Lax embeddings of the Hermitian Unital

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Abstract

In this paper, we prove that every lax generalized Veronesean embedding of the Hermitian unital $\mathcal{U}$ of $\mathsf{PG}(2,\mathbb{L})$, $\mathbb{L}$ a quadratic extension of the field $\mathbb{K}$ and $|\mathbb{K}| \geq 3$, in a $\mathsf{PG}(d,\mathbb{F})$, with $\mathbb{F}$ any field and $d \geq 7$, such that disjoint blocks span disjoint subspaces, is the standard Veronesean embedding in a subgeometry $\mathsf{PG}(7,\mathbb{K}')$ of $\mathsf{PG}(7,\mathbb{F})$ (and $d = 7$) or it consists of the projection from a point $p \in \mathcal{U}$ of $\mathcal{U} \setminus \{p\}$ from a subgeometry $\mathsf{PG}(7,\mathbb{K}')$ of $\mathsf{PG}(7,\mathbb{F})$ into a hyperplane $\mathsf{PG}(6,\mathbb{K}')$. In order to do so, when $|\mathbb{K}| > 3$ we strongly use the linear representation of the affine part of $\mathcal{U}$ (the line at infinity being secant) as the affine part of the generalized quadrangle $\mathsf{Q}(4,\mathbb{K})$ (the solid at infinity being non-singular); when $|\mathbb{K}| = 3$, we use the connection of $\mathcal{U}$ with the generalized hexagon of order 2.

Keywords: lax embedding, Hermitian unital, standard Veronesean embedding.

1 Introduction

The theory of embeddings of point-line geometries related to spherical buildings is now rather well developed when taking as points the elements of the type corresponding with the first vertex of the diagram, in natural (Bourbaki) order. Especially the theory of embeddings of generalized quadrangles and polar spaces is well understood, and much about existence is known for the exceptional diagrams. The dual polar spaces are also getting into good shape thanks to recent and ungoing work of De Bruyn, Pasini, Cardinali. By work of the second author and Thas and Steinbach, some partial results on generalized hexagons are available. However, it seems hard to establish the embedding rank for the ordinary split Cayley hexagons, and the triality hexagons and their duals. This question is intimately related to the generating rank of these geometries, which also remains a mystery. The main obstacle to deal with embeddings of these generalized hexagons is the fact that the classical methods used for quadrangles and polar spaces do not work, since slicing with subspaces does not necessarily produce subhexagons—it does when one imposes the rather strong condition of being *flat*, but this restriction, producing most of the partial results mentioned above, is too restrictive to be used for the general case, although it yields nice and complete results. One way to minimize the main obstacle is to gather as much embedding information as possible for as many substructures as possible—and not only for subhexagons. A successful example of this approach has been obtained in [7], where the classification of generalized Veronesean embeddings of finite projective planes is used to classify certain non-flat embeddings of the split Cayley hexagons of characteristic 2 in projective spaces of dimension 12.

With an eye on the triality hexagons, the second author jointly with some others [1] looked at *lax* generalized Veronesean embeddings of projective planes—or more general, projective spaces. The reason is that a triality hexagon contains a split Cayley hexagon as a subhexagon, but not as a *full* subhexagon. So these subhexagons are laxly embedded! Another important subgeometry of the split Cayley hexagon, or rather of its dual, is the Hermitian unital. In characteristic 3, the classical embedding of the split Cayley hexagon gives rise to a Veronesean embedding of such unital. Hence the second author jointly with De Wispelaere and Huizinga, started to investigate Veronesean embeddings of these unitals in a general framework, see [6]. In the present paper, we continue this job, shifting the emphasis to the lax case, in order to deal later with the triality hexagons in characteristic 3. Of course, this restriction on the characteristic does not simplify the arguments one uses, and so we treat the question in full generality.

So far for motivation. Let us now formulate the problem in question.

The classical Hermitian unital $\mathcal{U}$ (for precise definitions, see below) admits a Veronesean embedding in 7-dimensional space, i.e., the points of $\mathcal{U}$ are points of a 7-dimensional projective space, and the blocks of $\mathcal{U}$ are planar conics. This representation is essentially unique, as shown in [6], and no such representation exists in higher dimensional projective space. In a certain sense, this embedding is a *full* one, since conics cannot be properly contained in other conics. The corresponding non-full or *lax* notion is when blocks are planar arcs, or, even more generally, just planar point sets. We call the corresponding embeddings *generalized lax Veronesean*. For projective planes, and more generally, projective spaces, generalized lax Veronesean embeddings are classified, under the only condition that they occur in high enough dimensional projective space, see [1]. For reasons explained in the previous paragraphs, we would now like to do the same for the Hermitian unital, where "high enough dimension" means "dimension at least 7". However, unlike projective planes, blocks in unitals do not always meet, and this seems to cause insuperable problems, unless one assumes that disjoint blocks span disjoint subspaces (a block is allowed to span a line instead of a plane). This seems to be a natural condition, and it gives rise to the following natural and general embedding problem: suppose one has a morphism α of a point-line geometry into the point-line geometry of a projective space, and suppose that α is not injective, but it is injective on pairs of points in the most general position. What can we say? In our setting, we will be confronted with this question for the point-line geometry arising from the orthogonal generalized quadrangle $\mathsf{Q}(4,\mathbb{K})$ by deleting a (full) grid, see below. In fact, we will have a slightly more restricted situation (we will know that certain sets of points—the Hermitian conics—are contained in a plane), which the more will show that such a question is highly non-trivial. We were unable to solve the problem without this condition although we neither found counter examples.

The paper is structured as follows. In Section 2, we introduce notation, define the Hermitian unital and review its linear representation in 4-space. In Section 3, we introduce generalized lax Veronesean embeddings and state our Main Result, which we then prove In Section 4.

2 The Hermitian unital and its linear representation

Let $\mathbb{L}$ be a quadratic Galois extension of the field $\mathbb{K}$ and let $\mathsf{PG}(2,\mathbb{L})$ be the projective plane over $\mathbb{L}$. A *Hermitian curve* of $\mathsf{PG}(2,\mathbb{L})$ is the set of the absolute points with respect to a non-degenerate *Hermitian sesquilinear form*. Every line of $\mathsf{PG}(2,\mathbb{L})$ intersects such a curve in at most one point or in a *Baer subline*, i.e. a set of points isomorphic to a projective line $\mathsf{PG}(1,\mathbb{K})$. Let $\mathcal{P}$ be the set of points of the curve and $\mathcal{B}$ the set of sublines arising from the nontrivial intersection of the curve with non-tangent lines and I be the natural incidence relation, then $\mathcal{U}=(\mathcal{P},\mathcal{B},I)$ is a *linear space* also called the *Hermitian unital* and the elements of $\mathcal{B}$ are sometimes also called *blocks*, since when $\mathbb{K}$ is the finite field $\mathbb{F}_q$ the unital is a $2-(q^3+1,q+1,1)$ design.

If the Hermitian sesquilinear form considered above is degenerate (i.e. the defining matrix is singular), then the set of singular points with respect to it are either a *Hermitian cone* (i.e. when the the matrix associated to the form has rank 2) or a line (i.e. when the the matrix associated to the form has rank 1). In particular, a Hermitian cone is also called a *Baer subpencil* because it is a subset of a pencil through a point p such that its intersection with a line not through p is a Baer subline. The point p is referred to as the *vertex* of the pencil.

Let $\mathsf{PG}(3,\mathbb{K})$ be the 3-dimensional projective space over the field $\mathbb{K}$, consider as underlying 4-dimensional vector space $\mathbb{L}\times\mathbb{L}$ and let $\mathcal{S}=\{\ell_\infty\}\cup\{\ell_a, a\in\mathbb{L}\}$ be the following line partition of $\mathsf{PG}(3,\mathbb{K})$: $\ell_\infty=\{(0,x),x\in\mathbb{L}\}$ and $\ell_a=\{(x,ax),x\in\mathbb{L}\}$. This is a generalization of the so called *Desarguesian spread* of finite projective spaces, and, as in the finite case, it gives rise to an incidence structure isomorphic to the affine plane $\mathsf{AG}(2,\mathbb{L})$ (see [3] for the finite case). Embed $\mathsf{PG}(3,\mathbb{K})$ as a hyperplane "at infinity" in $\mathsf{PG}(4,\mathbb{K})$; the points of the incidence structure are the ones of $\mathsf{AG}(4,\mathbb{K})=\mathsf{PG}(4,\mathbb{K})\setminus\mathsf{PG}(3,\mathbb{K})$, the lines are the planes of $\mathsf{PG}(4,\mathbb{K})$ intersecting $\mathsf{PG}(3,\mathbb{K})$ in a line of $\mathcal{S}$ and the incidence is the natural one. In [4], the author describes how the Hermitian unital $\mathcal{U}$ is represented in this setting. The following can be extracted from this reference, or follows from it in a rather straightforward way. Let $\mathsf{Q}(4,\mathbb{K})$ be a non-singular parabolic quadric of $\mathsf{PG}(4,\mathbb{K})$ intersecting the hyperplane at infinity in a hyperbolic quadric $\mathsf{Q}^+(3,\mathbb{K})$ that shares a regulus of lines with $\mathcal{S}$ and let $\mathsf{AQ}(4,\mathbb{K})$ be the affine part of $\mathsf{Q}(4,\mathbb{K})$. If $\mathcal{U}$ intersects the line at infinity in a subline B, then the points of $\mathcal{U}$ correspond to the points of a suitable $\mathsf{AQ}(4,\mathbb{K})$. If a subline intersects B non-trivially, then it corresponds to a line of $\mathsf{AQ}(4,\mathbb{K})$. Otherwise it corresponds to a conic of $\mathsf{AQ}(4,\mathbb{K})$, that we will call *Hermitian conic*, contained in a plane intersecting the hyperplane at infinity in a line of $\mathcal{S}$ not belonging to $\mathsf{Q}^+(3,\mathbb{K})$. We remark that in $\mathcal{U}$ every two points are *collinear*, that means that there is a block containing both. But in $\mathsf{AQ}(4,\mathbb{K})$ we say that two points are *collinear* if and only if there is a line contained in $\mathsf{AQ}(4,\mathbb{K})$ joining them. In $\mathsf{AQ}(4,\mathbb{K})$ there are two kinds of affine hyperbolic quadrics, also called *grids*, the grids of Type 1 and the grids of Type 2, according to whether the grid intersects the hyperplane at infinity in two intersecting lines or a conic. Also, a grid consists of two families of lines, called *(opposite) reguli*, such that if the grid is of Type 1, then two lines meet in one point if they belong to two different reguli, they are skew otherwise; if the grid is of Type 2, then two lines belonging to the same regulus are skew and a line belonging to a regulus $\mathcal{R}_1$ intersects in one point all the lines of the regulus $\mathcal{R}_2$ but one. Also, we recall that $\mathsf{Q}(4,\mathbb{K})$ is a *generalized quadrangle*, hence for every point P and every line L such that $P \notin L$, there exists a unique line through P intersecting L. This implies in $\mathsf{AQ}(4,\mathbb{K})$ that for every P and L as before there is at most one line through P intersecting L.

Finally, we take a look at Baer subpencils. A Baer subpencil through the point $p \in \mathcal{U}$, viewed as a Hermitian curve in $\mathsf{PG}(2,\mathbb{L})$, containing the tangent at p to $\mathcal{U}$ will be referred to as an *affine Baer subpencil* with vertex p, whereas a *projective Baer subpencil* with vertex p does not contain the tangent line at p to $\mathcal{U}$. A *transversal* of a projective Baer subpencil is a block of $\mathcal{U}$ that does not pass through p and that intersects all the blocks of the pencil. Every point distinct from p on some block of a projective Baer subpencil belongs to a transversal. This contrasts with the fact that there is no block intersecting at least three different blocks of a given affine Baer subpencil. An affine Baer subpencil which contains the block on the line at infinity of $\mathsf{AG}(2,\mathbb{L})$, corresponds in $\mathsf{AQ}(4,\mathbb{K})$ to an affine quadratic cone with vertex a point at infinity. Likewise, a projective Baer subpencil which contains the block on the line at infinity of $\mathsf{AG}(2,\mathbb{L})$, corresponds in $\mathsf{AQ}(4,\mathbb{K})$ to all affine lines of a regulus belonging to a grid of Type 1; the transversals are the lines belonging to the opposite regulus. Also, a projective Baer subpencil with affine vertex for which the block contained in the line at infinity of $\mathsf{AG}(2,\mathbb{L})$ is a transversal, corresponds in $\mathsf{AQ}(4,\mathbb{K})$ with a cone with affine vertex, and every such cone arises in this way. It follows that any such cone without its affine vertex is partitioned into Hermitian conics. Note that also a grid of Type 2 is partitioned into Hermitian conics, as the 3-space spanned by it intersects $\mathsf{PG}(3,\mathbb{K})$ in a plane, which contains a unique line of the spread (since the spread is a regular spread, it is also a *dual* spread—hence a spread of the dual projective space—as is easily verified).

3 Lax embeddings and the Main Result

Let $\mathcal{M}_{n+1}(\mathbb{L})$ be the vector space of the square matrices of order $n+1$ over the field $\mathbb{L}$ and let $\mathcal{H}_{n+1}$ be the set of Hermitian matrices of rank 1. In [5], the authors prove for the finite case that $\mathcal{H}_{n+1}$ is the algebraic variety of $\mathsf{PG}(n^2+2n,\mathbb{K})$ corresponding to a particular case of the variety $\mathcal{V}_{r,t}$ introduced in [9] with $t=2$ and it is called the *Hermitian Veronesean* of index n of $\mathsf{PG}(n^2+2n,\mathbb{K})$. Also, $\mathcal{H}_{n+1}$ is the image of an injective map θ defined on $\mathsf{PG}(n,\mathbb{L})$ and called the *Hermitian embedding.* Under the action of θ, the points of $\mathsf{PG}(n,\mathbb{L})$ correspond to the points of $\mathcal{H}_{n+1}$, the lines of $\mathsf{PG}(n,\mathbb{L})$ correspond to elliptic quadrics $\mathsf{Q}^-(3,\mathbb{K})$ contained in $\mathcal{H}_{n+1}$ and sublines correspond to conics contained in the elliptic quadrics. Finally, the points of a hyperplane section of $\mathcal{H}_{n+1}$ correspond to the points of a Hermitian variety of $\mathsf{PG}(n,\mathbb{L})$. We are interested in the case $n=2$: this case is quite remarkable because if $\mathbb{K}$ is the finite field $\mathbb{F}_q$ and $q\equiv 2$ or $0 \bmod 3$, then a suitable hyperplane section of $\mathcal{H}_3$ is the unitary ovoid (see [8]) of the hyperbolic quadric $\mathsf{Q}^+(7,q)$ and is directly related to triality, see [12]. In fact, the embedding we are interested in is the restriction of θ to the Hermitian unital $\mathcal{U}$ of $\mathsf{PG}(2,\mathbb{L})$ and by a slight abuse of notation we will call it θ as well. Hence, we get an embedding of $\mathcal{U}$ in a $\mathsf{PG}(7,\mathbb{K})$ such that the image of a block is a plane conic and if two planes intersect in a point, then it must be a point of $\mathcal{U}^\theta$. This embedding is called in [6] the *standard Veronesean embedding* of $\mathcal{U}$. More in general, a *Veronesean embedding* of a linear space $(\mathcal{P},\mathcal{L},I)$ in a projective space $\mathsf{PG}(d,\mathbb{F})$ is such that the image of $\mathcal{P}$ generates $\mathsf{PG}(d,\mathbb{F})$ and the image of the points of a line is a plane oval. In the case of Veronesean embedding of $\mathcal{U}$, we have the following result:

Theorem 3.1 (See [6])**.** *Every Veronesean embedding of the Hermitian unital of* $\mathsf{PG}(2,\mathbb{K})$, $|\mathbb{K}|\geq 3$, *in a* $\mathsf{PG}(d,\mathbb{F})$, *with* $\mathbb{F}$ *any field and* $d\geq 7$, *is the standard Veronesean embedding and hence* $d=7$ *and* $\mathbb{F}\cong\mathbb{K}$.

Following the terminology of [1], we will say that an injective map $\theta:\mathcal{U}\longrightarrow\mathsf{PG}(d,\mathbb{F})$, with $\mathbb{F}$ any field, such that collinear points are mapped onto coplanar ones and $\mathsf{PG}(d,\mathbb{F})$ is generated by the image, is a *lax generalized Veronesean embedding* of $\mathcal{U}$ in $\mathsf{PG}(d,\mathbb{F})$.

Our main result is the following:

Main Result 3.2. *Every lax generalized Veronesean embedding of the Hermitian unital $\mathcal{U}$ of $\mathsf{PG}(2,\mathbb{L})$, $\mathbb{L}$ a quadratic extension of the field $\mathbb{K}$ and $|\mathbb{K}| \geq 3$, in a $\mathsf{PG}(d,\mathbb{F})$, with $\mathbb{F}$ any field and $d \geq 7$, such that disjoint blocks span disjoint subspaces of $\mathsf{PG}(d,\mathbb{F})$, either is the standard Veronesean embedding in a subgeometry $\mathsf{PG}(7,\mathbb{K}')$ of $\mathsf{PG}(7,\mathbb{F})$ (and $d = 7$) or consists of the projection from a point $p \in \mathcal{U}$ of $\mathcal{U} \setminus \{p\}$ from a subgeometry $\mathsf{PG}(7,\mathbb{K}')$ of $\mathsf{PG}(7,\mathbb{F})$ into a hyperplane $\mathsf{PG}(6,\mathbb{K}')$, together with an arbitrary point p' playing the role of p in $\mathcal{U}$, and lying outside the $\mathbb{F}$-span of $\mathsf{PG}(6,\mathbb{K}')$, where $\mathbb{K}' \cong \mathbb{K}$.*

Remark Compared with the main result of [1], the hypotheses we assume are somewhat stronger, but this is due to the fact that the lines of the projective planes pairwise intersect in a point, whereas there are blocks of the unital with empty intersection. We refer to the introduction for more comments on the hypotheses.

4 Proof of the Main Result

Let θ be a lax generalized Veronesean embedding of $\mathcal{U}$ in $\mathsf{PG}(d,\mathbb{F})$ such that disjoint blocks span disjoint subspaces. From now on we identify the points and the blocks of $\mathcal{U}$ with their image in $\mathsf{PG}(d,\mathbb{F})$. We fix the following convention for the notation: we denote by capital letters points and lines of $\mathsf{AQ}(4,\mathbb{K})$ and by small letters the points of $\mathcal{U}$ and of $\mathsf{PG}(d,\mathbb{F})$.

We first show that the image of collinear points spans a plane.

Lemma 4.1. *Every block of $\mathcal{U}$ spans a plane of $\mathsf{PG}(d,\mathbb{F})$.*

Proof. Suppose, by way of contradiction, that a block B is contained in a line of $\mathsf{PG}(d,\mathbb{F})$. We project $\mathcal{U} \setminus B$ from $\langle B\rangle$ onto a suitable $(d-2)$-dimensional subspace. Assume that this projection is not injective and let p_1, p_2 be two points with same projection. Consider the block D through p_1, p_2. Then $\langle D\rangle$ meets $\langle B\rangle$ in at least one point, and so our hypothesis implies that B and D meet in $\mathcal{U}$, say in the point p_3. If the points p_1, p_2, p_3 were not collinear, then B would be contained in $\langle D\rangle$, a contradiction with our hypothesis, because it is always possible to find a block B' through a point of B disjoint from D and we would get $\langle B'\rangle \cap \langle D\rangle \neq \emptyset$. If we now consider two disjoint blocks B_1 and B_2 containing p_1 and p_2 respectively, and intersecting B nontrivially (such blocks certainly exist!), then we see that the planes spanned by B_1 and B_2 share a common point of the plane generated by B, p_1 and p_2. This is again a contradiction, and we conclude that the projection is injective.

Now, Lemma 1 of [6] implies that $d - 2 \leq 4$, hence $d \leq 6$, a contradiction. The lemma is proved. □

We now divide our proof in three main parts. In the first part, we prove for $|\mathbb{K}| > 3$, that either all blocks are plane arcs, or there exists a point $p \in \mathcal{U}$ such that all the blocks

B through p consist of the point p and the points of $B \setminus \{p\}$ lying on a line not through p, and then all blocks not through p are plane arcs. We call this the *reduction step*. In the second part, we do the same for $|\mathbb{K}| = 3$. In the last part, we then finish the proof.

We note that the case $|\mathbb{K}| = 3$ roughly consumes one third of the space. We consider that this is worth the trouble since it uses a beautiful connection with the generalized hexagons of order 2, and since along the way, we can prove in a synthetic way some facts that were previously only shown by computer (see the digression in Subsection 4.2).

4.1 Part 1: reduction step for $|\mathbb{K}| > 3$

In this subsection, in order to avoid double work, we sometimes still allow $|\mathbb{K}| = 3$, in which case we will clearly say so.

Due to Lemma 4.1, we can talk about the plane π_B generated by the block B, for any block B of $\mathcal{U}$. Our hypothesis easily implies that the plane π_B containing the block B does not contain any other point of the unital. Hence we can consider the projection from π_B of the points off B as a map $\alpha : \mathsf{AQ}(4, \mathbb{K}) \longrightarrow \mathsf{PG}(d-3, \mathbb{F})$ such that the image of α generates $\mathsf{PG}(d-3, \mathbb{F})$ and collinear points either map onto a unique point or onto *some* subset of points of a line of $\mathsf{PG}(d-3, \mathbb{F})$. Hence, when we consider the image of a block B of $\mathcal{U}$ (or of a line L or a Hermitian conic C of $\mathsf{AQ}(4, \mathbb{K})$) under θ, we will refer to it as B (or L, or C respectively), whereas, if we consider its projection from a fixed plane, then we will refer to it as B^α (or L^α, or C^α respectively).

Let us fix a plane π_0 containing the block B_0 and project from it. We get the following three different cases.

$\boxed{\textbf{Case 1 } (|\mathbb{K}| \geq 3): \quad \langle \mathcal{U} \setminus B_0 \rangle \cap \pi_0 = \emptyset.}$

This implies that every block B intersecting B_0 in the point p is such that all the points of $B \setminus \{p\}$ lie on a unique line skew with π_0 and hence, as it is proved in [6], $\dim\langle \mathcal{U} \setminus B_0 \rangle = 4$ and $d = 7$. However, if π_1 and π_2 are two planes containing two disjoint blocks not meeting B_0, then π_1 and π_2 are certainly not disjoint, contradicting our hypothesis.

$\boxed{\textbf{Case 2 } (|\mathbb{K}| \geq 4): \quad \langle \mathcal{U} \setminus B_0 \rangle \cap \pi_0 = \{p\}.}$

Lemma 4.2. *This case cannot occur.*

Proof. If $p \notin B_0$, then again every block intersecting B_0 in a point consists of points on a line and a point on B_0. Hence, by Lemma 1 of [6], we get that $\dim\langle \mathcal{U} \setminus B_0 \rangle = 4$ and $d = 6$, a contradiction.

If $p \in B_0$, then $\mathcal{U} \setminus B_0$ is an embedding of $\mathsf{AQ}(4, \mathbb{K})$ in a $\mathsf{PG}(d-2, \mathbb{F})$ in such a way that the points contained in a line not intersecting a fixed line at infinity L_∞ are embedded in a line of $\mathsf{PG}(d-2, \mathbb{F})$, while the points on lines intersecting L_∞ span at most a plane. A grid of Type 1 is such that it either contains the line L_∞ (meaning that there is a regulus of affine lines all intersecting L_∞ in a point) or it has only one affine line intersecting L_∞.

Let Q be a grid of Type 1 not containing L_∞. It is easy to see that such a grid (viewed as set of points of $\mathsf{AQ}(4,\mathbb{K})$) spans a space of dimension at most three. Let L be an affine line of Q not intersecting L_∞ and such that the unique affine line of Q intersecting L_∞ at infinity belongs to the regulus of L. If $\mathsf{AQ}(4,\mathbb{K}) \subseteq \langle Q\rangle$, then $d-2 \leq 3$, clearly a contradiction. So there must be a line intersecting L, possibly at infinity, not contained in $\langle Q\rangle$. If all the lines intersecting L in affine points are contained in $\langle Q\rangle$, then there must be a line M intersecting L at infinity such that $M \not\subset \langle Q\rangle$. Let $R \in M$ be such that $R \notin \langle Q\rangle$. Let $M' \neq M$ be a line through R and not intersecting L_∞. Then M', L and M are contained in a grid of Type 2 and all the points of $M' \setminus \{R\}$ are contained in $\langle Q\rangle$. This is possible only if M' spans a plane, hence the point at infinity of M' belongs to L_∞. Since there is only one line of a regulus with this property, the other lines of the regulus of M' span lines of $\mathsf{PG}(d-2,\mathbb{F})$ and hence $M \setminus \{R\}$ is contained in Q. Since the line M intersect L at infinity, M spans a line and so we get a contradiction. Hence there must be a line M intersecting L in an affine point such that $M \not\subset \langle Q\rangle$. Suppose that all the lines meeting L in an affine point not contained in $\langle Q\rangle$ intersect L_∞. Let M' be such a line. The grid of Type 1 containing L and M' but not L_∞ is such that all the lines of the regulus of M' except M' are contained in $\langle Q\rangle$, but this easily leads to a contradiction. Hence there must be a line M intersecting L in an affine point and disjoint from L_∞, i.e. $\langle M\rangle$ is a line. There are two grids of Type 1 containing the lines M and L and none of the two can contain L_∞, since both M and L do not meet L_∞. At least one of the two intersects Q in L and in an affine line meeting L and let Q' be such a grid. Hence we have $\dim\langle Q'\rangle = 3$ and $\dim\langle Q, Q'\rangle = 4$. Let P be a point not contained in Q nor Q'. Suppose that the line joining P and L is such that it intersects L in an affine point P_L. Let R be the intersection point of the lines contained in $Q \cap Q'$. Suppose that $P_L \neq R$. Then the cone with vertex P_L contains the line L, the line of Q meeting L in P_L, namely N, and the line of Q' meeting L in P_L, namely N' and we have $N \neq N'$. The line N does not meet L_∞ because the line of Q that intersects L_∞ is in the same regulus of L. Clearly, the line N' cannot be a line of Q, and we have also that $N' \not\subset \langle Q\rangle$, since if N' was contained in $\langle Q\rangle$, we would have two lines of Q' of the same regulus (hence skew) in $\langle Q\rangle$, so $Q' \subset \langle Q\rangle$, a contradiction. So every Hermitian conic of the cone with vertex P_L has a point on N, a point on L and one on N' that cannot be contained in $\langle Q\rangle$ and it is then contained in the span of $\langle N, L, N'\rangle$. Hence the cone is contained in $\langle Q, Q'\rangle$ and also P is. Now let P be collinear with either the point at infinity of L, or R. Then there is a line N'' through P such that N'' is not the line meeting L and does not intersect L_∞. It follows that $\langle N''\rangle$ is a line and all its point except at most two (one collinear with R and one collinear with the point at infinity of L) are contained in $\langle Q, Q'\rangle$. So also $P \in \langle Q, Q'\rangle$, because we have

$|\mathbb{K}| \geq 4$. Hence we have proved that $\mathsf{AQ}(4,\mathbb{K}) \subset \langle Q, Q' \rangle$. Since $\dim\langle Q, Q' \rangle \leq 4$, we have $d-2 \leq 4$, a contradiction.

So this case can not occur. $\square$

$\boxed{\textbf{Case 3}: \quad \dim(\langle \mathcal{U} \setminus B_0 \rangle \cap \pi_0) \geq 1 \text{ for every block } B_0.}$

We begin by enumerating some easy and direct consequences of the fact that disjoint blocks span disjoint planes.

(I) The images of two skew lines L and M such that their points at infinity are not contained in the same line of the spread (i.e. the corresponding blocks in $\mathsf{PG}(7,\mathbb{K})$ meet B_0 in two different points) span two spaces $\langle L^\alpha \rangle$ and $\langle M^\alpha \rangle$ such that, without loss of generality, either $\langle L^\alpha \rangle$ is a point and $\langle L^\alpha \rangle \nsubseteq \langle M^\alpha \rangle$, with the latter a line, or $\langle L^\alpha \rangle$ and $\langle M^\alpha \rangle$ are two lines with at most one common point.

(II) If C is a Hermitian conic of $AQ(4,\mathbb{K})$, then α is injective on it and $\dim\langle C^\alpha \rangle = 2$. Since every pair of non-collinear points of $AQ(4,\mathbb{K})$ is contained in a Hermitian conic, this implies that $P_1^\alpha \neq P_2^\alpha$ for every two distinct non-collinear points P_1 and P_2.

(III) Let P be an arbitrary point of $\mathsf{AQ}(4,\mathbb{K})$. Then all the points $R \in \mathsf{AQ}(4,\mathbb{K})$ such that $R^\alpha = P^\alpha$ must lie on a unique line.

We can prove the following lemmas.

Lemma 4.3. *Let Q be a grid of Type 1. The image of Q under α spans a space of dimension at most three, and if one of the lines has as image a point, then the image of Q spans a plane.*

Proof. Let L_1 and L_2 be two opposite lines of Q such that the line at infinity intersecting them is not a line of the spread. If L_1^α is a point, then by **(I)**, $\langle L_2^\alpha \rangle$ is a line which does not contain L_1^α. Hence the image of Q is contained in the plane $\langle L_1^\alpha, L_2^\alpha \rangle$. If a line of the other regulus, say M, maps on a point, then all the lines of the regulus of L_1 map on distinct lines through the point M^α and no other point of those lines maps on M^α, by **(I)** and **(III)**. Hence another line of the regulus of M maps on a line that meets all the lines of the regulus of L in a point distinct from M^α, hence their images are contained in a plane. Suppose now that $\langle L_1^\alpha \rangle$ and $\langle L_2^\alpha \rangle$ are two lines, which, by **(I)**, are distinct. If $\langle L_1^\alpha \rangle \cap \langle L_2^\alpha \rangle$ is not a point of the image of both L_1 and L_2 (this includes the empty set!), then clearly $Q^\alpha \subseteq \langle L_1^\alpha, L_2^\alpha \rangle$. Suppose now $\langle L_1^\alpha \rangle \cap \langle L_2^\alpha \rangle = P_1^\alpha = P_2^\alpha$ for some $P_1 \in L_1$ and $P_2 \in L_2$. Then P_1 and P_2 lie on the same line M. Since no other point of L_1 and L_2 can map on $P_1^\alpha = P_2^\alpha$, all the lines meeting L_1 and L_2 distinct from M map on the plane $\langle L_1^\alpha, L_2^\alpha \rangle$, so $\dim\langle Q^\alpha \rangle \leq 3$. $\square$

Lemma 4.4. *Suppose $|\mathbb{K}| \geq 4$. Let Q be a grid of Type 2, then the image of Q under α spans a space of dimension at most three, and if one of the lines has as image a point, then the image of Q spans a plane.*

Proof. Let L be a line of Q such that L^α is a point and let $\mathcal{R}_1$ be the regulus of L. By **(I)**, any line L' of $\mathcal{R}_1$ is such that $L^\alpha \notin L'^\alpha$. The lines intersecting both L and L' in an affine point are mapped on the plane $\pi_{L'} = \langle L^\alpha, L'^\alpha \rangle$; their images have to be distinct by **(I)** and no point of them can be mapped on L^α by **(III)**. This readily implies that the plane $\pi_{L'}$ is independent of the choice of L', and we denote it by π. So, varying L' we see that only the line M of Q intersecting L at infinity could possibly be mapped outside π. But every point of M is contained in a line of $\mathcal{R}_1$ that has at least two points with distinct images mapped on $\langle L^\alpha, L'^\alpha \rangle$; hence every point of M is mapped on $\langle L^\alpha, L'^\alpha \rangle$, which is a plane.

From now on, we can assume that the images of all the lines of Q generate a line. Suppose that there exist at least two lines L and L' of the regulus $\mathcal{R}_1$ mapped on two mutually skew lines. The lines intersecting both L and L' in an affine point are mapped in $\langle L^\alpha, L'^\alpha \rangle$. Let M be the line of Q not belonging to $\mathcal{R}_1$ and intersecting L in a point at infinity and suppose that some point P_0 of M maps outside $\langle L^\alpha, L'^\alpha \rangle$. Let L_0 be the line of $\mathcal{R}_1$ through P_0. Let $\mathcal{R}_2$ be the regulus of Q opposite to $\mathcal{R}_1$ and let P'_0 be the point of L_0 such that the line of $\mathcal{R}_2$ through P'_0 meets L' at infinity. Then all the points of L_0 other then P_0 and P'_0 are mapped onto the same point of $\langle L^\alpha, L'^\alpha \rangle$, say P^α. Since $|\mathbb{K}| \geq 4$, there are at least two points mapped on P^α, hence there are at least two lines of $\mathcal{R}_2$ mapped onto lines through P^α and meeting both L^α and L'^α. So these cannot be skew, yielding a contradiction. Hence $M^\alpha \subseteq \langle L^\alpha, L'^\alpha \rangle$. Similarly, the line of Q not belonging to $\mathcal{R}_1$ and intersecting L' in a point at infinity maps entirely in $\langle L^\alpha, L'^\alpha \rangle$.

So we are reduced to the case that the images of the lines of $\mathcal{R}_1$ generate lines that pairwise intersect. Of course, we may assume the same for the other regulus $\mathcal{R}_2$. Then either all images of the lines of $\mathcal{R}_i$, $i = 1, 2$, are contained in a plane, or they are lines through a common point. In the former case, the lemma is proved, so we may assume that the images of all lines of the regulus $\mathcal{R}_i$, $i = 1, 2$, are lines sharing a common point P_i. If $P_1 \neq P_2$, then we consider a line N_1 of $\mathcal{R}_1$ whose image is not incident with P_2. We then see that all lines of $\mathcal{R}_2$ are contained in the plane $\langle N_1^\alpha, P_2 \rangle$, except possibly the unique line of $\mathcal{R}_2$ meeting N_1 in a point at infinity. But adding the image of that line, the image of everything is contained in a 3-space and we are done.

Hence $P_1 = P_2$ and the images of all lines of Q go through P_1. Let L be a line of Q and let $R \in L$ be such that $R^\alpha \neq P_1$. Let M be the other line of Q containing R, then $M^\alpha \subset \langle L^\alpha \rangle$. By **(I)**, there can not be any other line of the same regulus as M that is mapped on $\langle L^\alpha \rangle$, hence all the other points of L are mapped on P_1. We can reason in the same way for M, hence also all the points of $M \setminus \{R\}$ are mapped on P_1, yielding a contradiction by **(III)**. □

Lemma 4.5. *The image under α of a cone K with an affine vertex spans a space of dimension at most three, and if one of the lines has as image a point, then the image of K spans a plane.*

Proof. Let P be the vertex of K and let C be a Hermitian conic contained in K. If $P^\alpha \notin C^\alpha$, then clearly $K^\alpha \subset \langle P^\alpha, C^\alpha \rangle$. If $P^\alpha \in C^\alpha$, then there is at most one line of the cone that does not map on the plane $\langle C^\alpha \rangle$, and hence K maps into a 3-space.

If some line L of K maps on a point, then clearly the image of K is contained in $\langle C^\alpha \rangle$. □

In order to proceed to the main step of the proof, we need to refine Lemma 4.3 in case α is neither injective nor constant on the set of points of some line of the grid.

Lemma 4.6. *Suppose $|\mathbb{K}| \geq 4$. Let Q be a grid of Type 1 and let M be a line of Q not belonging to the same regulus as the spread line contained in Q. If α is neither injective nor constant on the set of points of M, then the image of Q under α spans a space of dimension three.*

Proof. Let P_1 and P_2 be two points of M such that $P_1^\alpha = P_2^\alpha$. Note that M^α is not a point, and that the two lines L_1 and L_2 of Q intersecting M in P_1 and P_2 have at infinity two points not contained in a common line of the spread. Hence, by **(I)**, their images are two distinct lines, spanning a plane π. Suppose that Q^α is contained in the plane π.

Suppose first that every line intersecting M in an affine point maps on π. Then, since the image of $\mathsf{AQ}(4,\mathbb{K})$ generates a space of dimension at least 4, there exist two points R_1 and R_2 of $\mathsf{AQ}(4,\mathbb{K})$ collinear with the point at infinity of M such that $\langle R_1^\alpha, R_2^\alpha \rangle$ is skew to π. If R_1 and R_2 were collinear in $\mathsf{AQ}(4,\mathbb{K})$, then the span of the image of any grid of Type 2 through R_1 and M must contain the plane π (by our assumption that all lines meeting M in an affine point map into π, and the ones of the grid cannot all go into the image of M) and $\langle R_1^\alpha, R_2^\alpha \rangle$; so it would be a 4-space, contradicting Lemma 4.4. Hence R_1 and R_2 are not collinear. For every line L through R_i not meeting M at infinity, all points except R_i are collinear with an affine point of M, $i = 1, 2$, so $(L \setminus R_i)^\alpha$ is a unique point on π. It follows that, if there exist two lines K_1 and K_2 through R_1 and R_2, respectively, such that $K_1 \cap K_2$ is an affine point, then $(K_1 \setminus R_1)^\alpha = (K_2 \setminus R_2)^\alpha$, contradicting **(III)**. So we may assume that the two cones with vertices R_1 and R_2 meet only at infinity. Then we choose two lines K_1, K_2 through R_1, R_2, respectively, not intersecting one another, not intersecting M (at infinity) and such that the grid Q_0 through K_1 and K_2 is of Type 2. By the assumption made on R_1 and R_2, the line of Q_0 through R_i distinct from L_i meets L_j at infinity, $\{i, j\} = \{1, 2\}$. Hence there are at least two lines N_1 and N_2 of Q_0 meeting

K_1 and K_2 in affine points distinct from R_1 and R_2. But since $(K_i \setminus R_i)^\alpha$, $i = 1, 2$, is a unique point, N_1^α and N_2^α contradict **(I)**.

Hence there is at least one line, say L, intersecting M in an affine point, say P_L, such that $L^\alpha \nsubseteq \pi$. Our aim is to show the claim that the whole image of $\mathsf{AQ}(4, \mathbb{K})$ is contained in the 3-space $\Sigma := \langle \pi, L^\alpha \rangle$, which will be the final contradiction. In order to prove this claim, we proceed by showing that every grid or cone G through L and M is mapped into Σ. For the time being, we assume that G does not share a line at infinity with Q. Hence Q and G share, in addition to M, also a line L' intersecting M in an affine point $P_{L'}$. We distinguish some cases.

If $L'^\alpha = P_{L'}^\alpha$, then by the previous lemmas, the image of G is contained in the plane $\langle M^\alpha, L^\alpha \rangle$. If $\langle L'^\alpha \rangle$ is a line and $\langle L'^\alpha \rangle \neq \langle M^\alpha \rangle$, then $\dim\langle M^\alpha, L'^\alpha, L^\alpha \rangle = 3$, and so it contains the image of G; hence $G^\alpha \subseteq \Sigma$. Finally, assume that $\langle L'^\alpha \rangle = \langle M^\alpha \rangle$. Then we have to distinguish the different cases.

Suppose first that G is a cone. Let R be a point of L that does not map on π. Then R is contained in a Hermitian conic C that has a point P on M and a point R' on L'. The three points R, R', P map on three distinct points that can not lie on the same line. Consequently $C^\alpha \subseteq \langle L^\alpha, M^\alpha \rangle$. If no point of C maps on the image P_L^α of the vertex of the cone, then clearly the image of the cone is contained in the plane $\langle L^\alpha, M^\alpha \rangle$, and hence in Σ. Suppose now that there is a point of C that maps on P_L^α. Then it is not the point R of L. But then no point of L except for P_L maps on P_L^α as otherwise we would have two non-collinear points with the same image (contradicting **(II)**). So we can (re-)choose for R every point of L distinct from P_L and we conclude by the foregoing that every Hermitian conic on G maps into Σ, and hence so does the whole cone G.

Suppose secondly that G is a grid of Type 1. If no point of L' maps on P_L^α, then obviously the image of the grid is contained in the plane $\langle L'^\alpha, L^\alpha \rangle \subseteq \Sigma$. In the other case $P_{L'}$ necessarily maps onto P_L^α, and no other point of L' does (by the injectivity of α on non-collinear points). But again, this implies that the image of G is contained in Σ, as the only line of G belonging to the same regulus as M and for which the intersections with L and L' do not map to distinct points, is M itself. But M is trivially mapped into Σ.

Finally, suppose that G is a grid of Type 2. Then, reasoning as before, all the lines intersecting both L and L' in an affine point map into $\langle M^\alpha, L^\alpha \rangle$. Let M' and M'' be the lines of G intersecting L' and L, respectively, in a point at infinity. Suppose, by way of contradiction, that there exists a point $R \in M''$ not mapped into the plane $\langle M^\alpha, L^\alpha \rangle$ and let N be the other line of G through R. Then all the points of $N \setminus \{R, M' \cap N\}$ are mapped on a unique point S^α. If $S^\alpha \in M^\alpha$, then all the lines meeting both L and L' in an affine point would be mapped on $\langle M^\alpha \rangle$, contradicting **(I)**. Hence the line N must meet M at infinity and our argument implies that R is the only point of M'' not mapped on the plane $\langle M^\alpha, L^\alpha \rangle$. But then all the other points of M'' are mapped on a unique point of L'. Now each of these points is collinear with a point of M, so we get distinct lines of the regulus of L mapped on $\langle L'^\alpha \rangle$, and this is again in contradiction with **(I)**. Consequently, also M'' is mapped into the plane $\langle M^\alpha, L^\alpha \rangle$.

Likewise, if M' was not mapped into the plane $\langle M^\alpha, L^\alpha \rangle$, then the only point of that line that would not be mapped in that plane is the point (which we can again denote by R) collinear with the point at infinity of M. All the other points of the grid would be mapped into that plane. In particular all points of $M' \setminus \{R\}$ map onto one point T^α. This implies that $P_{L'} \in \{P_1, P_2\}$, as otherwise the lines of G through P_1, P_2 distinct from M meet M' in affine points and are hence mapped onto the same line, contradicting **(I)**.

Now, the point R is contained in a unique Hermitian conic C completely contained in G and in this case we have that all the points of the conic distinct from R are mapped into a line of the plane $\langle M^\alpha, L^\alpha \rangle$. We distinguish between two cases.

- (i) Suppose C contains two distinct points of M and L', say P and P', respectively. Then $(C \setminus \{R\})^\alpha \subset \langle M^\alpha \rangle$. Let $S \in C \setminus \{R, P, P'\}$ be arbitrary. Let L''' and M''' be the lines of G through S. Then $(L''' \cup M''') \cap (M \cup L')$ is a pair of non-collinear affine points $\{P'', P'''\}$, with $P''^\alpha \neq P'''^\alpha$ and both on M^α. Since also S^α belongs to M^α, at least one of the lines L''' or M''' maps onto $M^\alpha = L'^\alpha$, contradicting **(I)**.

- (ii) Suppose that C contains the point $M \cap L'$. Recall that $P_{L'} \in \{P_1, P_2\}$, and we may thus assume $P_{L'} = P_1$. The line L'_2 of G distinct from M through P_2 also contains a point of C, and so we see that $C \setminus \{R\}$ is mapped into $\langle L_2'^\alpha \rangle$, which also contains T^α. Let N' be an arbitrary line of the regulus of L'_2, but distinct from the one through R, and distinct from L'. Then N'^α contains T^α, and it also contains the image of a point of $C \setminus \{R\}$. These two cannot be the same, in view of **(II)**. Hence $\langle N'^\alpha \rangle = \langle L_2'^\alpha \rangle$, contradicting **(I)**.

So we have shown that G maps into Σ.

Now we let G be the grid of Type 1 containing L and M and sharing a line at infinity with Q. To complete the proof of our claim, we only have to show that G maps into Σ. Suppose, by way of contradiction, that this is not the case. Let L'_i, $i = 1, 2$, be the line of G different from M and incident with P_i. Then no point of $L_i \setminus \{P_i\}$ is mapped onto P_i^α (by **(III)**) and the images of L'_1 and L'_2 are distinct (by **(I)**). Hence G^α is contained in $\langle M^\alpha, L_1'^\alpha, L_2'^\alpha \rangle$. This implies that at least one of L'_1, L'_2 is mapped outside Σ. Without loss of generality, we may assume that L'_1 is mapped outside Σ. Then all points of $L'_1 \setminus \{P_1\}$ are mapped outside Σ. Let R_1 and R_2 be two such points.

Consider a grid Q' of Type 2 containing M and L'_1. If $L \subset Q'$, then Q' is a grid of Type 2 containing L and M, so from what we have already proved, we would get $Q'^\alpha \subset \Sigma$

and this is a contradiction. Let $N \neq L_1$ be a line of Q' intersecting M (also at infinity), then L, N, M are contained in a cone with affine vertex or in a grid, say G', containing L and M. If $G' = G$, then Q' and G would share three distinct lines: M, N and L, and so $G = Q'$, clearly a contradiction (Q' is a grid of Type 2 and G is a grid of Type 1). So we have that $G'^{\alpha} \subset \Sigma$, and hence $N^{\alpha} \subset \Sigma$. Then, all the lines of Q' intersecting M (also at infinity), except for L_1', are mapped into Σ. So if N_i, $i = 1, 2$, is the other line of Q' through R_i, then all the points of $N_i \setminus \{R_i\}$ are mapped on the same point $P_i'^{\alpha} \in \Sigma$. Consequently the lines of Q' distinct from L_1' intersecting both N_1 and N_2 in an affine point—and there are at least two such lines in view of $|\mathbb{K}| \geq 4$— map onto the line $\langle P_1'^{\alpha}, P_2'^{\alpha} \rangle$, which contradicts **(I)**.

In this way, we have proved our claim that the image of $\mathsf{AQ}(4, \mathbb{K})$ is contained in $\langle L^{\alpha}, \pi \rangle$, a final contradiction. The lemma is proved. □

The next proposition is the main step of our proof:

Proposition 4.7. *Let $|\mathbb{K}| \geq 4$. Either the map α is injective on the set of points of $\mathsf{AQ}(4, \mathbb{K})$, or there exists a point $p \in \mathcal{U} \setminus B_0$ such that every block through p consists of a set of points lying on a unique line plus the point p not on that line.*

Proof. If α is either injective or constant on the set of points of any line of $\mathsf{AQ}(4, \mathbb{K})$, then α is a so called *linear projective stacking*, and Lemma 1 of [6] implies that α is injective. Hence we may assume that there is a line M on which α is neither constant, nor injective.

We use the same notation as in Lemma 4.6. So we have the line M with two points P_1, P_2 mapped onto the same point $P_1^{\alpha} = P_2^{\alpha}$, and L_1, L_2 are two lines of a grid Q of Type 1 through M incident with P_1, P_2, respectively (where the line at infinity of Q meeting M is a spread line). Let π be the plane $\langle L_1^{\alpha}, L_2^{\alpha} \rangle$. As shown in the previous proof, all the lines of the regulus of M but M map into π, hence M does not map on π and there exists a point $R \in M$ such that $R^{\alpha} \notin \pi$. The point R is contained in another line of Q, say L_3. Our assumptions imply that L_3^{α} intersects π in just one point P^{α}. Hence all the lines of the regulus of M but M must map into lines in π through P^{α}. So, if a line of the other regulus does not map into π, then a unique point of it maps outside π and into M^{α}. We now claim that only L_3 does not map completely into π.

Suppose by way of contradiction that some other line L_4 of Q does not map completely into π. Then $R_4 := L_4 \cap M$ maps onto some point R_4^{α} of $M^{\alpha} \setminus \{P_1^{\alpha}\}$ and all other points of L_4 map into the same point P_4^{α} in π. By **(III)**, $P_4^{\alpha} \neq P^{\alpha}$. Hence every line of Q distinct from M but belonging to the same regulus as M maps onto the line $N^{\alpha} = \langle P^{\alpha}, P_4^{\alpha} \rangle$. The line N^{α} is skew to $\langle M^{\alpha} \rangle$, and so every line L of the regulus of L_1 maps onto a pair of points: $L \cap M$ maps onto some point of M^{α}, and the other points of L map onto one point P_L^{α} of N^{α}. Let Q^* be any grid of Type 1 containing M sharing with Q no line at infinity (i.e., the line at infinity of Q^* in the same regulus as M is a spread line). The grids Q and Q^* share, apart from M, a second line L^*, concurrent with M. All lines of Q^* in the same regulus as M map onto lines through $P^\alpha_{L^*}$, which are all distinct by **(I)**. Note also that, by **(III)**, the only points of Q^* mapped onto $P^\alpha_{L^*}$ are incident with L^*. Without loss of generality, we may suppose that $L^* \neq L_1$ (otherwise rename L_2 as L_1). Also, without loss of generality, we may suppose that $L^* \neq L_3$. Let L_1^* and L_3^* be the lines of Q^* distinct from M and incident with P_1 and R, respectively. Then the foregoing implies that both L_1^* and L_3^* map into distinct lines of the plane π^* spanned by the images of the lines of Q^* belonging to the same regulus as M, but distinct from M. Hence π^* contains M^α and $P^\alpha_{L^*}$ and so belongs to the 3-space $\langle M^\alpha, \pi\rangle$.

This implies that all points of $\mathsf{AQ}(4,\mathbb{K})$ not collinear with the point at infinity of M are mapped into the 3-space $\langle M^\alpha, \pi\rangle$. Let S be a point of $\mathsf{AQ}(4,\mathbb{K})$ collinear with the point at infinity of M, and let N_i, $i = 3, 4$, be the line through S meeting L_i (possibly at infinity). Since the points at infinity of L_3 and L_4 are collinear, one of the points $L_3 \cap N_3$ and $L_4 \cap N_4$ is affine, say $L_3 \cap M_3$. Then M and N_3 are contained in a grid Q_S of Type 2, and Q_S also contains L_3. If S^α were not contained in $\langle M^\alpha, \pi\rangle$, then, since all other affine points of N_3 are contained in there, all these points would map to a common point R_3^α. But this includes the intersection with L_3, hence $R_3^\alpha = P^\alpha$. As all but one points of L_3 are also mapped onto P^α, we obtain a contradiction with **(III)**. Hence z, and consequently the whole image of $\mathsf{AQ}(4,\mathbb{K})$, is contained in a 3-space, a contradiction. Our claim is proved.

Consequently, the only point of Q not mapped into π is R, and so the points of each line of Q through R distinct from R are mapped onto a single point. Varying Q, we see that this is true for all lines through R. The image of all points collinear with R except for R coincides with the image of every Hermitian conic in it, and is hence contained in a plane π'. Using similar arguments as before, it is easy to see that there is at least one grid of Type 1 containing R such that the image of it without R is not contained in π'. Without loss of generality we may assume that the image of $Q \setminus \{R\}$ is not contained in π'; hence $\pi \neq \pi'$.

Notice that no line of Q is mapped into the line $\langle P_1^\alpha, P^\alpha\rangle$, as this would imply that all lines of Q not through R are mapped into $\langle P_1^\alpha, P^\alpha\rangle$, and so all the lines but one of both reguli would be mapped on a single line, which contradicts **(I)**. This implies that the image of every line not through R not intersecting Q at infinity and not meeting L_3 or M, is contained in $\langle \pi, \pi'\rangle$. Since every point is contained in such a line (because $|\mathbb{K}| \geq 4$), this implies that the image of $\mathsf{AQ}(4,\mathbb{K}) \setminus \{R\}$ is contained in the 3-space $\langle \pi, \pi'\rangle$. Consequently R^α lies outside $\langle \pi, \pi'\rangle$, which implies in turn that the image of every block through R consist of a set of points lying on a unique line plus the point R^α.

This completes the proof of the proposition.

□

Corollary 4.8. *Let $|\mathbb{K}| \geq 4$. If there exists a point $p \in \mathcal{U}$ such that all the blocks B through p consist of the point p and the points of $B \setminus \{p\}$ lying on a line not through p, then $d = 7$ and $\dim\langle \mathcal{U} \setminus \{p\}\rangle = 6$. Also, if B_0 is an arbitrary block through p, then the corresponding projection α is injective on the set $\mathcal{U} \setminus B_0$.*

Proof. Suppose that $\dim\langle \mathcal{U} \setminus \{p\}\rangle \geq 7$, then we can consider another lax generalized Veronesean embedding of $\mathcal{U}$. Indeed, we can consider $\mathcal{U} \setminus \{p\}$ embedded as before and the point p mapped not in $\langle \mathcal{U} \setminus \{p\}\rangle$. Hence we have a lax generalized Veronesean embedding of $\mathcal{U}$ in some $\mathsf{PG}(d', \mathbb{F})$ with $d' \geq 8$, and clearly, this embedding also satisfies the condition that the planes generated by disjoint blocks are disjoint themselves.

Now project from a block through p on a $(d'-3)$-dimensional space and we are again in the hypothesis of Case 3. If α is injective, then by [6], $d' - 3 = 4$, a contradiction. Consequently, we have a point $r \in \mathsf{PG}(d', \mathbb{F})$ such that every block B through r consists of the point r and the points of $B \setminus \{r\}$ lying on a line not through r. But if we consider the block through p and r we easily get a contradiction.

Likewise, if we consider a block B_0 through p and the corresponding projection α, then Proposition 4.7 implies that either α is injective or there is a point $r \in \mathcal{U} \setminus B_0$ such that all points except for r of every block through r are collinear. Considering the block through p and r, we obtain a contradiction in the latter possibility. □

The following gathers some immediate consequences of the previous results.

Corollary 4.9. *Let $|\mathbb{K}| \geq 4$. Either there is a point $p \in \mathcal{U}$ such that all the blocks B through p consist of the point p and the points of $B \setminus \{p\}$ lying on a line not through p, or every block is a plane arc. In either case $d = 7$. Moreover, we have $B \subset \langle \mathcal{U} \setminus B\rangle$ for a block B if and only if B is a plane arc*

4.2 Part 2: reduction step for $|\mathbb{K}| = 3$

Many arguments above are not valid for the case $|\mathbb{K}| = 3$. Hence, for this case, we have to give alternative proofs. There is a tight connection with the generalized hexagons of order $(2, 2)$, and we will use this to derive a number of properties. Also, we will show that the condition $d \geq 7$ cannot be dispensed with by providing a counterexample for $d = 6$ and $|\mathbb{K}| = 3$. We will construct that counterexample explicitly using the connection with the hexagons.

Recall that a *generalized hexagon* (of order (s, t)) is a point-line structure $\mathcal{H}$ (with $s + 1$ points on each line and $t + 1$ lines through each point) containing no ordinary m-gons, for $m < 6$, and such that every two elements (points, lines or flags) are contained in an ordinary hexagon. Recall also that $\mathsf{H}(2)$ is the generalized hexagon constructed as follows. Let ρ be a Hermitian polarity of $\mathsf{PG}(2,9)$ and let $\mathcal{U}$ be the corresponding unital. The points of $\mathsf{H}(2)$ are the self-polar (non-degenerate) triangles of $\mathsf{PG}(2,9)$ with respect to ρ, and the lines of $\mathsf{H}(2)$ are the blocks of $\mathcal{U}$, hence the non-tangent lines to $\mathcal{U}$. If we identify a non-tangent line with its image under ρ, then incidence is inclusion made symmetric. This definition is due to Tits [12], and an explicit treatment is given in Section 1.3.12 of [13]. From that reference, and from the fact that the automorphism group of $\mathcal{U}$ acts as a rank 4 group on the blocks of $\mathcal{U}$ (since it acts as a rank 4 group on the lines of $\mathsf{H}(2)$), one deduces the following connections:

(i) Two lines of $\mathsf{H}(2)$ are opposite (i.e. at distance 6 in the incidence graph) if and only if the corresponding blocks of $\mathcal{U}$ intersect.

Concerning non-opposite lines in $\mathsf{H}(2)$, there are two kinds: two different lines can intersect or be at distance 4 in the incidence graph. This distinction can be seen in $\mathcal{U}$ as follows.

(ii) Two lines of $\mathsf{H}(2)$ are at distance 4 in the incidence graph if and only if the corresponding blocks of $\mathcal{U}$ are transversals of a common projective Baer subpencil.

There is another way to distinguish pairs of blocks of $\mathcal{U}$ that come from pairs of lines of $\mathsf{H}(2)$ at distance 2 or 4 in the incidence graph.

(iii) Two lines of $\mathsf{H}(2)$ are at distance 4 in the incidence graph if and only if in the affine generalized quadrangle $\mathsf{AQ}(4,3)$ corresponding to one block, the other block is a Hermitian conic contained in an affine cone (and in this case it is contained in exactly two cones and in one grid of Type 2); two lines of $\mathsf{H}(2)$ meet if and only if in the affine generalized quadrangle $\mathsf{AQ}(4,3)$ corresponding to one block, the other block is a Hermitian conic contained in an at least two distinct grids of Type 2 (and in this case it is contained in exactly two grids of Type 2 and in no cone).

So given a fixed line of $\mathsf{H}(2)$, the blocks of $\mathcal{U}$ corresponding to the lines of $\mathsf{H}(2)$ equal to or concurrent with that fixed line form a *spread* of $\mathcal{U}$, i.e. a set of disjoint lines partitioning the point set of $\mathcal{U}$. In fact, one can see from the definition of $\mathsf{H}(2)$ above that this spread can also be obtained as follows: consider the block B of $\mathcal{U}$ corresponding to the fixed line, and then take all blocks of $\mathcal{U}$ whose support in $\mathsf{PG}(2,9)$ is incident with the polar point of the support of B. The block B will be referred to as the *base block* of the spread.

We can also interpret the points of $\mathcal{U}$ in $\mathsf{H}(2)$. Indeed, the set of blocks through a point corresponds to a set of 9 mutualy opposite lines of $\mathsf{H}(2)$. This is exactly a (distance-3) spread of $\mathsf{H}(2)$, and there are indeed 28 of them. They are obtained by intersecting the standard representation of $\mathsf{H}(2)$ on $Q(6,2)$ with an elliptic hyperplane. Note that these spreads contain 9 lines and can be naturally given the structure of the smallest unital.

Digression—From the foregoing it follows that a spread of $\mathcal{U}$ corresponds to a set of seven non-opposite lines of $\mathsf{H}(2)$. But it is an elementary combinatorial exercise to show that, in $\mathsf{H}(2)$, the only such sets are the ones obtained from a fixed line by considering the lines at distance at most 2 from it in the incidence graph. Hence, we derive uniqueness of spreads of $\mathcal{U}$. It is now also easy to see that a *resolution* (i.e. a partition of the blocks of $\mathcal{U}$ in spreads) corresponds to a (distance-3) spread of $\mathsf{H}(2)$. And it is an easy combinatorial exercise to show that $\mathsf{H}(2)$ contains exactly 28 spreads, which are all Hermitian spreads. This provides computer-free proofs of the following results. Betten, Betten, and Tonchev [2] exhaustively search by computer for spreads and resolutions in certain unitals on 28 points. In particular, they find exactly 63 spreads of the Hermitian unital on 28 points (all equivalent) and 28 resolutions (all equivalent). So there is a natural equivalence between the points of $\mathcal{U}$ and the resolutions: to each point p we attach the following resolution: each spread contains as base block a block through p.

Now consider a line L of $\mathsf{H}(2)$ with corresponding block B of $\mathcal{U}$. Let $C_1, C_2, D_1, D_2, E_1, E_2$ be the blocks of $\mathcal{U}$ corresponding with the lines of $\mathsf{H}(2)$ that intersect L, with the convention that the lines corresponding to the blocks C_1, C_2 (D_1, D_2 and E_1, E_2, respectively) also meet. The foregoing implies that for each $(i,j,k) \in \{1,2\}^3$, there is a point $p_{i,j,k} \in B$ such that C_i, D_j, E_k are the transversals of a projective Baer subpencil $\mathfrak{B}_{i,j,k}$ with vertex $\{p_{i,j,k}\}$, and we have $p_{i,j,k} = p_{i',j',k'}$ if and only if $i+i' = j+j' = k+k' = 3$. Since a line in $\mathsf{H}(2)$ not opposite L and not concurrent with L is always opposite either C_1, C_2, D_1, D_2, or C_1, C_2, E_1, E_2, or D_1, D_2, E_1, E_2, we see that each block of $\mathcal{U}$ not meeting B, and not belonging to $\{C_1, C_2, D_1, D_2, E_1, E_2\}$ meets either C_1, C_2, D_1, D_2, or C_1, C_2, E_1, E_2, or D_1, D_2, E_1, E_2 in unique points. We refer to this property as (*).

From now on, we assume that $\mathcal{U}$ is embedded in some $\mathsf{PG}(d,\mathbb{F})$, with $d \geq 7$, in such a way that blocks are planar point sets, disjoint blocks span disjoint subspaces (and then, by Lemma 4.1, blocks span planes), $\mathcal{U}$ spans $\mathsf{PG}(d,\mathbb{F})$, and, additionally,

(Hyp1) for no point $p \in \mathcal{U}$, all blocks through p consist of p and three collinear points of $\mathsf{PG}(d,\mathbb{L})$, and

(Hyp2) not all blocks of $\mathcal{U}$ are planar 4-arcs.

We are now ready to prove some lemmas, which will culminate in the non-existence of such an embedding. The first two lemmas hold independently of (Hyp1) and (Hyp2).

Lemma 4.10. *Any Baer subpencil $\mathfrak{B}$ spans either a 5-space or a 6-space. If it spans a 6-space, and if the vertex is contained in the span of two transversals, then there is exactly one point of the third transversal not contained in that span.*

Proof. Since two transversals span disjoint planes, we readily deduce that $\mathfrak{B}$ spans at least a 5-space. Now suppose that the vertex x of $\mathfrak{B}$ does not belong to the 5-space generated by two transversals T_1, T_2. Every block of $\mathfrak{B}$ (through x) contains two points of $T_1 \cup T_2$, and these generate, together with x, a unique plane which must then contain all points of the block. It follows that $\mathfrak{B}$ is entirely contained in $\langle T_1, T_2, x\rangle$, which is 6-dimensional.

Now suppose that $x \in \langle T_1, T_2\rangle$ and that the third transversal T_3 is not contained in $\langle T_1, T_2\rangle$. Consider a point $y \in T_3$ not contained in $\langle T_1, T_2\rangle$. The block through x and y contains three collinear points in $\langle T_1, T_2\rangle$, say $x, x_1 \in T_1$ and $x_2 \in T_2$. If y' were another point of T_3 not contained in $\langle T_1, T_2\rangle$, then we would obtain three collinear points $x, x_1' \in T_1$ and $x_2' \in T_2$. But then, in the plane $\langle x, x_1, x_1'\rangle$, the lines $\langle x_1, x_1'\rangle$ and $\langle x_2, x_2'\rangle$ would meet in a point that belongs to both $\langle T_1\rangle$ and $\langle T_2\rangle$, a contradiction.

The assertion follows. □

Lemma 4.11. (*i*) *It cannot happen that all but exactly two points of $\mathcal{U}$ are contained in a given 6-dimensional subspace of $\mathsf{PG}(d, \mathbb{F})$.*

(*ii*) *It cannot happen that all but exactly three points of $\mathcal{U}$ are contained in a given 5-dimensional subspace V of $\mathsf{PG}(d, \mathbb{F})$.*

Proof. (*i*) Let V be a 6-space containing $\mathcal{U} \setminus \{x_1, x_2\}$, for $x_1, x_2 \in \mathcal{U}$, $x_1 \neq x_2$. Let $y \in \mathcal{U}$ be such that it is not contained in the block B through x_1, x_2. Let B_1, B_2 be the two other transversals of the projective Baer subpencil $\mathfrak{B}$ with vertex y and transversal B. Since $\langle \mathfrak{B}\rangle$ is at most 6-dimensional, the space $W =: \langle y, B_1, B_2\rangle$ is 5-dimensional. But then Lemma 4.10 implies that at most one point is outside $W \subseteq V$, contradicting the fact that both x_1, x_2 lie outside V.

(*ii*) Let V be a 5-space containing $\mathcal{U} \setminus \{x_1, x_2, x_3\}$, for $x_1, x_2, x_3 \in \mathcal{U}$, $x_1 \neq x_2 \neq x_3 \neq x_1$. Since $d \geq 7$, we may assume without loss that $\langle V, x_1\rangle \neq \langle V, x_2\rangle$ and $x_3 \notin \langle V, x_1\rangle$. Hence $x_2, x_3 \notin \langle V, x_1\rangle$, which is 6-dimensional, contradicting (*i*).

□

Corollary 4.12. *Under the hypothesis* (Hyp1)*, it cannot happen that all but at most two points of $\mathcal{U}$ are contained in a given 6-dimensional subspace of $\mathsf{PG}(d, \mathbb{F})$, and it cannot happen that all but at most three points of $\mathcal{U}$ are contained in a given 5-dimensional subspace V of $\mathsf{PG}(d, \mathbb{F})$.*

Proof. We only have to exclude the possibility that exactly one point x of $\mathcal{U}$ is not contained in some 6-space V. But in this case, all blocks through x must have three collinear points in V, contradicting (Hyp1). $\square$

Lemma 4.13. *Under the hypotheses* (Hyp1) *and* (Hyp2)*, we have that there exists some projective Baer subpencil contained in a* 5*-space of* $\mathsf{PG}(d, \mathbb{F})$*.*

Proof. Suppose by way of contradiction that every projective Baer subpencil spans a 6-space.

By (Hyp2) there is a block B of $\mathcal{U}$ such that B is not a 4-arc in $\langle B \rangle$. Then there is some point $p \in B$ with the property that $B \setminus \{p\}$ is a set of three collinear points in $\mathsf{PG}(d, \mathbb{F})$. Let $x \in B \setminus \{p\}$, and let B' be any block of $\mathcal{U}$ through p, with $B' \neq B$, and which we can choose in such a way that $B' \setminus \{p\}$ does not consist of three collinear points (using (Hyp1)). Let $\mathfrak{B}$ be the projective Baer subpencil with vertex x and transversal B'.

Let T_1, T_2 be the other two transversals. Clearly $x \in \langle T_1, T_2 \rangle$. By Lemma 4.10, there is a unique point $y \in B'$ not contained in $\langle T_1, T_2 \rangle$. But $y \neq p$ as $B' \setminus \{y\}$ is a set of three collinear points. Hence for the block $C \in \mathfrak{B}$ through x and y holds that $C \setminus \{y\}$ is a set of three collinear points. But then, just as in the proof of Lemma 4.10, the plane $\langle C \setminus \{y\}, B \setminus \{p\} \rangle$ contains a point of the intersection $\langle T_1 \rangle \cap \langle T_2 \rangle$.

This contradiction concludes the proof of the lemma. $\square$

From now on, we will assume that the projective Baer subpencil with vertex $p_{1,1,1}$ and transversals C_1, D_1, E_1 is contained in 5-space V.

Lemma 4.14. *Not all of* $p_{1,1,2}, p_{1,2,1}, p_{2,1,1}$ *are contained in* V*.*

Proof. Suppose, by way of contradiction, that B is entirely contained in V. Lemma 4.10 implies that each projective Baer subpencil $\mathfrak{B}_{1,1,2}, \mathfrak{B}_{1,2,1}, \mathfrak{B}_{2,1,1}$ contains at most one point not in V. Since the union of these Baer subpencils, together with $p_{1,1,1,}$, which is in V, is the whole point set of $\mathcal{U}$, we have obtained a contradiction to Corollary 4.12. $\square$

Lemma 4.15. *At most one of* $p_{1,1,2}, p_{1,2,1}, p_{2,1,1}$ *is contained in* V*.*

Proof. By Lemma 4.14, we may assume that $p_{1,1,2}$ and $p_{1,2,1}$ are contained in V and $p_{2,1,1}$ is not. Note that at most one point of each of D_2 and E_2 is not contained in V, hence by Corollary 4.12, not all points of C_2 are contained in V. But not all points of D_2 are contained in V as otherwise Lemma 4.10 applied to $\mathfrak{B}_{2,2,1}$ implies that at most one point of C_2 does not belong to V and so there are at most three points of $\mathcal{U}$ not belonging to V (one on C_2, one on E_2, and $p_{2,1,1}$), contradicting Corollary 4.12. Hence exactly one point d_2 of D_2 is not contained in V, and, similarly, exactly one point e_2 of E_2 is not contained in V.

Now, since $d \geq 7$, at least one of d_2, e_2 does not belong to $W := \langle V, p_{2,1,1}\rangle$. Suppose, without loss, that d_2 does not belong to W. Let x be a point on C_2 not lying on the block through d_2 and e_2. We claim that x belongs to V. Indeed, the block through d_2 and x contains two more points either on $C_1 \cup D_1$, or in $E_1 \cup \{p_{1,1,2}\}$, or in $(E_2 \setminus \{e_2\}) \cup \{p_{1,1,1}\}$. But these two points are always contained in V, and if x was not contained in V, then the line $\langle d_2, x\rangle$ would be skew to V (as $x \in W$ and $d_2 \notin W$), contradicting the fact that the block is a planar set of points. Our claim is proved.

Since there is at most one point of C_2 also in the block through d_2 and e_2, at most one point of C_2 is not contained in V. Hence, by Corollary 4.12, *exactly* one point c_2 is not contained in V. By the foregoing, the block through d_2, e_2 contains c_2, and hence also $p_{1,1,1}$. Consequently the block through $p_{2,1,1}$ and d_2 does not contain e_2 and hence contains two points of $C_1 \cup (E_2 \setminus \{e_2\})$. But again, the line $\langle p_{2,1,1}, d_2\rangle$ is skew to V, a contradiction.

We conclude that this situation cannot happen, proving the lemma. $\square$

Lemma 4.16. *None of $p_{1,1,2}, p_{1,2,1}, p_{2,1,1}$ is contained in V.*

Proof. By the previous lemmas we may assume that $p_{2,1,1} \in V$ and $p_{1,2,1}, p_{1,1,2} \notin V$.

Now $W := \langle V, B\rangle$ is 6-dimensional, hence $\mathcal{U} \setminus C_2$ is contained in W (since the projective Baer subpencils $\mathfrak{B}_{1,2,1}$ and $\mathfrak{B}_{1,1,2}$ are contained in W). But at most, and hence exactly, one point c_2 of C_2 is not contained in W. But now all blocks through c_2 have three collinear points in W, contradicting (Hyp1). $\square$

Lemma 4.17. *At least one of $p_{1,1,2}, p_{1,2,1}, p_{2,1,1}$ is contained in V.*

Proof. Suppose, by way of contradiction, that none of $p_{1,1,2}, p_{1,2,1}, p_{2,1,1}$ is contained in V.

If all of $p_{1,1,2}, p_{1,2,1}, p_{2,1,1}$ were contained in a common 6-space W containing V, then B would be contained in W. But then $\mathcal{U}$, as the union of the projective Baer subpencils $\mathfrak{B}_{1,1,2}$, $\mathfrak{B}_{1,2,1}$, $\mathfrak{B}_{2,1,1}$ and the point $p_{1,1,1}$, is contained in W, a contradiction.

Define $W_1 = \langle p_{2,1,1}, V\rangle$, $W_2 = \langle p_{1,2,1}, V\rangle$ and $W_3 = \langle p_{1,1,2}, V\rangle$. By the previous paragraph, not all of these subspaces are equal. So we can distinguish two cases.

First suppose that $W_1 \neq W_2 \neq W_3 \neq W_1$. Then, by Corollary 4.12, there is a least one point of $C_2 \cup D_2 \cup E_2$ which does not belong to V. Without loss, we may assume that some point $c_2 \in C_2$ does not belong to V. The block through c_2 and $p_{1,1,2}$ contains a point of V on E_1, and a point d_2 of D_2. If $d_2 \in V$, then $\langle c_2, p_{1,1,2}\rangle$ must meet V, contradicting $W_1 \neq W_3$. Hence $d_2 \in W_2 \setminus V$. Similarly the point $e_2 \in E_2$ on the block through c_2 and $p_{1,2,1}$ belongs to $W_3 \setminus V$.

Since the blocks meeting B cannot contain a triangle (as is clear from the fact that they form $\mathsf{AQ}(4,3)$), the block F through d_2 and e_2 does not meet B, and hence, by (*), has two points in $D_1 \cup E_1 \subseteq V$. Again, this contradicts the fact that F is a planar set of points and $W_2 \neq W_3$.

Hence, since $d \geq 7$, we may assume that $W_1 = W_2 \neq W_3$. Using Lemma 4.10, we see that $\mathfrak{B}_{2,1,1}$ and $\mathfrak{B}_{1,2,1}$ are contained in W_1, consequently only $p_{1,1,2}$ and E_2 are not contained in W_1.

If $p_{1,1,1}$, C_2 and D_2 generated a 6-space, then this 6-space would coincide with $\langle \mathfrak{B}_{2,2,2} \rangle$, and so $E_2 \subseteq W_1$. This would imply that $p_{1,1,2}$ is the only point outside W_1, contradicting Corollary 4.12. Hence $p_{1,1,1}$, C_2 and D_2 generate a 5-space (contained in W_1) and Lemma 4.10 yields that at most one point of E_2 lies outside W_1. As before, (Hyp1) implies that exactly one point e_2 of E_2 is contained in $W_3 \setminus V$, and Lemma 4.11(i) shows that this cannot occur.

The lemma is proved. $\square$

Hence we have shown that either there is a (unique) point p in $\mathcal{U}$ such that every block through p consists of p plus three points on a line, or every block is a 4-arc. In the latter case, the projection of $\mathcal{U} \setminus B$ from $\langle B \rangle$, for any block B, is a linear projective stacking (see the beginning of the proof of Proposition 4.7) of $\mathsf{AQ}(4,3)$, and hence an injective embedding by Lemma 1 of [6]. In this case $d = 7$. In the former case, we can prove the following lemma.

Lemma 4.18. *If there is a (unique) point p in $\mathcal{U}$ such that every block through p consists of p plus three points on a line of $\mathsf{PG}(d,\mathbb{F})$, then $\mathcal{U} \setminus \{p\}$ spans a 6-space and $d = 7$. Also, every block not through p is a 4-arc.*

Proof. Let B be any block through p and let L be the line of $\mathsf{PG}(d,\mathbb{F})$ containing three points of B. Now project $\mathcal{U} \setminus B$ from L (onto some skew subspace of dimension $d-2$). The arguments given in the proof of Lemma 4.1 can here be repeated verbatim to show that this projection is injective (just do not use p for one of the points x_1, x_2 in that proof). But it is a linear projective stacking of $\mathsf{AQ}(4,3)$ because of our assumption on p. Hence by Lemma 1 of [6], it is an embedding and the image generates a subspace of dimension at most 4. Consequently, $\mathcal{U} \setminus \{p\}$ generates a subspace of dimension at most 6, hence exactly 6, and $d = 7$. Let $\{x_1, x_2, x_3, x_4\} \not\ni p$ be a block such that $x_1 \in \langle x_2, x_3 \rangle$. Since B was arbitrary, we may suppose $x_1 \in L$. But then $\langle L, x_2 \rangle = \langle L, x_3 \rangle$, contradicting the injectivity of the projection of $\mathcal{U} \setminus B$ from L. $\square$

Now we prove a result that it is needed in the next section.

Proposition 4.19. *The image of a grid of Type 2 is at most 3-dimensional.*

Proof. Let Q be a grid of Type 2, let L_1 and L_2 be two skew lines of Q, let M_1 and M_2 be the two lines of Q intersecting both L_1 and L_2 in an affine point, and let M_3 and M_4

be the lines intersecting only L_1 and L_2 respectively in an affine point. If P is a point of M_i, $i = 3, 4$, not contained in L_{i-2}, then clearly $\langle Q^\alpha \rangle \subseteq \langle L_1^\alpha, L_2^\alpha, P^\alpha \rangle$ (we recall that α is injective). Hence, if $\langle L_1^\alpha \rangle \cap \langle L_2^\alpha \rangle \neq \emptyset$, then $\dim \langle Q^\alpha \rangle \leq 3$. Suppose now that the images of skew lines of Q lie on skew lines, so $\Sigma = \langle L_1^\alpha, L_2^\alpha \rangle$ is a solid and suppose by contradiction that $Q^\alpha \nsubseteq \Sigma$. Let $P \in M_3$ be such that $P \notin L_1$, let $R \in M_4 \cap L_2$ and let C be a conic of Q through them. The conic C has a point on the line M_1 and a point on the line M_2; there are two choices for C and it is easy to check that, for exactly one choice, C does not contain any point at infinity. Let C be that conic. Hence the points of $(C \setminus \{P\})^\alpha$ are all contained in Σ. If $(C \setminus \{P\})^\alpha$ lies in a line, then the lines $\langle M_1^\alpha \rangle$ and $\langle M_2^\alpha \rangle$ lie in the plane $\langle L_1^\alpha, (C \setminus \{P\})^\alpha \rangle$, a contradiction. So $(C \setminus \{P\})^\alpha$ spans a plane and, since $P^\alpha \notin \Sigma$, $\langle C^\alpha \rangle$ is a solid. A quadratic cone in $\mathsf{AQ}(4,3)$ with vertex not in Q intersects Q in a conic that has no points at infinity, since a point at infinity is contained in exactly two affine lines, which must belong to Q if the point at infinity belongs to Q. Let R_1 be a point not in Q and collinear with R. The quadratic cone with vertex R_1 intersects Q in a conic C' through R; suppose that this conic intersects M_3 in the point $M_3 \cap L_1$. Then, consider the point $R_2 \in RR_1$ distinct from R and R_1: it is also a vertex of a quadratic cone and this cone intersects Q in a conic tangent to C' (since $\mathsf{Q}(4,q)$ is a generalized quadrangle), and so it will intersect M_3 in a point distinct from $M_3 \cap L_1$; without loss of generality, we may assume that this is P.

So, we can assume that the conic C considered before is contained in a quadratic cone with an affine vertex, and hence it is contained in two quadratic cones, say K_1 and K_2. We have proven that $\langle C^\alpha \rangle$ is a solid and since the image of a cone lies in a space of dimension at most three, we obtain $\langle K_1^\alpha, K_2^\alpha \rangle = \langle C^\alpha \rangle$. Let C_i be the conic at infinity of K_i. Let S be a point not contained in $K_1 \cup K_2$ and let K_S be the quadratic cone with vertex S. If the conic at infinity of K_S does not coincide with C_1 or C_2 or does not have two points on C_1 and two points on C_2, then $S^\alpha \in \langle C^\alpha \rangle$. Suppose that S is such that K_S intersects the space at infinity in C_i, for some $i = 1, 2$, and let S' be any affine point collinear with S. The cone $K_{S'}$ has only one point on C_i, so, by the above, $S'^\alpha \in \langle C^\alpha \rangle$ and so also S^α belongs to $\langle C^\alpha \rangle$.

There are exactly 24 affine points. Hence they determine 12 conics at infinity. We already have C_1 and C_2 and then, as we have shown before, there must be two conics intersecting C_1 in only a point, for every point of C_1. Consequently there are at most two conics at infinity that may intersect both C_1 and C_2. Denote them by C_3 and C_4 (if they exist; if only one of them exists, then the argument is similar) and let P_i, $i = 1, \ldots, 4$, be the vertices of the cones containing them. The points P_i^α are the only ones that may not be in $\langle C^\alpha \rangle$. But through every P_i, there must be at least one line that does not contain any P_j, for all $j \neq i$. Hence $P_i^\alpha \in \langle C^\alpha \rangle$ and so $\mathsf{AQ}(4,3)^\alpha \subseteq \langle C^\alpha \rangle$, clearly a contradiction. □

4.3 Part 3: end of the proof

Suppose now either

(ASS1) that the projection from the plane spanned by any block B is injective from $\mathcal{U} \setminus B$ onto some 4-space $\mathsf{PG}(4,\mathbb{F})$, or

(ASS2) that there exists a point p such that all the blocks B through p consist of the point p and the points of $B \setminus \{p\}$ lying on a line of $\mathsf{PG}(7,\mathbb{F})$ not through p.

In both cases we have $d = 7$. We claim that $\mathcal{U}$ is contained in a projective subspace $\mathsf{PG}(7,\mathbb{K}')$ such that $\mathbb{K}' \cong \mathbb{K}$ and every block of $\mathcal{U}$ corresponds to a plane conic in $\mathsf{PG}(7,\mathbb{K}')$.

First we consider the projection α from the span of an arbitrary block B, which contains p in case (ASS2). We show that the projection of $\mathcal{U} \setminus B$ itself is contained in a 4-space over $\mathbb{K}' \cong \mathbb{K}$. We identify every point of $\mathcal{U} \setminus B$ with the corresponding point in $\mathsf{PG}(4,\mathbb{F})$ (and so blocks different from B get identified with certain point sets—collinear if the block meets B, not collinear otherwise). First, let D be a block different from B but meeting B in $\mathcal{U}$ non-trivially, say in x. Let E be another block of $\mathcal{U}$ also containing x, and suppose E is contained in the affine Baer subpencil $\mathfrak{A}$ determined by B and D. If the line L_D supporting D is skew to the line L_E supporting E, then all the points P lying in a grid of Type 2 together with D and E are contained in $\langle L_E, L_D\rangle$ (since the image of a grid is at most 3-dimensional by Lemmas 4.3, 4.4 and Proposition 4.19). Let $R \in \mathsf{AQ}(4,\mathbb{K})$ be a point contained in a cone with two lines at infinity together with D and E, and let M be a line through R not in that cone. All the points of $M \setminus \{R\}$ are contained in $\langle L_E, L_D\rangle$ by the foregoing, and so does R. Hence $\mathsf{AQ}(4,\mathbb{K})$ would be contained in a 3-dimensional space, a contradiction. We conclude that L_D and L_E meet in a point, and so do all lines supporting a block from the affine Baer subpencil $\mathfrak{A}$.

If $|\mathbb{K}| = 3$, then L_D and L_E are the only two lines of $\mathfrak{A}$ and we say that $L_D \cap L_E$ is the point at infinity of these two lines.

Now assume that $|\mathbb{K}| > 3$. We claim that all the lines L_F, with $F \in \mathfrak{A}$, meet in a common point. Indeed, suppose by way of contradiction that this is not the case. Then they are contained in the same plane π. Let F be a line of $\mathsf{AQ}(4,\mathbb{K})$ intersecting E in an affine point, and denote by L_F the affine span $\langle F\rangle$. Then all the points P which are contained in a grid of Type 2 together with E and F are contained in the solid $\langle \pi, L_F\rangle$. Let R be a point not contained in a grid of Type 2 together with E and F. Let G be a grid of Type 1 containing E and F and not containing R (such a grid exists since there are precisely two grids of Type 1 containing E and F, and R is contained in at most one of them). Let L_∞ be the line at infinity of $\mathsf{AQ}(4,\mathbb{K})$ belonging to G and incident with the point at infinity of E. Let H be the line of $\mathsf{AQ}(4,\mathbb{K})$ through R and meeting L_∞ in a point at infinity. Then no affine point of H is contained in G, exactly one affine point of

H is contained in the other grid of Type 1 containing E and F, and exactly one point of H is collinear with the intersection of E and F (by the choice of R, amongst the latter two points there is R of course). Hence there are at least two points left on H contained in a grid of Type 2 together with E and F. So the line L_H is contained in $\langle \pi, L_F \rangle$ and so is R. This is again a contradiction. Our claim is proved.

So with each affine Baer subpencil $\mathfrak{B}$ containing B corresponds a point $x_{\mathfrak{B}}$ and we will identify these points with the points at infinity of the lines of the affine quadrangle $\mathsf{AQ}(4, \mathbb{K})$. We will also denote $x_{\mathfrak{B}}$ by x_D, for $D \in \mathfrak{B}$. Note that there is some abuse of language here and that in principle, it could happen that the points x_D and x_E coincide, although they are two different points in the projective extension $\mathsf{Q}(4, \mathbb{K})$ of $\mathsf{AQ}(4, \mathbb{K})$. We will prove later that this can never happen.

Now we note that

$(*)$ *no pair of grids of Type 1 having a common line at infinity is contained in a solid*, and

$(**)$ *no grid of Type 1 is contained in a plane of* $\mathsf{PG}(7, \mathbb{F})$.

Indeed, we first show $(*)$. Let Q and Q' be two grids of Type 1 having a common line at infinity contained in a solid Σ of $\mathsf{PG}(7, \mathbb{F})$. Let p be any point of $\mathsf{AQ}(4, \mathbb{K})$. Then there are at least four lines of $\mathsf{AQ}(4, \mathbb{K})$ through p, and it is easy to see that at least one of them has at least two points in common with $Q \cup Q'$. This implies that p is contained in Σ and hence $\mathsf{AQ}(4, \mathbb{K})$ is contained in Σ, a contradiction.

Now we show $(**)$. Assume by way of contradiction that some grid Q of Type 1 is contained in a plane π of $\mathsf{PG}(7, \mathbb{F})$. Let D, E be two affine lines of Q belonging to the same regulus. Let e be the point at infinity of E and let E' be another affine line through e. Then the grid of Type 1 containing D and E' generates in $\mathsf{PG}(7, \mathbb{F})$ a subspace Σ of dimension at most 3 having D and e in common with π. This implies that π and Σ generate a subspace of dimension at most 3, contradicting $(*)$.

Identifying affine Baer subpencils with their unique vertex, viewed as a point at infinity of $\mathsf{AQ}(4, \mathbb{K})$, we now claim that the map $\mathfrak{B} \mapsto x_{\mathfrak{B}}$ is injective on each line at infinity of $\mathsf{AQ}(4, \mathbb{K})$. Indeed, suppose on the contrary that for two lines D, E of $\mathsf{AQ}(4, \mathbb{K})$, which have collinear but distinct points at infinity, we have $x_D = x_E$. Then the grid of Type 1 defined by E and D is contained in a plane, contradicting $(**)$. The claim is proved.

Now we claim that the map $\mathfrak{B} \mapsto x_{\mathfrak{B}}$ maps collinear points to collinear points. Indeed, let L be the set of points of a line at infinity (in $\mathsf{PG}(7, \mathbb{F})$) in a grid Q of Type 1. Let Q' be another grid of Type 1 containing L and an affine line of Q, say D. By $(*)$, Q and Q' intersect in a plane and so $\dim\langle L \rangle \leq 2$. Suppose that $\dim\langle L \rangle = 2$, then $D \subseteq \langle L \rangle$ and this would be true for every line of the regulus of Q containing D. Hence Q would be contained in a plane, a contradiction. We conclude that L spans a line.

Next we claim that the map $\mathfrak{B} \mapsto x_{\mathfrak{B}}$ is injective. Indeed, suppose on the contrary that for two lines D, E of $\mathsf{AQ}(4, \mathbb{K})$, which have non-collinear points at infinity, we have $x_D = x_E$. Then it is easily seen that all points at infinity (this structure forms a grid G!) are contained in a plane π. But then this plane is contained in the solid spanning any grid of Type 1. Considering two grids of Type 1 containing the same arbitrary affine line, this leads to the contradiction that every affine line must be contained in π. Our claim is proved.

It follows now that $\mathsf{Q}(4, \mathbb{K})$ is laxly embedded in $\mathsf{PG}(4, \mathbb{F})$. We now show that it is also *polarized*, i.e., for every point x of the quadrangle, the points collinear in the quadrangle to x are contained in a solid of $\mathsf{PG}(4, \mathbb{F})$. This already holds for all affine points of $\mathsf{AQ}(4, \mathbb{K})$ by Lemma 4.5, but in order to prove it for the points at infinity, it is just as much trouble to state and prove the following independent and general result.

Lemma 4.20. *Let* $\mathsf{Q}(4, \mathbb{K})$*,* $|\mathbb{K}| \geq 2$*, be laxly embedded in* $\mathsf{PG}(4, \mathbb{F})$*, then either the embedding is polarized, or* $|\mathbb{K}| = 3$ *and for every point* x *of* $\mathsf{Q}(4, \mathbb{K})$*, the set of points of* $\mathsf{Q}(4, \mathbb{K})$ *collinear with* x *generates* $\mathsf{PG}(4, \mathbb{F})$*.*

Proof. The proof is completely similar to the finite case, see Theorem 5.2 in [11], but we have to provide different appropriate references. First we treat the small cases. For $|\mathbb{K}| = 2$, the three lines through every point must generate a solid. For $|\mathbb{K}| = 3$, the proof of Theorem 5.1 in [11] applies verbatim. Alternatively, we can argue as in the general case below.

Let $|\mathbb{K}| \geq 3$. Consider an arbitrary line L of $\mathsf{Q}(4, \mathbb{K})$, and a plane π of $\mathsf{PG}(4, \mathbb{F})$ skew to L. Every line of $\mathsf{Q}(4, \mathbb{K})$ concurrent with L is projected from L into π onto some point. Every grid containing L is likewise projected onto a line. This yields an embedding of the dual of the affine plane $\mathsf{AG}(2, \mathbb{K})$ into π. By the dual of Lemma 1 of [1], this embedding either extends to an embedding of the projective closure $\mathsf{PG}(2, \mathbb{K})$ of $\mathsf{AG}(2, \mathbb{K})$ in π, or $|\mathbb{K}| = 3$, $\mathbb{F}$ contains a nontrivial root of unity and the set of affine points of no line of $\mathsf{PG}(2, \mathbb{F})$ that does not belong to $\mathsf{AG}(2, \mathbb{F})$ is mapped onto a collinear set of points of π. This implies that either all lines through an arbitrary point of L are contained in a solid, or $|\mathbb{K}| = 3$ and for no point on L this is the case.

The lemma is proved. $\square$

Applied to our situation, where we know that the condition of being polarized is satisfied for some points, this lemma means that the embedding of $\mathsf{Q}(4, \mathbb{K})$ in $\mathsf{PG}(4, \mathbb{F})$ is polarized. Clearly, by looking in solids spanned by grids, lines of $\mathsf{PG}(4, \mathbb{F})$ either intersect $\mathsf{Q}(4, \mathbb{K})$ in 0, 1, 2 points, or all points of a line of $\mathsf{Q}(4, \mathbb{K})$. Hence we can apply the main result of [10] to conclude that there is a subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$ such that $\mathsf{Q}(4, \mathbb{K})$ is fully embedded in a subspace $\mathsf{PG}(4, \mathbb{K}')$.

In particular we see that all blocks of $\mathcal{U}$ not intersecting B are conics in planes of $\mathsf{PG}(4,\mathbb{K}')$. By varying B, we conclude that every block (not through p in case (ASS2)) is a conic in some plane isomorphic to $\mathsf{PG}(2,\mathbb{K})$ of $\mathsf{PG}(7,\mathbb{F})$. Hence we can talk about the (unique) tangent at a point to such conic, and consequently, to the tangent line at a point of $\mathcal{U}$ to a block containing that point.

From now on we are in a position to follow the proof of the full case, see [6], but with additional arguments and references where appropriate. We first assume that all blocks are plane arcs.

Consider a point x on B, and a blok $D \neq B$ through x. The tangent line at x to D is projected from $\langle B\rangle$ onto a point at infinity of $\mathsf{AQ}(4,\mathbb{K})$ on the line at infinity corresponding with x as in the Bose-Bruck-André representation of $\mathcal{U}$. Hence all the tangents at x to blocks of $\mathcal{U}$ containing x are contained in a 4-space, which also contains B, but no other block through x. Considering the block D instead of B, and intersecting the thus obtained 4-space with the first one results in a 3-space Σ_x, containing all tangents at x to blocks of $\mathcal{U}$ containing x.

Likewise, let $\mathfrak{B}$ be an affine Baer subpencil containing B such that all members of $\mathfrak{B}$ contain x. Then the tangents at x to the members of $\mathfrak{B}$ are contained in a plane $\pi_{\mathfrak{B}}$. Since the affine Baer subpencil in x defines the structure of an affine plane on the set of blocks through x, we see that the projection from x of the set of tangents at x to blocks through x onto a plane π_x of Σ_x not through x, yields an affine plane $\mathcal{A}$. Let T be the tangent line at x to B, and let x_T be its projection from x onto π_x. Then the projection of $\mathcal{A}$ from x_T onto some line of $\mathcal{A}$ not through x_T yields a projective subline over $\mathbb{K}'$ (as this projection can be identified with a line at infinity of $\mathsf{AQ}(4,\mathbb{K})$). Hence, by Lemma 4.20, the projective closure of $\mathcal{A}$ embeds as a subplane in π_x, defined over the subfield $\mathbb{K}'$ of $\mathbb{F}$.

Now consider a point y not on B. The projection of all blocks through y intersecting B from $\langle B\rangle$ onto $\mathsf{PG}(4,\mathbb{F})$ is a quadratic cone inside a solid of $\mathsf{PG}(4,\mathbb{K}')$. But since this projection coincides with the projection of the tangents at y to these blocks, and since these tangents themselves also span a 3-space Σ_y, the projection restricted to Σ_y is bijective. Now we translate this situation to x (we let x play the role of y). Consider a projective Baer subpencil $\mathfrak{P}$ in x, say all members of $\mathfrak{P}$ intersect the block D, and assume $B \in \mathfrak{P}$. Then the projection of $\mathfrak{P}$ onto π_x is a conic in $\mathcal{A}$. Moreover, let Θ be a 6-space of $\mathsf{PG}(7,\mathbb{F})$ not containing x, but containing π_x, then the projection of D onto Θ is a conic is some projective plane π_D over $\mathbb{K}'$ (as the projection of D union its tangent at $D \cap B$ from $\langle B\rangle$ is a projective line over $\mathbb{K}'$), skew to π_x. Now, considering the grid of $\mathsf{AQ}(4,\mathbb{K})$ defined by $\mathfrak{P}$, we find for each point $z \in B \setminus \{x\}$ a block D_z containing z and intersecting each member of $\mathfrak{P}$. In projection from x onto Θ, this yields a set S of points with the following structure: through every point u of S, there is a projective subline $\lambda_u \subseteq S$ over $\mathbb{K}'$ (this amounts to the projection of a block of $\mathfrak{P}$ union its tangent at x), and there is a conic $C_u \subseteq S$ contained in a subplane over $\mathbb{K}'$, such that S generates a 5-space Λ of Θ, and every pair of distinct conics also generate this 5-space. Also, every line of S intersects every conic of S. This structure can be interpreted and referred to as a *Segre variety of a plane conic and a projective line*, both defined over $\mathbb{K}$. We now show that S is contained in a unique subspace of Λ defined over $\mathbb{K}'$. In essence, we do not need all our data. More exactly, we have the following lemma.

Lemma 4.21. *Let S be a Segre variety of a plane conic and a projective line, both defined over $\mathbb{K}$, and embedded in the projective space $\mathsf{PG}(5, \mathbb{F})$. Suppose that S is not contained in a hyperplane, suppose that there are two conics C, C' of S contained in a subplane over the subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$, and suppose that one projective line of S is a projective subline in $\mathsf{PG}(5, \mathbb{F})$ over the same subfield $\mathbb{K}'$. Then S is contained in a subspace over the subfield $\mathbb{K}'$.*

Proof. Consider the two conics C, C' of S in subplanes $\pi_{\mathbb{K}}, \pi'_{\mathbb{K}}$ over $\mathbb{K}'$, and let the line L of S consist of the points of a projective subline of $\mathsf{PG}(5, \mathbb{F})$ over $\mathbb{K}'$ (and we denote that subline also by L). By choosing appropriate coordinates, we can put C, C', L in a projective subspace $\mathsf{PG}(5, \mathbb{K}')$. We now show that every other point of S is also contained in $\mathsf{PG}(5, \mathbb{K}')$. Indeed, let $x \in S \setminus (C \cup C' \cup L)$. Let C_x be the conic of S containing x and let y be the intersection of C_x with L. Then x is contained in a line M of $\mathsf{PG}(5, \mathbb{F})$ intersecting both C and C'. Hence, M is also a line of $\mathsf{PG}(5, \mathbb{K}')$ (but not all points on M belong to $\mathsf{PG}(5, \mathbb{K}')$, of course!). We consider two other lines N, N' of $\mathsf{PG}(5, \mathbb{F})$ spanned by lines of S, different from $\langle L \rangle$ and M. Both N and N' are also lines of $\mathsf{PG}(5, \mathbb{K}')$. Since no three points of C and C' are collinear, the space η generated by y, N, N' is 4-dimensional and meets M in a unique point x', which, by our foregoing remarks, belongs to $\mathsf{PG}(5, \mathbb{K}')$. But η also contains the conic C_x of S. This implies $x = x'$ and the lemma is proved. □

So S is contained in a subspace $\mathsf{PG}(5, \mathbb{K}')$ of Λ.

Inspired by the last part of Section 3.3 in [6], we now consider a block F through x not contained in $\mathfrak{P}$. It is easy to see that there exists a projective Baer subpencil $\mathfrak{P}_F$ intersecting $\mathfrak{P}$ in two members D_1, D_2. The projection F' from x onto Θ of $F \setminus \{x\}$ is a line not contained in Λ (because the projection from B of $F \setminus \{x\}$ is not contained in the projection from B of $\mathfrak{P} \setminus \{B\}$—the latter being a grid of Type 1 in $\mathsf{AQ}(4, \mathbb{K})$). Now F' together with the projection of the tangent line at x to F determines a unique subline L_F over $\mathbb{K}'$. And L_F together with $\mathsf{PG}(5, \mathbb{K}')$ defines a unique subspace $\mathsf{PG}(6, \mathbb{K}')$ over $\mathbb{K}'$ of Θ.

The projection of all points on members of $\mathfrak{P}_F$ except x from x onto Θ is contained in a unique subspace $\mathsf{PG}(5, \mathbb{K}')^*$, which is, however, uniquely determined by the projective closure of $\mathcal{A}$, and the three sublines corresponding to (the projections of) D_1, D_2, F.

Hence $\mathsf{PG}(5,\mathbb{K}')^*$ is contained in $\mathsf{PG}(6,\mathbb{K}')$. Hence the projection of $\mathfrak{P}_F$ (taking out x) is contained in $\mathsf{PG}(6,\mathbb{K}')$. This now holds by the same argument for every projective Baer subpencil through x having three elements in common with $\mathfrak{P}\cup\mathfrak{P}_F$. But, as argued in the last paragraph of Section 3.3 of [6], such pencils cover the whole set of blocks through x.

We conclude that the projection of $\mathcal{U}\setminus\{x\}$ from x is contained in $\mathsf{PG}(6,\mathbb{K}')$.

Now observe that, since the projection from x onto Θ of $\mathfrak{P}$ is contained in Λ, the union of the members of $\mathfrak{P}$ generate a 6-space of $\mathsf{PG}(7,\mathbb{F})$. In fact, this holds for every projective Baer subpencil, and we can now consider one, say $\mathfrak{P}^*$, that is entirely contained in $\mathcal{U}\setminus\{B\}$. We can now re-choose π_x and Θ such that Θ contains $\mathfrak{P}^*$. Essentially since blocks contain skeletons of planes (a *skeleton* of a projective space of dimension d is a set of $d+2$ points no $d+1$ of which contained in a hyperplane), $\mathfrak{P}^*$ uniquely determines $\mathsf{PG}(6,\mathbb{K}')$.

Now we pick a point x' on B distinct from x. Then x,x' and $\mathsf{PG}(6,\mathbb{K}')$ are contained in a unique subspace $\mathsf{PG}(7,\mathbb{K}')$. We can project $\mathcal{U}\setminus\{x'\}$ from x' onto Θ. By the foregoing, this projection lands entirely in $\mathsf{PG}(6,\mathbb{K}')$. Now let z be any point of $\mathcal{U}\setminus\{x,x'\}$. The lines xz and $x'z$ of $\mathsf{PG}(7,\mathbb{F})$ bot contain a point of $\mathsf{PG}(6,\mathbb{K}')$ and hence are both contained in $\mathsf{PG}(7,\mathbb{K}')$. Consequently so is their intersection z.

We have shown that $\mathcal{U}$ is contained in $\mathsf{PG}(7,\mathbb{K}')$ such that every block is a conic in a plane. We can now apply the main result of [6] to conclude the proof of our main result in this case.

Next we assume (ASS2). In this case the projection of $\mathcal{U}\setminus\{p\}$ can be identified with $\mathcal{U}\setminus\{p\}$ itself. Consider a projective Baer subpencil $\mathfrak{B}$ with vertex p and transversal D. Then, by the previous (the part where we project from a block through p), we obtain a Segre variety of a conic and an affine line, both defined over $\mathbb{K}$, and embedded in some $\mathsf{PG}(5,\mathbb{F})$ in such a way that each conic is a conic in some subplane over the subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$, and similarly for each affine line. Just as in Lemma 4.21, one can show that this whole structure is embedded in a $\mathsf{PG}(5,\mathbb{K}')$. Moreover, projecting it from one of its affine lines, we obtain a grid structure where one line is missing; hence the missing conic is uniquely determined and the structure can be canonically completed to a Segre variety of a conic and a projective line embedded naturally in $\mathsf{PG}(5,\mathbb{K}')$. Now continuing as in the case (ASS1), we see that the structure $\mathcal{U}\setminus\{p\}$ is unique in some $\mathsf{PG}(6,\mathbb{K}')$. Since also the projection of standard Veronesean embedding of $\mathcal{U}$ from one of its points has that very same structure, we conclude by uniqueness that $\mathcal{U}\setminus\{p\}$ is isomorphic to such a projection. Adding p arbitrarily in $\mathsf{PG}(7,\mathbb{F})\setminus\langle\mathsf{PG}(6,\mathbb{K}')\rangle$, our main result follows.

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