$\mathcal{L}(x) + \mathfrak{N}(x) = Hf(t) \tag{1}$$

where $\mathcal{L}(x)$ is a linear differential operator, $\mathfrak{N}(x)$ is a nonlinear N_D-dimensional vector function, $f(t)$ is an N_R-dimensional input vector and H is an $N_D \times N_R$ constant real matrix.

The aim of this study is to design the excitation, $f(t)$, so as to identify the nonlinear part of the model, $\mathfrak{N}(x)$, employing knowledge of the linear operator $\mathcal{L}(x)$. The analysis will be based on the concept of selective sensitivity. Results will be developed for several polynomial forms of the nonlinear function $\mathfrak{N}(x)$.

The structure of $\mathcal{L}(x)$ is:

$$\mathcal{L}(x) = \sum_{n=0}^{N_L} A_n \frac{d^n x(t)}{dt^n} \tag{2}$$

where each A_n is a real, constant $N_D \times N_D$ matrix. For a linear elastic system the linear part of the model may take the form:

$$\mathcal{L}(x) = M\ddot{x}(t) + C\dot{x}(t) + Kx(t) \tag{3}$$

where M, C, and K are inertia, damping, and stiffness matrices, respectively.

Now consider a measurement equation which expresses the fact that only N_M measurements are made. The output $y(t)$ is related to the state vector $x(t)$ by:

$$y(t) = Gx(t) \tag{4}$$

where G is a real constant $N_M \times N_D$ real matrix.

The linear part of the model, $\mathcal{L}(x)$, depends on an r_p-dimensional vector p of linear-model parameters. Similarly, $\mathfrak{N}(x)$ depends on an r_Q-dimensional vector q of nonlinear model parameters as well as, possibly, on p. Let us define an $(r_P + r_Q)$-dimensional parameter vector $\rho^T = (p^T, q^T)$, where the superscript T implies matrix transposition. We will assume that G and H are independent of the model parameters p and q.

We will denote the Laplace transforms of $x(t)$, $y(t)$, $f(t)$, and $\mathfrak{N}(x)$, by $\xi(s)$, $\psi(s)$, $\phi(s)$, and $\theta(s)$, respectively.

Now define a "dynamic stiffness matrix" of the linear part of the model, $\Gamma(s)$, in terms of the Laplace transform of $\mathcal{L}(x)$, for zero initial conditions:

$$\mathfrak{J}[\mathcal{L}(x)] = \Gamma(s)\xi(s) \tag{5}$$

where $\mathfrak{J}[\bullet]$ is the Laplace transform operator. $\Gamma(s)$ is a complex $N_D \times N_D$ matrix and, we assume, nonsingular for at least one value of s. For example, if $\mathcal{L}(x)$ is given by Eq. (3) then $\Gamma(s)$ becomes:

$$\Gamma(s) = s^2 M + sC + K \tag{6}$$

In this case $\Gamma(s)$ is invertible provided that s is not an eigenvalue of $\Gamma(s)$.

Now the Laplace transformation of Eq. (1) can be expressed:

$$\xi(s) = \Gamma^{-1}(s)H\phi(s) - \Gamma^{-1}(s)\theta(s) \tag{7}$$

Employing Eq. (4) one obtains the Laplace transform of the output as:

$$\psi(s) = G\Gamma^{-1}(s)H\phi(s) - G\Gamma^{-1}(s)\theta(s) \tag{8}$$

This is not strictly an input-output relation, since the input is ϕ but the state, $x(t)$, appears in both the output $\psi(s)$ and in $\theta(s)$.

3 Sensitivity to Model Parameters

In this section we investigate the variation of the frequency-domain output, $\psi(s)$, with the model parameters. This prepares the ground for designing the excitation function, based on the concept of selective sensitivity [13–15].

Evaluation of the sensitivity is obtained by differentiating $\psi(s)$ with respect to a model parameter, assuming that G, H, and $\phi(s)$ are independent of the model parameters but that $\Gamma(s)$, $\psi(s)$, and $\theta(s)$ depend on ρ. Employing the relation:

$$\frac{\partial \Gamma^{-1}}{\partial \rho_m} = -\Gamma^{-1}\frac{\partial \Gamma}{\partial \rho_m}\Gamma^{-1} \tag{9}$$

one finds from Eq. (8):

$$\frac{\partial \psi(s)}{\partial \rho_m} = GW_m(\theta - H\phi) - G\Gamma^{-1}\frac{\partial \theta}{\partial \rho_m}, \quad m = 1, \ldots, r_P + r_Q \tag{10}$$

where we define:

$$W_m \equiv \Gamma^{-1}\frac{\partial \Gamma}{\partial \rho_m}\Gamma^{-1}, \quad m = 1, \ldots, r_P + r_Q \tag{11}$$

Now define the *sensitivity of the response* to model parameter ρ_m as the inner product of this vector with itself:

$$S(\rho_m) \equiv \left(\frac{\partial \psi}{\partial \rho_m}\right)^\dagger \left(\frac{\partial \psi}{\partial \rho_m}\right) \tag{12}$$

$$= (\theta - H\phi)^\dagger W_m^\dagger G^T G W_m (\theta - H\phi) - (\theta - H\phi)^\dagger W_m^\dagger G^T G \Gamma^{-1}\left(\frac{\partial \theta}{\partial \rho_m}\right) - \left(\frac{\partial \theta}{\partial \rho_m}\right)^\dagger \Gamma^{-1\dagger} G^T G W_m(\theta - H\phi) + \left(\frac{\partial \theta}{\partial \rho_m}\right)^\dagger \Gamma^{-1\dagger} G^T G \Gamma^{-1}\left(\frac{\partial \theta}{\partial \rho_m}\right) \tag{13}$$

where the superscript † implies matrix conjugate transposition. $S(\rho_m)$ is non-negative. A large value for $S(\rho_m)$ implies that the output varies strongly with variation of the mth model parameter. Conversely, if $S(\rho_m)$ is small, then the response is comparatively insensitive to ρ_m. We seek an excitation which causes the sensitivity to be non-zero for only a single model parameter. The system response to this excitation is used for estimating the single corresponding model parameter. This estimation is insensitive to inaccuracy in knowledge of the remaining model parameters. This is of course differential insensitivity: the response is strictly invariant only for small fluctuations of the remaining model parameters.

We are particularly interested in the sensitivity to the nonlinear model parameters, q. The argument will center on the fact that $\Gamma(s)$, the linear dynamic stiffness matrix, is independent of q. This implies that W_m vanishes for nonlinear model parameters ρ_m, $m > r_P$:

$$W_m = 0,\ m > r_P \tag{14}$$

Consequently, the sensitivity to a nonlinear model parameter q_m is simply:

$$S(q_m) = \theta_m^\dagger \Gamma^{-1\dagger} G^T G \Gamma^{-1} \theta_m \tag{15}$$

where we define:

$$\theta_m = \frac{\partial \theta}{\partial q_m},\ m = 1, \ldots, r_Q \tag{16}$$

We will henceforth be interested only in the sensitivity to nonlinear model parameters. In denoting the sensitivity we will adopt the abbreviated notation of referring only to the index of the parameter as a subscript: $S_m \equiv S(q_m)$. Furthermore, let us define:

$$V(s) \equiv \Gamma^{-1\dagger} G^T G \Gamma^{-1} \tag{17}$$

We note the following points. (*1*) $S(q_m)$ is real and non-negative. In particular, $S(q_m)$ is positive semi-definite in θ_m. (*2*) $V(s)$ is hermitian and positive semi-definite. (*3*) $V(s)$ is independent of the nonlinear part of the model, q. (*4*) The derivation of Eqs. (13) and (15) is independent of the structure of the nonlinear part of the model.

4 Selective Sensitivity for a Quadratic Non-Linearity

In this section we consider a quadratic nonlinearity and we design the excitation so that the output is sensitive to a single nonlinear model parameter and insensitive to infinitesimal variations of all the remaining nonlinear model parameters. The derivation of this excitation will require specification of the nonlinear part of the model. In section 5 a cubic nonlinear term will be examined, and in section 6 we will study a hybrid quadratic-cubic nonlinearity.

4.1 The Quadratic Non-Linear Term. Let us consider a particular choice of the nonlinear model:

$$\mathfrak{N}(x) = (xq^T)x = (q^T x)x = (xx^T)q \tag{18}$$

Thus the ith element of the vector function $\mathfrak{N}(x)$ is:

$$\mathfrak{N}_i(x) = x_i \sum_{j=1}^{N_D} q_j x_j \tag{19}$$

For this model, the derivatives with respect to nonlinear parameters are:

$$\frac{\partial \mathfrak{N}}{\partial q_\alpha} = x_\alpha x \tag{20}$$

Consequently, recalling the definition in Eq. (16), one finds:

$$\theta_\alpha = \mathfrak{I}(x_\alpha x),\ \alpha = 1, \ldots, r_Q \tag{21}$$

4.2 The State Vector. We wish to find an excitation $f(t)$ which causes the sensitivity to the αth nonlinear model parameter to be non-zero, and the sensitivities to the remaining nonlinear model parameters to vanish. This form of "selectively sensitive" excitation will provide the basis for an adaptive multi-hypothesis estimation of the nonlinear model parameters, to be discussed in section 7 and illustrated in section 8. The approach will be, first, to find the state vector which this excitation must induce, and then (in the next subsection) to find the excitation itself.

Let $v(s)$ be a complex vector which is the Laplace transform of a real vector $z(t)$:

$$v(s) = \mathfrak{I}[z(t)] \tag{22}$$

Our approach will be as follows. First find the excitation which causes θ_α to equal $v(s)$:

$$\theta_\alpha = v(s) \tag{23}$$

We will find this excitation by noting that condition (23) uniquely determines the state vector $x(t)$. Once we derive an expression for this state vector, the excitation is calculated from Eq. (1).

Having found the state vector which is consistent with Eq. (23), the second step of the argument is to choose the vector $v(s)$ so that S_α is non-zero and so that S_β vanishes, for all $\beta \neq \alpha$. This will be based on observing, in Eq. (15), that the sensitivity $S(q_\alpha)$ is a quadratic function of θ_α.

To begin, we let $v(s)$ be an arbitrary complex N_D-vector which is Laplace-invertible to a vector $z(t)$, where $z_\alpha(t)$ is positive. Recall that we seek selective sensitivity to q_α.

To find the state vector determined by $v(s)$, we note that Eqs. (21) and (23) imply:

$$\mathfrak{I}[x_\alpha x] = v(s) \tag{24}$$

Inverting this to the time domain, one can readily demonstrate that this relation is satisfied by the following state vector:

$$x(t) = \pm \frac{1}{\sqrt{z_\alpha(t)}} z(t) \tag{25}$$

The excitation, $f(t)$, which produces this state vector is obtained by substituting this $x(t)$ into Eq. (1). This is discussed further in subsection 4.3.

Recall that the sensitivity to q_α is, from Eq. (15), simply $\theta_\alpha^\dagger V \theta_\alpha$. Consequently, if the excitation causes the state to behave as Eq. (25), then Eq. (23) holds and the sensitivity to q_α is:

$$S_\alpha = v^\dagger V u \tag{26}$$

To assure that the sensitivity S_α is positive, we must ascertain that $v(s)$ is not an eigenvector of $V(s)$ whose associated eigenvalue is zero.

What will be the sensitivity to a different nonlinear model parameter, q_β? In other words, what is the value of:

$$S_\beta = \theta_\beta^\dagger V \theta_\beta \tag{27}$$

when Eq. (25) describes the behavior of the system? Employing Eq. (21) (replacing α by β) and Eq. (25) one finds:

$$\theta_\beta = \mathfrak{I}[x_\beta x] = \mathfrak{I}\left[\frac{z_\beta}{z_\alpha} z\right] \tag{28}$$

Now, suppose we choose $v(s)$ or, more directly, $z(t)$, so that:

$$z(t) = \zeta(t) r \tag{29}$$

where r is a real positive N_D-vector with $r_\alpha > 0$ and $\zeta(t)$ is a real positive scalar function. Then

$$\frac{z_\beta(t)}{z_\alpha(t)} = \frac{r_\beta}{r_\alpha},\ \beta = 1, \ldots, N_D \tag{30}$$

Combining Eqs. (27)–(30), the sensitivity to q_β becomes:

$$S_\beta = \frac{r_\beta^2}{r_\alpha^2} v^\dagger V u = \frac{r_\beta^2}{r_\alpha^2} S_\alpha \tag{31}$$

The numbers r_β, $\beta \neq \alpha$, may be freely chosen, so the relative sensitivity, S_β / S_α, may be made arbitrarily small for each $\beta \neq \alpha$.

For example, we may choose:

$$r_\beta = \begin{cases} 1 & , \quad \beta = \alpha \\ \epsilon & , \quad \beta \neq \alpha \end{cases} \tag{32}$$

where ϵ is a small positive number. Then the sensitivity to q_β is much less than the sensitivity to q_α:

$$\frac{S_\beta}{S_\alpha} = \epsilon^2, \ \beta \neq \alpha \tag{33}$$

In fact, we are free to chose $\epsilon = 0$, which causes the response to be sensitive to q_α and differentially insensitive to all other nonlinear model parameters.

Choosing $\epsilon = 0$, the vector function $z(t)$ which determines the state vector becomes, in light of Eq. (29):

$$z(t) = \zeta(t) e^\alpha \tag{34}$$

where $\zeta(t)$ is an arbitrary positive real scalar function and e^α is a standard unit vector: unity in the αth position and zero elsewhere. The sensitivity to q_α can be made arbitrarily large by proper choice of $\zeta(t)$.

Combining Eqs. (25) and (34) yields the following expression for the state vector:

$$x(t) = \pm\sqrt{\zeta(t)}\, e^\alpha \tag{35}$$

where $\zeta(t)$ is an arbitrary real positive scalar function.

Returning to the most general choice of $z(t)$, Eq. (29), the state vector, Eq. (25), becomes:

$$x(t) = \sqrt{\frac{\zeta(t)}{r_\alpha}}\, r \tag{36}$$

4.3 The Excitation Function. The choice of Eq. (32) with $\epsilon = 0$ seems clearly preferable over other possibilities for r, since it yields complete differential insensitivity to all nonlinear model parameters other than q_α. However, this choice of r, and the resulting state vector $x(t)$ [Eq. (35)] are acceptable only if an excitation function $f(t)$ exists which is consistent with Eq. (1) at each instant in time. At each instant, Eq. (1) can be thought of as a linear equation in the unknown vector $f(t)$. It is well known [19, p. 906], that a necessary and sufficient condition for the existence of a solution of such an equation can be expressed in terms of the ranks of the matrices involved. In our case, a necessary and sufficient condition for the existence of a vector $f(t)$ is that the input matrix H and the augmented matrix $\{H, \mathcal{L}[x(t)] + \mathfrak{N}[x(t)]\}$ have the same rank at each value of t:

$$\text{rank}[H] = \text{rank}\{H, \mathcal{L}[x(t)] + \mathfrak{N}[x(t)]\} \tag{37}$$

The flexibility in choosing r and $\zeta(t)$ enables one considerable freedom in seeking to satisfy this condition. Nevertheless, when H is not invertible, the general problem of selecting $\zeta(t)$ and r to satisfy Eq. (37) and to provide adequate selective sensitivity is rather difficult.

We will consider an approximate method for selecting the excitation, rather than addressing this general problem. Choose a positive function $\zeta(t)$ and a real vector r (with $r_\alpha > 0$) yielding adequate selective sensitivity according to Eq. (31). Now calculate $x(t)$ from Eq. (36). Let $g(t)$ be the vector function obtained by substituting this state vector into the lefthand side of Eq. (1):

$$g(t) \equiv \mathcal{L}[x(t)] + \mathfrak{N}[x(t)] \tag{38}$$

If H is square and nonsingular then the excitation is found exactly from Eq. (1) as:

$$f(t) = H^{-1} g(t) \tag{39}$$

If H is not invertible, we construct an excitation in a specified time interval $[0, T]$ which approximately produces the state response of Eq. (36). Represent the excitation vector by a truncated Fourier cosine series:

$$f(t) = \sum_{n=0}^{N_F} f^n \cos n\pi t/T \tag{40}$$

where f^n is a vector of Fourier coefficients of the nth harmonic. Expand $g(t)$ similarly:

$$g(t) = \sum_{n=0}^{\infty} g^n \cos n\pi t/T \tag{41}$$

We will choose f^n so that Hf^n approximates g^n, by minimizing the quadratic form:

$$J_n = (Hf^n - g^n)^T (Hf^n - g^n) \tag{42}$$

Furthermore, the magnitude of f^n is constrained by:

$$f^{nT} f^n = w_n^2 \tag{43}$$

The values of w_n, $n = 0, \ldots, N_F$, determine the relative contribution of the $N_F + 1$ harmonics to the total synthesized excitation, Eq. (40).

Adjoining the constraint, Eq. (43), to J_n with a Lagrange multiplier λ and then differentiating with respect to f^n one obtains the optimal choice of the nth vector of Fourier coefficients:

$$f^n = (H^T H + \lambda I)^{-1} H^T g^n \tag{44}$$

The unknown multiplier λ is chosen to satisfy the constraint, Eq. (43), which becomes:

$$g^{nT} H (H^T H + \lambda I)^{-2} H^T g^n = w_n^2 \tag{45}$$

5 Selective Sensitivity for a Cubic Non-Linearity

5.1 The Cubic Nonlinear Term. The method of the previous section can be applied to a cubic nonlinearity of analogous form. Let the nonlinear model parameters be stored in a square $N_D \times N_D$ matrix Q. Let the nonlinear vector function $\mathfrak{N}(x)$ be:

$$\mathfrak{N}(x) = [x^T Q x] x = [x x^T] Q x \tag{46}$$

We lose no generality by assuming Q to be a symmetric matrix. The ith element of the vector function $\mathfrak{N}(x)$ is:

$$\mathfrak{N}_i(x) = [x^T Q x] x_i = x_i \sum_{j,k=1}^{N_D} q_{jk} x_j x_k \tag{47}$$

For this model, the derivatives with respect to the nonlinear parameters are:

$$\frac{\partial \mathfrak{N}_i}{\partial q_{\alpha\beta}} = x_\alpha x_\beta x_i \tag{48}$$

In analogy to Eq. (16) we define:

$$\theta_{\alpha\beta} = \frac{\partial \theta}{\partial q_{\alpha\beta}} \tag{49}$$

For this cubic model the ith element of $\theta_{\alpha\beta}$ is:

$$(\theta_{\alpha\beta})_i \equiv \frac{\partial \theta_i}{\partial q_{\alpha\beta}} = \Im(x_\alpha x_\beta x_i) \tag{50}$$

5.2 The State Vector. As in section 4, we wish to find an excitation $f(t)$ which causes the sensitivity to the $\alpha\beta$th nonlinear model parameter to be non-zero, and the sensitivities to the remaining non-linear model parameters to vanish. The procedure is much the same.[1] First we find the state vector which the excitation must induce; then we find excitation.

Let $v(s)$ be a complex vector which is the Laplace transform of $z(t)$. Choose $z(t)$ so that neither $z_\alpha(t)$ nor $z_\beta(t)$ vanish

[1] It will be noted that extension to higher orders of nonlinearity is also possible.

and so that $v(s)$ is not an eigenvector of $V(s)$ with zero as associated eigenvalue. Then the state vector is completely[2] determined by the condition:

$$\theta_{\alpha\beta} = v(s) \quad (51)$$

To find the state vector consistent with this relation, note that Eq. (51) is equivalent to:

$$\mathfrak{Z}[x_\alpha x_\beta x_i] = v_i(s),\ i = 1, \ldots, N_D \quad (52)$$

or, inverting to the time domain:

$$x_\alpha x_\beta x_i = z_i(t) \quad (53)$$

This relation is satisfied by:

$$x(t) = \frac{1}{(z_\alpha z_\beta)^{1/3}} z(t) \quad (54)$$

Now we wish to choose $z(t)$ to obtain selective sensitivity to $q_{\alpha\beta}$ and differential insensitivity to other nonlinear model parameters. Employing a method analogous to that of section 4, we note that, based on Eq. (54):

$$(\theta_{mn})_i = \mathfrak{Z}[x_m x_n x_i] = \mathfrak{Z}\left[\frac{z_m z_n}{z_\alpha z_\beta} z_i\right] \quad (55)$$

Choose $z(t)$ as:

$$z(t) = \zeta_\alpha(t) e^\alpha + \zeta_\beta(t) e^\beta \quad (56)$$

where $\zeta_\alpha(t)$ and $\zeta_\beta(t)$ are nonvanishing real scalar functions which we are free to choose. Then $(\theta_{mn})_i$ vanishes for every i if either m or n differ from both α and β. Thus θ_{mn} is identically zero if $\{m, n\} \not\subseteq \{\alpha, \beta\}$. But the sensitivity to q_{mn} is $S_{mn} = \theta_{mn}^\dagger V \theta_{mn}$. As a result, the sensitivities to nonlinear model parameters other than $q_{\alpha\beta}$ vanish:

$$S_{mn} = 0 \text{ if } \{m, n\} \not\subseteq \{\alpha, \beta\} \quad (57)$$

Furthermore, the sensitivity to $q_{\alpha\beta}$ can be made arbitrarily large by appropriate choice of $\zeta_\alpha(t)$ and $\zeta_\beta(t)$. We have thus obtained selective sensitivity to $q_{\alpha\beta}$, $q_{\alpha\alpha}$ and $q_{\beta\beta}$.

Substituting Eq. (56) into Eq. (54) results in the following expression for $x(t)$:

$$x(t) = \zeta_\alpha^{2/3}(t)\zeta_\beta^{-1/3}(t)e^\alpha + \zeta_\alpha^{-1/3}(t)\zeta_\beta^{2/3}(t)e^\beta \quad (58)$$

where $\zeta_\alpha(t)$ and $\zeta_\beta(t)$ are arbitrarily positive functions.

5.3 The Excitation. The method of subsection 4.3 can be used here without modification to select an excitation which causes the state to approximate Eq. (58). In Eq. (38), $g(t)$ is evaluated with $x(t)$ from Eq. (58), and then Eqs. (40), (44), and (45) determine the desired excitation.

6 Selective Sensitivity for a Hybrid Nonlinearity

In this section we derive an excitation which provides nearly complete selective sensitivity to parameters of a nonlinear model formed as a combination of the quadratic and cubic terms studied in sections 4 and 5.

6.1 The Hybrid Nonlinear Function. The nonlinear component of the model is formed by adding together Eqs. (18) and (46):

$$\mathfrak{N}[x(t)] = (q^T x)x + (x^T Q x)x \quad (59)$$

Thus the nonlinear model parameters are stored in the vector q and in the matrix Q. Derivatives of $\mathfrak{N}$ with respect to these parameters are:

$$\frac{\partial \mathfrak{N}}{\partial q_\alpha} = x_\alpha x \quad (60)$$

$$\frac{\partial \mathfrak{N}}{\partial q_{\alpha\beta}} = x_\alpha x_\beta x \quad (61)$$

The Laplace transforms of these quantities are:

$$\theta_\alpha = \mathfrak{Z}(x_\alpha x) \quad (62)$$

$$\theta_{\alpha\beta} = \mathfrak{Z}(x_\alpha x_\beta x) \quad (63)$$

6.2 Selectivity for q_α. We will now indicate the state vector which must be elicited in order to yield sensitivity to the nonlinear model parameter q_α. It will be seen that the response is insensitive to all other elements of q, and insensitive to all elements of Q except $q_{\alpha\alpha}$.

Let the state behave as in Eq. (35):

$$x(t) = \sqrt{\zeta(t)}\, e^\alpha \quad (64)$$

where $\zeta(t)$ is a positive real function. The excitation which (approximately) induces this state response is derived in subsection 4.3. We already know from subsection 4.2 that the sensitivity to q_β vanishes for all $\beta \neq \alpha$.

What is the sensitivity to q_{mn} when the excitation causes the state to follow Eq. (64)? In other words, what is the value of $\theta_{mn}^\dagger V \theta_{mn}$? Employing Eq. (64) in Eq. (63) shows that θ_{mn} vanishes if either m or n differs from α. But $S(q_{mn}) = \theta_{mn}^\dagger V \theta_{mn}$. From this we conclude that the sensitivity vanishes to all the model parameters in Q other than $q_{\alpha\alpha}$:

$$S(q_{mn}) = 0 \text{ if } \{m, n\} \not\subseteq \{\alpha\} \quad (65)$$

and

$$S(q_{\alpha\alpha}) \neq 0 \quad (66)$$

6.3 Selectivity for $q_{\alpha\beta}$. We will now indicate the state vector which must be induced in order to yield sensitivity to the nonlinear model parameter $q_{\alpha\beta}$. It will be seen that the response is insensitive to all elements of Q other than $q_{\alpha\alpha}$ and $q_{\beta\beta}$, and insensitive to all elements of q except q_α and q_β.

Let the state behave as in Eq. (58):

$$x(t) = \zeta_\alpha^{2/3}(t)\zeta_\beta^{-1/3}(t)e^\alpha + \zeta_\alpha^{-1/3}(t)\zeta_\beta^{2/3}(t)e^\beta \quad (67)$$

where $\zeta_\alpha(t)$ and $\zeta_\beta(t)$ are arbitrary positive functions. The excitation which (approximately) induces this state response is derived in subsection 5.3. We already know from subsection 5.2 that the sensitivity to q_{mn} vanishes for all $\{m, n\} \not\subseteq \{\alpha, \beta\}$.

What is the sensitivity to q_m when the excitation causes the state to follow Eq. (67)? In other words, what is the value of $\theta_m^\dagger V \theta_m$? Employing Eq. (67) in Eq. (62) shows that θ_m vanishes if m differs from both α and β. Thus the sensitivity to all the model parameters in q other than q_α and q_β vanishes:

$$S(q_m) = 0 \text{ if } m \notin \{\alpha, \beta\} \quad (68)$$

and

$$S(q_\alpha) \neq 0,\ S(q_\beta) \neq 0 \quad (69)$$

7 Identification

To summarize our discussion up to now of the three different polynomial nonlinearities, we have found a state vector which, if induced by an excitation, causes zero differential sensitivity to (usually) only one[3] non-linear model parameter. We have also shown how to calculate the excitation for achieving this state vector. It now remains to formulate a method for identifying the value of the nonlinear model parameter to which the response is sensitive.

We will not attempt to estimate q_α directly. Rather, an adaptive, multihypothesis decision method will be employed.

7.1 Motivation. To motivate this approach let us point out that calculation of the excitation function needed for obtaining selective sensitivity to q_α requires knowledge of q_α itself,

[2]Some sign variations (±) are possible as in Eq. (25), but we will ignore them.

[3]Sensitivity to 2 or 3 non-linear model parameters occurs in the cubic and hybrid non-linear models.

as well as of the other non-linear model parameters, in order to calculate $g(t)$ in Eq. (38). To emphasize that the excitation is a function of the current estimate $\hat{q}$ of the nonlinear model parameters, let us denote the excitation as $f(t; \hat{q})$. The response that is elicited by $f(t; \hat{q})$ is then dependent on both $\hat{q}$ and on the actual value of q. We will indicate this as $x(t; \hat{q}, q)$.

When the current estimate is completely correct ($\hat{q} = q$), the state response will be precisely as anticipated in subsection 4.2, either Eq. (35) or (36) (depending on the choice of $z(t)$). Let us denote this anticipated state as $\hat{x}(t)$. However, if the model-estimate is erroneous ($\hat{q} \neq q$) then the state response will differ from $\hat{x}(t)$.

The difference between $\hat{x}(t)$ and $x(t)$ is a measure of the accuracy of the estimate, $\hat{q}$. In a moment we will define specific measures of accuracy and illustrate how they are used to estimate model parameters. Before doing so, let us point out the role played by selective sensitivity. By definition, if the response is selectively insensitive to a given model parameter, q_β, then the response is invariant to infinitesimal changes in q_β. When the estimate of the model is correct ($\hat{q} = q$) an excitation can be found which causes complete insensitivity to any particular set of model parameters. Conversely, as an erroneous estimate improves, the insensitivity also improves. So, while the magnitude of the discrepancy between $\hat{x}(t)$ and $x(t)$ represents the accuracy of the model-estimate, it in fact will tend to be sensitive only to the accuracy of $\hat{q}_\alpha$, and insensitive to the other model-parameter estimates. Complete insensitivity to all q_β, $\beta \neq \alpha$, will occur only when the estimate is fully correct. However, as $\hat{q}$ approaches q, the effect of inaccuracy in estimates of model parameters other than q_α will diminish. In this way we see that, by choosing an excitation which induces selectively insensitive response, we are able to partially "immunize" the estimation of q_α from imprecise knowledge of the remaining model parameters. This then allows us to separate the estimation of q_α from estimation of the remaining model parameters. In other words, the task of estimating the r_Q-dimensional vector q simplifies to r_Q nearly independent 1-dimensional estimations.

7.2 Adaptive Multihypothesis Estimation.

We now formulate the multihypothesis algorithm for estimating the αth nonlinear model parameter. The system will be excited for a duration $[0, T]$ and the response, $y(t)$, will be sampled at N_S instants, $t_1, \ldots, t_{N_S}$. Let $\hat{x}(t)$ be the anticipated state vector, based on the model-estimate $\hat{q}$ used to generate the excitation. The performance of this estimate can be evaluated by:

$$J_1(\hat{q}) = \frac{1}{N_S} \sum_{n=1}^{N_S} \| y(t_n) - G\hat{x}(t_n) \| \tag{70}$$

where $\|\cdot\|$ is the Euclidean vector norm. $J_1(\hat{q})$ is a non-negative function which, in the absence of noise, vanishes when the estimate, $\hat{q}$, is correct. In subsection 7.3 we will define additional useful measures of performance. These performance measures are called "decision functions."

If we were to repeat these measurements, each time with a different value of the estimate of q_α (the other estimates being held constant) and with the corresponding excitation, the decision function $J_1(\hat{q})$ would vary, somewhat as in Fig. 1. Our earlier discussion leads us to expect that the correct value of q_α is near the value which minimizes $J_1(\hat{q})$.

The adaptive multihypothesis search for the correct q_α is performed as follows. Before beginning the nth stage of the adaptation the estimated nonlinear model is $\hat{q}^{(n-1)}$. Likewise, at the outset of the nth stage, knowledge of the value of q_α constrains it to a domain $\mathfrak{D}^n$. N_H different values are chosen from this domain. Now N_H "hypothesized" models are constructed by replacing the αth element of $\hat{q}^{(n-1)}$ by each of these N_H selections from $\mathfrak{D}^n$. Let us denote these hypothesized models

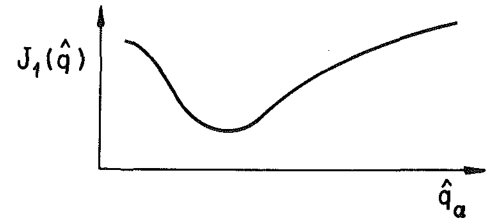

Fig. 1 Schematic portrayal of the variation of the decision function versus the estimate of the αth nonlinear model parameter

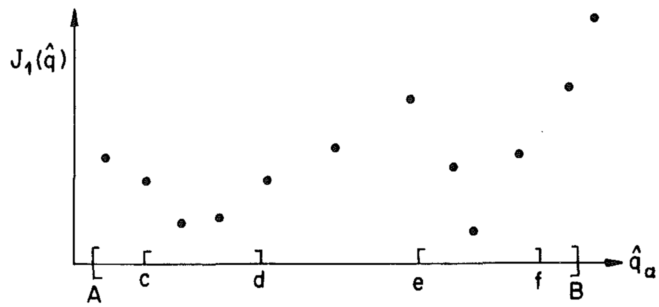

Fig. 2 Schematic representation of the measured decision function, illustrating up-dating of the estimated domain of q_α

by $h^1, \ldots, h^{N_H}$. That is, h^m equals $\hat{q}^{(n-1)}$ except in its αth element, where h^m equals the mth selection from $\mathfrak{D}^n$.

Now N_H excitation functions are constructed, one for each hypothesis: $f(t, h^1), \ldots, f(t, h^{N_H})$. These excitations are chosen to induce responses which are selectively sensitive to q_α and insensitive to the other model parameters. The corresponding anticipated responses are $\hat{x}(t; h^1), \ldots, \hat{x}(t; h^{N_H})$.

We now apply each of these excitations to the system, from the same initial conditions, and sample the output N_S times on each run. The decision function is evaluated for each of these periods of excitation: $J_1(h^1), \ldots, J_1(h^{N_H})$. These N_H numbers are a sample from the graph shown schematically in Fig. 1.

The nth stage of the excitation is completed by updating the domain containing q_α. For example consider Fig. 2, which schematically shows the values of $J_1(\hat{q})$ based on 12 hypotheses. The domain of q_α-values at the outset of the nth stage was the interval $\mathfrak{D}^n = [A, B]$. After these measurements it is reasonable to reduce the domain to two intervals: $\mathfrak{D}^{n+1} = \{[c, d], [e, f]\}$.

We have now completed the specification of a hierarchical, adaptive, multihypothesis procedure for estimating q_α. The sequence of domains, $\mathfrak{D}^1, \mathfrak{D}^2, \ldots$, and the sequence of collections of hypothesized models $\{h^1, \ldots, h^{N_H}\}$ chosen at each stage from these domains, form a hierarchy of hypothesized models. The viability of each of these modes is evaluated by the decision function $J_1(\hat{q})$.

The hierarchical structure is important as a means for efficiently resolving the value of q_α to desired accuracy. For example, suppose q_α is initially supposed to fall in the interval $\mathfrak{D}^1 = [0, 1]$, and suppose it is desired to estimate q_α to an accuracy of 0.0001. One need not test the ten-thousand hypotheses which $\mathfrak{D}^1$ offers. A few score hypotheses will most likely suffice: about 20 hypotheses from $\mathfrak{D}^1$ will resolve the value of q_α to an accuracy of about $1/20 = 0.05$; 20 hypotheses from the sub-intervals defining $\mathfrak{D}^2$ will achieve a resolution of about $1/20^2 = 0.0025$; finally 20 hypotheses from $\mathfrak{D}^3$ will narrow the range of q_α to an interval of about $1/20^3 = 0.000125$.

The adaptivity of the identification procedure is manifested in the selection of the model hypotheses at each stage. This selection is contingent upon the updated domain $\mathfrak{D}^n$ which is defined from the results of the previous stage. These model hypotheses then generate the excitations which are applied to the system at each stage.

The selective sensitivity plays a crucial role, as discussed in subsection 7.1, by reducing the dimension of the search procedure. While the number r_Q of nonlinear model parameters may be large, the estimation is performed for each parameter separately. Repeating a hierarchically-adaptive 1-dimensional search r_Q times is simpler than a single r_Q-dimensional search. The advantage in this reduction of dimensionality is not only in decreasing the number of hypotheses to be tested, but also in ameliorating the problems of ill-conditioning which often arise in high-dimensional model-estimation inverse problems. It must be recalled however that, while the output is selectively sensitivity to only a single model parameter, formulation of the excitation is based on the entire model.

7.3 Decision Functions. In the numerical example to follow we will demonstrate the use of six different decision functions. In addition to $J_1(\hat{q})$, we will employ the following functions:

$$J_2(\hat{q}) = \frac{1}{N_S} \sum_{n=1}^{N_S} \left\| y(t_n) - \frac{\|y(t_n)\|}{\|G\hat{x}(t_n)\|} G\hat{x}(t_n) \right\| \quad (71)$$

$$J_3(\hat{q}) = \frac{1}{N_S} \sum_{n=1}^{N_S} \left\| y(t_n) - \frac{\|y(t_n)\|}{\|G\hat{x}(t_n)\|} G\hat{x}(t_n) \right\| \Big/ \|y(t_n)\| \quad (72)$$

$$J_4(\hat{q}) = \frac{1}{N_S} \sum_{n=1}^{N_S} |y_\alpha(t_n) - [G\hat{x}(t_n)]_\alpha| \quad (73)$$

$$J_5(\hat{q}) = \frac{1}{N_S} \sum_{n=1}^{N_S} \frac{|y_\alpha(t_n) - [G\hat{x}(t_n)]_\alpha|}{|y_\alpha(t_n)|} \quad (74)$$

$$J_6(\hat{q}) = \frac{1}{N_S} \sum_{n=1}^{N_S} \frac{\|y(t_n) - G\hat{x}(t_n)\|}{\|y(t_n)\|} \quad (75)$$

8 Example: Nonlinear Beam Stiffness

8.1 The Beam Model. We will study the forced vibration of a clamped-free beam whose length, width, and height are $L = 1$, $w = 0.01$ and $h = 0.0025$ [m], respectively. The modulus of elasticity is $E = 2 \times 10^{11}$ [N/m^2] and the density is $\rho = 8.1 \times 10^3$ [kg/m^3]. The first three natural frequencies for linear flexural vibration in the vertical plane of this beam are 12.6, 78.9 and 221 [Hz].

The linear component of the beam vibration will be modelled with a simple finite-element method described by Gladwell [20]. The beam is divided into N_D mass points connected by weightless rigid arms. Flexure at each node is approximated by a torsional spring with stiffness k[Nm]. The value of k is chosen so that the end deflection under concentrated load at the free end equals the value obtained with a small-deflection continuous-beam model. Thus k is [13]:

$$k = \frac{(N_D + 1)(2N_D + 1)Ewh^3}{24N_D L} \quad (76)$$

The linear operator $\mathcal{L}(x)$ for the discrete deflection model is:

$$\mathcal{L}(x) = M\ddot{x}(t) + Kx(t) \quad (77)$$

where M is a diagonal mass matrix and K is a pentadiagonal stiffness matrix. Each diagonal element of M equals $1/N_D$ times the mass of the beam. The elements of K depend on the torsional stiffness k and the length of the rigid arms, L/N_D, as described by Gladwell. In our calculations $N_D = 20$.

The nonlinear part of the model will be quadratic, so $\mathfrak{N}(x)$ is given by Eq. (18).

We will suppose that lateral forces can be applied at each of the N_D nodes, so the input matrix H equals the identify matrix and $f(t)$ is an N_D-vector of forces. Deflection measurements will be performed at some or all of the nodes, and the input matrix G will be chosen accordingly. The excitation will be applied during an interval $[0, T]$ where $T = 0.93$ sec.

Uncertainty in the measured displacement vector $x(t)$ is simulated as additive white noise. If x_n^e is the exact value of the nth element of x, then the simulated noisy measurement of this displacement is:

$$x_n = x_n^e(1 + \gamma r) \quad (78)$$

where r is a uniform random variable on the interval $[-1, 1]$ and γ is a noise-amplitude parameter. The relative error of the measurement of x_n is $\sigma_n/x_n^e = \gamma/\sqrt{3}$, where σ_n is the standard deviation of x_n.

We will employ the method of selective sensitivity to define hierarchical adaptive excitations of the beam for identifying the nonlinear part of the model, assuming the linear part is known. This means that the parameter vector q, or part of it, will be identified. Several different examples will be performed. In all examples q contains only one non-zero element. In subsection 8.2 we consider the problem of determining the value of the non-zero element when its index is known. In subsection 8.3 we illustrate the determination of the index of the non-zero element of q when neither the index nor the amplitude are known. In subsection 8.4 we briefly consider the effect of measuring only some of the nodal deflections.

In all our examples the vector $x(t)$ is chosen to have the form of Eq. (34). The real positive function $\zeta(t)$ is:

$$\zeta(t) = 10^{-4}[1 + 10^{-8} - \cos(13.5t)] \quad (79)$$

8.2 Example 1. In this example we consider identification of the nonlinear part of the model when the true nonlinear model is:

$$q = q_\alpha e^\alpha \quad (80)$$

We presume that it is known that the nonlinear model is quadratic and that q has only one non-zero term. The amplitude q_α is unknown but the index of the non-zero term is known to be $\alpha = 5$. Deflections are measured at all 20 nodes.

The magnitude of q_α determines the relative contribution of the linear and nonlinear stiffness terms. The excitations applied to the beam produce deflection vectors which display small localized deflection. The linear stiffness term, for the parameters of our model, is of the order $10^5 x_n$, where $x_n = 0.001$[m] is a typical value for the local deflection. Similarly the nonlinear stiffness is approximately $q_\alpha x_n^2$. Thus the ratio of the nonlinear to the linear restoring forces is approximately (for $x_n = 0.001$):

$$\frac{q_\alpha x_n^2}{10^5 x_n} = q_\alpha 10^{-8} \quad (81)$$

If $q_\alpha = 10^8$ then the nonlinear and the linear terms are roughly equal. We will show that much smaller nonlinear perturbations to the nominal linear model can be identified. In the examples to follow we consider $q_\alpha = 10^4$ and $q_\alpha = 10^5$.

The adaptive multihypothesis procedure is as follows. A set of hypothesized q-vectors is postulated and an excitation vector for each hypothesis is applied, separately, to the beam. The deflections resulting from each excitation vector are measured and the 6 decision functions $J_1, \ldots, J_6$ are calculated.

In this subsection we suppose that we know the index of the non-zero element of q but not its amplitude. Consider 10 hypotheses $q_\alpha e^\alpha$ where $\alpha = 5$ (which is correct) and q_α varies between 10^4 and 3×10^5. The true value is $q_\alpha = 10^5$, indicating that the nonlinear contribution to the restoring force is on the order of one-thousandth of the linear term.

The resulting decision functions are plotted versus q_α in Fig. 3, based on simulated noise-free measurements. (The noise parameter γ is zero). The decision functions show a very clear minimum near the true value $q_\alpha = 10 \times 10^4$. Figure 4 shows further confirmation of this based on a second set of excitations, in which the magnitude of the hypothesized parameter

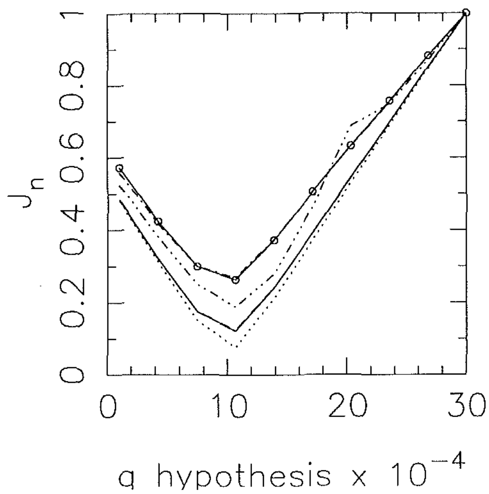

Fig. 3 Decision functions versus hypothesized value of q_α. $\alpha = 5$ (correct) and without noise ($\gamma = 0$). J_1: solid, J_2: dash-dash, J_3: dash-dot, J_4: dot-dot, J_5: dash-dot-dot-dot, J_6: solid line with circles

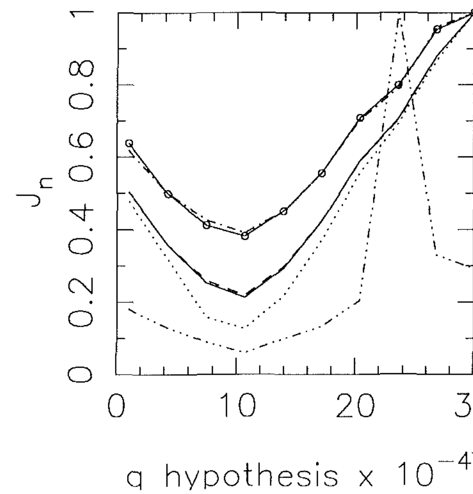

Fig. 5 Decision functions versus hypothesized value of q_α. $\alpha = 5$ (correct) and with 0.2 percent noise ($\gamma = 0.002$). Legend as in Fig. 3

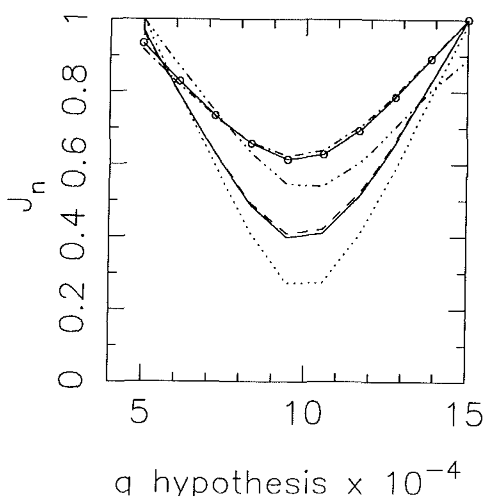

Fig. 4 Decision functions versus hypothesized value of q_α. $\alpha = 5$ (correct) and without noise ($\gamma = 0$). Legend as in Fig. 3

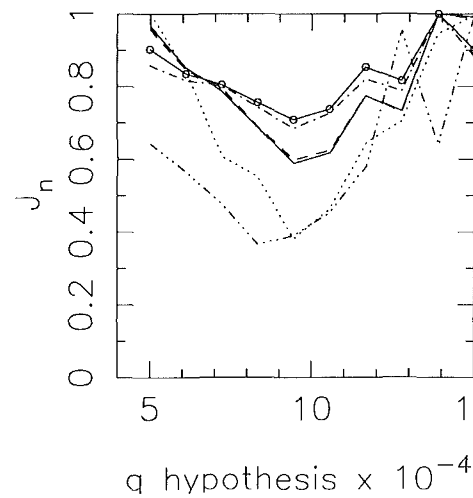

Fig. 6 Decision functions versus hypothesized value of q_α. $\alpha = 5$ (correct) and with 0.2 percent noise ($\gamma = 0.002$). Legend as in Fig. 3

vector varies over the more restricted range from 5×10^4 to 15×10^4. Again, all six decision functions point clearly to the correct value of q_α.

Figures 5 and 6 show the same sequence of multihypothesis tests, this time with about 0.2 percent additive measurement noise ($\gamma = 0.002$). The difference between these figures and Figs. 3 and 4 is marked. The noise substantially reduces the sharpness in the resolution of q_α. Furthermore we see that not all the decision functions perform with the same resolution. J_5, for example, (dash-dot-dot-dot) has a significantly blunter minimum than the other decision functions. Nevertheless, the correct value of q_α is very nearly achieved.

If, as in Fig. 7, the noise is increased to 0.5 percent ($\gamma = 0.005$) the resolution is further degraded, though a reasonable decision is still possible.

8.3 Example 2. Now consider a more difficult estimation problem. We know that the structure of the nonlinear term is $q = q_\alpha e^\alpha$. That is, q has only one non-zero term. However we know neither the value of q_α nor of α. The true index is $\alpha = 5$ and the true amplitude is $q_\alpha = 1 \times 10^4$, an order of magnitude lower than the previous example. We will perform a sequence of excitations for evaluating the index α of the non-zero element of q. Once the index has been identified, then the multihypothesis sequence employed in subsection 8.2 can be invoked here as well to estimate q_α. Deflections are measured at all 20 nodes.

The hypothesized q-vector is, as before, $q_\alpha e^\alpha$. The hypothesized value of the amplitude, q_α, assumes five different values between 7.444×10^3 and 20.333×10^3. Furthermore, five hypothesized indices are examined: $\alpha = 3, \ldots, 7$. For each of these hypotheses an excitation function is constructed and applied to the beam, the deflections are measured and the decision functions are evaluated. The measurement noise is about 1 percent of the signal ($\gamma = 0.01$).

Figures 8–12 display the results. In each figure the hypothesized amplitude is constant and the six decision functions are plotted on a logarithmic scale against the hypothesized index of the non-zero element of q. One sees that hypothesis $\alpha = 5$ (which is correct) is preferred in almost all the figures and by nearly all the decision functions.

8.4 Example 3. Let us briefly consider the effect of measuring deflections in fewer than all 20 nodes. Figure 13 shows decision function J_1 versus the hypothesized amplitude q_α, for hypothesized q-vectors with $\alpha = 5$. The true vector is $q = 10^4 e^5$. The noise is about 0.5 percent ($\gamma = 0.005$).

The solid line is based on measured deflections in all 20 nodes of the discrete beam model. The circles employ measurements at nodes 6, 12, and 18, while the $\times$s are obtained

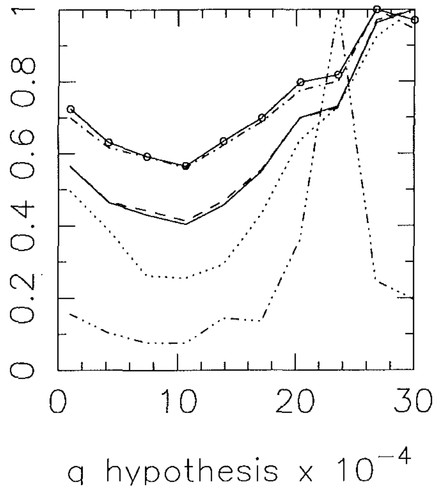

Fig. 7 Decision functions versus hypothesized value of q_α. $\alpha = 5$ (correct) and with 0.5 percent noise ($\gamma = 0.005$). Legend as in Fig. 3

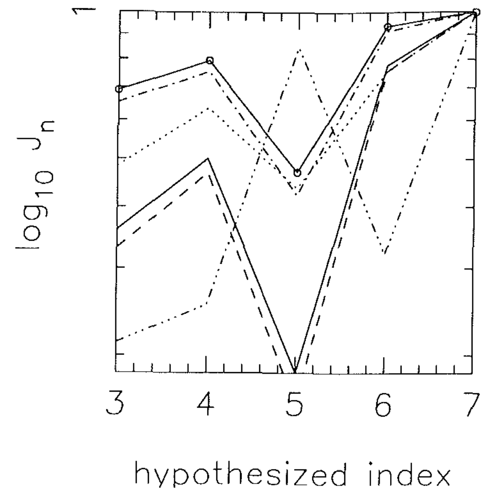

Fig. 10 Decision functions versus hypothesized index of the non-zero element of q. Hypothesized amplitude is $q_\alpha = 13.889 \times 10^3$. 1 percent noise ($\gamma = 0.01$). Legend as in Fig. 3

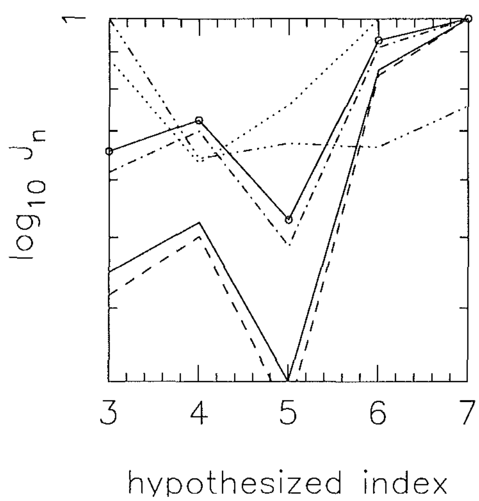

Fig. 8 Decision functions versus hypothesized index of the non-zero element of q. Hypothesized amplitude is $q_\alpha = 7.444 \times 10^3$. 1 percent noise ($\gamma = 0.01$). Legend as in Fig. 3

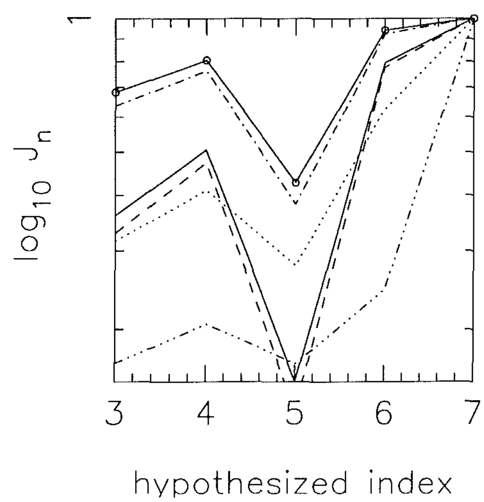

Fig. 11 Decision functions versus hypothesized index of the non-zero element of q. Hypothesized amplitude is $q_\alpha = 17.111 \times 10^3$. 1 percent noise ($\gamma = 0.01$). Legend as in Fig. 3

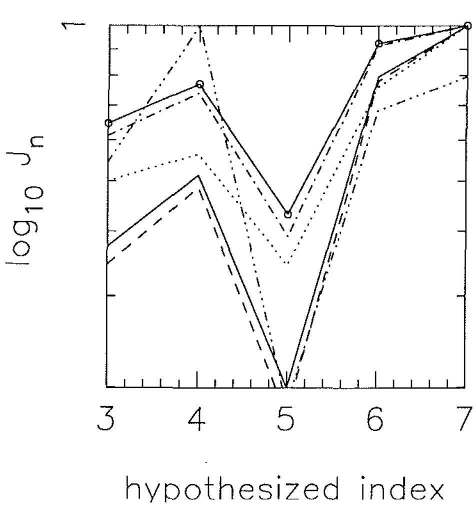

Fig. 9 Decision functions versus hypothesized index of the non-zero element of q. Hypothesized amplitude is $q_\alpha = 10.667 \times 10^3$. 1 percent noise ($\gamma = 0.01$). Legend as in Fig. 3

from measurements at nodes 8 and 16. The results in all cases are basically the same. This result, while encouraging, must be viewed with caution. The selective excitations are designed to produce substantially larger displacements at node 5 than at the other nodes. In our simulations, the noise is additive and proportional to the the size of the displacement. Nonproportional noise, whose standard deviation is independent of the magnitude of the displacement, would corrupt the measurements at nodes other than the 5th more severely than measurements at node 5. This would cause preferential degradation of decision functions which are based on partial measurements not including the 5th node.

9 Conclusions

We have developed a technique for identifying polynomial nonlinearities in dynamic systems, employing active vibration measurements. The identification is based on the method of selective sensitivity and involves an adaptive multihypothesis estimation procedure. The primary advantage of this method is the reduction of the ill-conditioning which often arises in high-dimensional inverse problems associated with system identification. Selective sensitivity enables the design of an

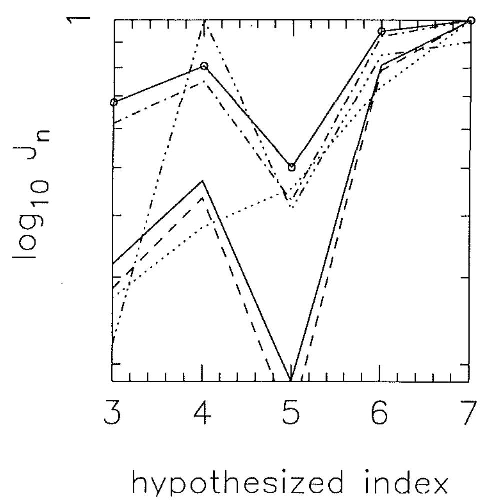

Fig. 12 Decision functions versus hypothesized index of the non-zero element of q. Hypothesized amplitude is $q_\alpha = 20.333 \times 10^3$. 1 percent noise ($\gamma = 0.01$). Legend as in Fig. 3

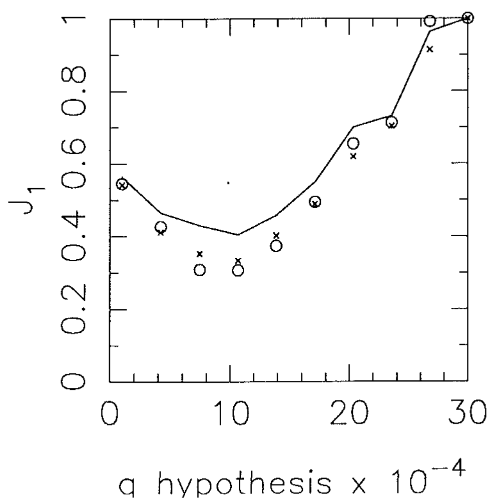

Fig. 13 Decision function J_1 versus hypothesized value of q_α between 1×10^3 and 30×10^3. $\alpha = 5$ (correct) and with 0.5 percent noise ($\gamma = 0.005$). Deflection measurement of all 20 nodes: solid line; only nodes 6, 12, 18: circles; only nodes 8 and 16: ×

excitation which causes the system-response to be selectively sensitive to a small set of model parameters and insensitive to all the remaining model parameters. The design of the input depends on the entire model. Consequently, the identification of the entire nonlinear model evolves as a sequence of low-dimensional estimation problems for which ill-conditioning is rarely an issue.

We have shown, for specific forms of quadratic, cubic and hybrid quadratic-cubic polynomial nonlinearities, how to design selectively sensitive excitations.

We have simulated the identification of quadratic stiffness nonlinearity in a uniform beam. Our examples suggest that the adaptive multihypothesis estimation seems to perform successfully even in the presence of low additive noise and with partial measurement.

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References

1 Moore, W., 1989, *Schrödinger: Life and Thought*, Cambridge University Press.

2 Yao, J. T. P., 1988, "Identification of Structural Damage in Civil Engineering," in H. G. Natke, ed., *Application of System Identification in Engineering*, Springer-Verlag, Vienna, pp. 349–390.

3 Natke, H. G., 1991, "Error Localization Within Spatially Finite-Dimensional Mathematical Models: A Review of Methods and the Application of Regularization Techniques," *Computational Mechanics*, Vol. 8, pp. 153–160.

4 Tomlinson, G. R., 1988, "Non-linearity in Dynamical Systems," in H. G. Natke, ed., *Application of System Identification in Engineering*, Springer-Verlag, Vienna, pp. 199–231.

5 Thompson, J. M. T., and Stewart, H. B., 1988, *Nonlinear Dynamics and Chaos*, John Wiley, New York.

6 Tomlinson, G. R., 1987, "Developments in the Use of the Hilbert Transform for Detecting and Quantifying Non-linearity Associated with Frequency Response Functions," *Mechanical Systems and Signal Processing*, Vol. 1, pp. 151–171.

7 Louis, A. K., 1989, *Inverse and Schlecht Gestellte Probleme*, B. G. Teubner, Stuttgart.

8 Bellman, R., 1961, *Adaptive Control Processes*, Princeton University Press, Princeton, N.J.

9 Schetzen, M., 1980, *The Volterra and Wiener Theories of Nonlinear Systems*, New York, John Wiley.

10 Bendat, J. S., 1990, *Nonlinear System Analysis and Identification From Random Data*, New York, John Wiley.

11 Billings, S. A., and Tsang, K. M., 1989, "Spectral Analysis for Nonlinear Systems: Part I: Parametric Non-Linear Spectral Analysis," *Mechanical Systems and Signal Processing*, Vol. 3, pp. 319–339.

12 Korenberg, M., Billings, S. A., Liu, Y. P., and McIlroy, P. J., 1988, "Orthogonal Parameter Estimation Algorithm for Non-Linear Stochastic Systems," *International Journal of Control*, Vol. 48, pp. 193–210.

13 Ben-Haim, Y., 1992, "Adaptive Diagnosis of Faults in Elastic Structures by Static Displacement Measurement: The Method of Selective Sensitivity," *Mechanical Systems and Signal Processing*, Vol. 6, pp. 85–96.

14 Ben-Haim, Y., and Prells, U., "Selective Sensitivity in the Frequency Domain, Part I: Theory," *Mechanical Systems and Signal Processing*, in press.

15 Ben-Haim, Y., and Prells, U., "Selective Sensitivity in the Frequency Domain, Part II: Applications," *Mechanical Systems and Signal Processing*, in press.

16 Natke, H. G., 1979, "Vergleich von algorithmen für die Anpassung des Rechenmodells einer schwingungsfähigen elastomechanischen struktur an versuchswerte," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 59, pp. 257–268.

17 Natke, H. G., 1988, "Rechner-und versuchsgestützte modellierung mechanischer systeme," *Zeitschrift der Universität Hannover*, Vol. 2, pp. 20–26.

18 Natke, H. G., and Zamirowski, M., 1990, "On Methods of Structure Identification for the Class of Polynomials Within Mechanical Systems," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 10, pp. 415–420.

19 Pearson, C. E., ed., 1974, *Handbook of Applied Mathematics*, Van Nostrand Reinhold.

20 Gladwell, G. M. L., 1986, *Inverse Problems in Vibration*, Martinus Nijhoff Publishers, Dordrecht, Holland.