Time-Frequency Signal Representations using Interpolations in Joint-Variable Domains

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***Abstract*—Time-frequency representations are a powerful tool for analyzing Doppler and microDoppler signals. These signals are frequently encountered in various radar applications. Data interpolators play a unique role in time-frequency signal representations under missing samples. When applied in the instantaneous autocorrelation domain over the time variable, the low-pass filter characteristic underlying linear interpolators lends itself to cross-terms reduction in the ambiguity domain. This is in contrast to interpolation performed over the lag variable or a direct interpolation of the raw data. We demonstrate the interpolator performance in both the time-domain and time-lag domain and compare it with sparse signal reconstruction, which exploits the local sparsity property assumed by most Doppler radar signals.**

***Index Terms*—Ambiguity domain, cross-terms reduction, interpolation, microDoppler, radar, time-frequency distribution.**

I. Introduction

INCOMPLETE, random, or nonuniform sampling of radar returns of moving targets can arise in data multiplexing, range ambiguity resolution, localization enhancement, noisy measurement removal, hardware simplification, sampling rate limitations, or logistical restrictions on data collections and acquisition schemes [1]-[4]. For nonstationary signals, missing or random samples introduce noise which clutters both the time-frequency (TF) and ambiguity domains and obscures desired signal information, especially the Doppler and microDoppler signatures [5], [6]. The latter arises in many applications, including urban radar and over the horizon radar [7]-[16].

In this paper, we show that interpolation applied in the instantaneous autocorrelation function (IAF) domain for data recovery has three-fold advantages:

(a) It reduces cross-terms by the virtue of the underlying low-pass filter characteristics when considered in the ambiguity domain. This property requires interpolation to be performed over the time variable instead of the lag variable of the IAF;

(b) It reduces noise as the result of the frequency bandlimited processing applied to noise, which is homogenously spread in the Fourier domain;

(c) It enhances signals auto-terms which lie in the vicinity of the origin.

This paper focuses on deterministic nonstationary signals rather than nonstationary random processes [17]. It considers uniform interpolations [18], [19] for data recovery applied in the time and time-lag domains under random sampling schemes. We compare the corresponding time-frequency distributions (TFDs) with the results obtained based on sparse reconstruction, which also uses the linear data model. This paper extends the work in [20] which reported initial results on the use of the interpolation techniques for computing TFDs of incomplete data. It puts linear interpolation and sparse reconstruction within the same data model, and provides comparison between interpolations performed in different domains and over different variables, including the IAF lag. Interpolator effects in the TF domain, which were noticed in [20], are further analyzed and explained to provide a better understanding of the role of interpolators in time-frequency analysis.

The paper is organized as follows. Section 2 briefly reviews TFDs and examines them from filtering and interpolation perspectives. Section 3 discusses the interpolations in the time domain and time-lag domain. Performance comparison is given in Section 4 using synthetic data.

II. Time-frequency distributions

The instantaneous autocorrelation function can be used to define the class of reduced interference distributions. For a signal $x \in C^N$, IAF is formulated over time n and lag m as,

$$R_{xx}(n,m) = x(n+m)x^*(n-m). \quad (1)$$

The Fourier transform (FT) of the IAF over time yields the ambiguity function (AF) defined over frequency p and lag m,

$$A(p,m) = \sum_{n=-N/2}^{N/2-1} R_{xx}(n,m) e^{-\jmath 2\pi np/N}, \quad (2)$$

whereas the FT over lag computes the Wigner-Ville distribution (WVD),

$$WD(n,k) = \sum_{m=-N/2}^{N/2-1} R_{xx}(n,m) e^{-\jmath 2\pi mk/N}. \quad (3)$$

WVD is a simple and efficient method for mono-component signals. However, for multi-component signals, WVD suffers from cross-terms induced by the bilinear data products of (1) [21], [22]. In order to reduce the presence of cross-terms, WVD is often smoothed using various kernels [23]. These kernels exhibit low pass filter characteristics when considered in the ambiguity domain and, as such, mitigate the cross-terms that are typically located distant from the origin. The signal auto-terms pass through or cluster around the origin

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and, therefore, are emphasized by the employed kernel. The resulting reduced interference distribution (RID) is obtained as the 2D Fourier transform of the filtered AF,

$$RID(n,k) = \sum_p \sum_m A(p,m)C(p,m)e^{-\jmath 2\pi np/N}e^{-\jmath 2\pi mk/N}. \tag{4}$$

C(p,m) is a kernel implementing some type of low pass filter. Equation (4) can also be represented in terms of convolution between the applied kernel and the bilinear data products,

$$RID(n,k) = \sum_l \sum_m x(l+m)x^*(l-m)c(n-l,m)e^{-\jmath 2\pi mk/N}. \tag{5}$$

Fast implementations of $RID(n,k)$ avoid the convolution process through spectrogram decomposition [24], [25]. Missing data samples in time map into missing data bilinear products [5]. It is important to note that the convolution process (5) indirectly interpolates the missing samples in the time-lag domain. However, it may be inappropriate to call these kernels true interpolators, since they change the values of the existing samples. This is in contrast with the commonly used interpolators which do not alter the data observations.

III. Interpolation in time and time-lag domain

The problem of estimating the signal $y(n)$ based on the given set of samples $x(n)$ is very common in signal processing. In general, this problem can be formulated in the form,

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{6}$$

where matrix **H** is a linear operator that maps vector **x** into vector **y**. For the considered interpolation problem, **H** contains the coefficients of the interpolation kernel. Alternatively, sparse reconstruction solves the inverse problem and can be used as a method for estimating the unknown samples, following the model $\mathbf{x} = \mathbf{B}\mathbf{y}$. In both cases, vector **y** is unknown; however, the following differences should be noted. In sparse reconstruction, the unknown vector **y** is sparse, while **B** is a dense matrix. Due to the underdetermined system of equations, iterative methods are used to solve for **y** which are clearly more complex than the simple convolution posed by (6). On the other hand, in linear interpolation, **H** is typically sparse and there is no need for iterative procedures. Various interpolation kernels have been designed in order to provide a good approximation of $y(n)$ given $x(n)$, including linear interpolator and nearest neighbor [18], [19]. The simplicity of classical interpolators motivates their use in the underlying TF signature estimation problem.

Interpolation can be regarded in terms of time-invariant and time-varying filtering. Time-invariant filters are associated with uniform downsampling. In this case, the same number of interpolated samples L lie in between any two consecutive observations. As a result, matrix **H** in (6) is Toeplitz. In the case of random undersampling, the number of samples interpolated between two consecutive observations would vary, amounting to time-varying filtering. In this case, matrix **H** is not Toeplitz and L becomes time-dependent, $L(n)$.

In applying interpolators to nonstationary signals, and prior to computing the time-frequency distributions, one has the option of interpolating the raw data or the bilinear data products. Without loss of generality, we use linear interpolator as a representative of classical interpolators. For the case when the sampling rate is increased $L+1$ times and new sampling period is denoted as T, the interpolator impulse and frequency responses are, respectively,

$$h(n) = \begin{cases} 1 - \frac{|n|}{L+1}, & |n| < L+1, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$H(\omega) = \frac{1}{L+1}[\frac{sin(\omega(L+1)T/2)}{sin(\omega T/2)}]^2, \tag{8}$$

which shows that the linear interpolator is a low pass filter. The coefficients of this filter depend on the number of missing samples L. Smaller L implies higher filter cutoff frequency.

A. Interpolation in time domain

When directly operating on the randomly sampled data, the interpolated data in the frequency domain become

$$Y(k) = \sum_p X(p)H(k,p), \tag{9}$$

which follows a time-varying low-pass filtering process. The exclusion of high frequency components due to filtering would depend on $L(n)$. In essence, fewer missing samples cause less truncated frequency spectrum. However, when dealing with undersampling of a critically sampled data, part of the high signal bandwidth will always be lost, leading to signal distortion.

B. Interpolation over the lag variable in time-lag domain

Interpolating the bilinear products over the lag variable is represented by

$$r_{yy_n}(m) = \sum_l h(m,l)r_{xx_n}(l), \tag{10}$$

where $r_{xx_n}(l)$ corresponds to the R_{xx} elements in (1) when observed at time n. This interpolation corresponds to the filtering process in the time-frequency domain,

$$WD_{yy_n}(k) = \sum_p WD_{xx_n}(p)H(k,p). \tag{11}$$

This process is similar to applying a mask in the time-frequency domain, a process commonly used for the separation of signal components. However, separation of nonstationary components is not the purpose of applying an interpolator in our problem. In the time-frequency domain, as in the time domain, linear interpolator causes loss of high frequency components and produces signal distortion.

C. Interpolation over the time variable in time-lag domain

Interpolation in the time-lag domain over the time variable can be written as,

$$r_{yy_m}(n) = \sum_l h(n,l)r_{xx_m}(l), \tag{12}$$

where $r_{xx_m}(l)$ corresponds to the R_{xx} elements in (1) when observed for a certain m. This procedure corresponds to the low-pass filtering in the ambiguity domain,

$$A_{yy_m}(k) = \sum_p A_{xx_m}(p)H(k,p). \tag{13}$$

Interpolation kernels will, as in the case of raw data, remove high frequency components. However, the loss of these components occurs in the ambiguity domain. Since the signal auto-terms are of low-pass characteristics with high power concentration at or near the origin, the interpolation filter is considered less harmful to the Doppler signature than that of the time-domain data.

This analysis sheds new light on the role of interpolators as a cross-terms removal method. However, this role only emerges when the data is randomly undersampled. Also, the difference between linear interpolators and classical time-frequency RID kernels is that the interpolators do not alter the observed samples, while maintaining the low-pass filtering characteristic. Therefore, out of three possible ways to interpolate the data, the one with the filtering effect in the ambiguity domain can be considered the most attractive for Doppler type signals.

IV. Simulations

In order to demonstrate the specific role of linear interpolators in different domains, we examine their performance for various polynomial phase signals. In all cases, WVD is computed using the interpolated data. For comparison, we include sparse reconstruction as well as WVD and Choi-Williams distribution (CWD) [23] which are applied directly to the incomplete data. Results show the advantages of using the linear interpolator in the time-lag domain over time axis. In all plots, frequency axis is normalized.

Example 1: We observe a sinusoidal FM signal where 40% of data samples are randomly missing,

$$x(n) = e^{\jmath 32\sin(2\pi n)}.$$

Fig. 1 (a) depicts a noisy WVD which is the consequence of missing samples in the time domain. The effects of applying the linear interpolator in different domains are shown in Fig. 1(b,c,d). Fig. 1(e) shows sparse reconstruction using Orthogonal Matching Pursuit (OMP) [26] over overlapping windows, similar to the work in [3]. WVD, obtained from the interpolated IAF over time, reveals the desired time-frequency signature, even though some noise is present. We can also observe that sparse reconstruction provides the least cluttered joint-variable representation, but also shows poor energy concentration along the signal instantaneous frequency.

Example 2: A multicomponent signal consisting of two chirps and a sinusoidal FM is considered,

$$x(n) = 1.5e^{\jmath 19.2\sin(2\pi n)+\jmath 64n} + e^{\jmath 8\pi(n-3)^2} + 1.5e^{-\jmath 27.2\pi n^2}.$$

Fig. 2 contains results when applying WVD and CWD to the data with and without missing samples. The strong presence of cross-terms in WVD (Fig. 2 (a)) can be successfully mitigated using the CWD (Fig. 2(b)). WVD and CWD of 65% of the data samples are shown in Fig. 2 (c,d), respectively. These results illustrate how missing samples introduce a significant level of noise in the time-frequency domain. Some of that noise can be removed by using standard time-frequency kernels which are low pass filters (Fig. 2 (d)). The smoothing effect of the kernel operation is visible, but it is inferior to the results obtained by sparse reconstruction or interpolation. Plots in Fig. 3 (a,b,c) pertain to WVD when applying interpolation to estimate the missing 35% of the samples. The benefits of applying the interpolation in the time-lag domain over the time variable are evident. Sparse reconstruction provides less cluttered representation, but with coarse resolution.

Example 3: In this example, we observe a multicomponent signal which has two components,

$$x(n) = e^{\jmath 4.8\pi(n-1)^3+\jmath 32n} + e^{\jmath 4.8\pi(n-1)^3-\jmath 96n}.$$

The phase of both components is a third degree polynomial. Signal is randomly undersampled and 50% of samples are missing. As in the previous example, we show the results when applying standard time-frequency distributions, namely WVD and CWD (Fig. 4). Compared with the distributions using full data (Fig. 4 (a,b)), missing samples cause noise which clutters both time-frequency distributions, as evident in Fig. 4 (c,d). Strong cross-term is visible even in WVD cluttered by noise (Fig. 4 (c)). WVDs obtained from applying the interpolator in different domains are displayed in Fig. 5. These results illustrate, as in the previous cases, the specific filtering role of the interpolator in the ambiguity domain (Fig. 5 (c)). However, cross-term in Fig. 5 (c) is not completely removed by the linear interpolator, indicating the dependence of the filter cutoff frequency on the number of missing samples. Higher number of missing samples will decrease the cutoff frequency of the corresponding filter, but at the cost of estimating more unknown points. Fig. 5 (d) shows the result corresponding to 35% of the data. Whereas less cross-terms are produced due to a narrower filter, the auto-terms are adversely affected due to the same reason.

V. Conclusion

In this paper, we considered nonstationary signals with missing or randomly undersampled data. We examined the role of linear interpolators for data recovery in the context of time-frequency signal representation. Whereas TFDs can be applied directly to the raw non-interpolated data, their performance improves when interpolation is performed prior to time-frequency signature estimation. It was shown that interpolation of the bilinear data products in the time-lag domain outperforms the interpolation of the raw data, owing to the underlying low-pass filtering behavior in the ambiguity domain and the resulting cross-terms reduction. The paper compared linear interpolation with sparse reconstruction, which makes use of the power concentration along the instantaneous frequency. Our simulations have shown that sparse reconstruction over overlapping windows, although cross-terms free, provides coarse signatures compared to interpolators.

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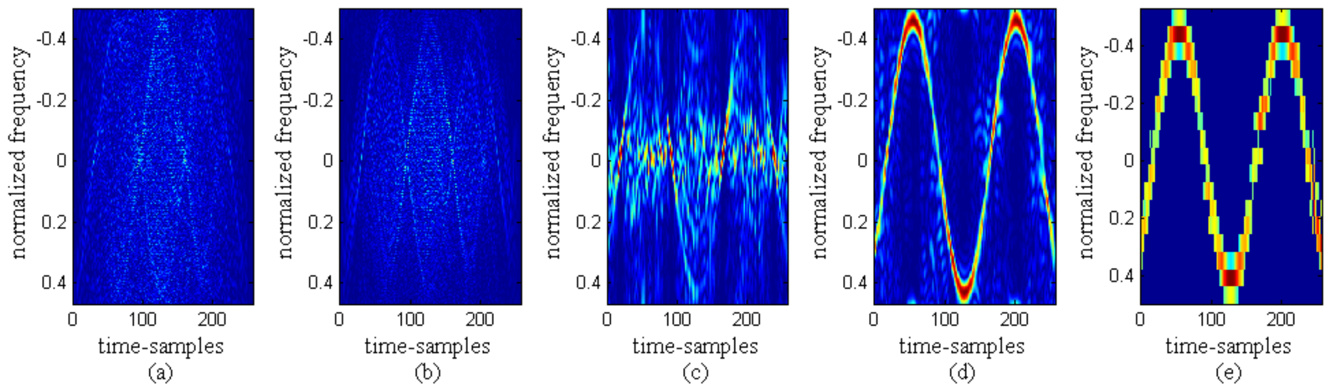

Fig. 1. Time-frequency representations of incomplete sinusoidal FM signal: (a) WVD obtained by using incomplete data; (b) WVD obtained after using linear interpolation in time domain; (c) WVD obtained after using linear interpolation along lag axis in the IAF; (d) WVD obtained after using linear interpolation along time axis in the IAF; (e) Time-frequency representation obtained by using sparse reconstruction over overlapping windows.

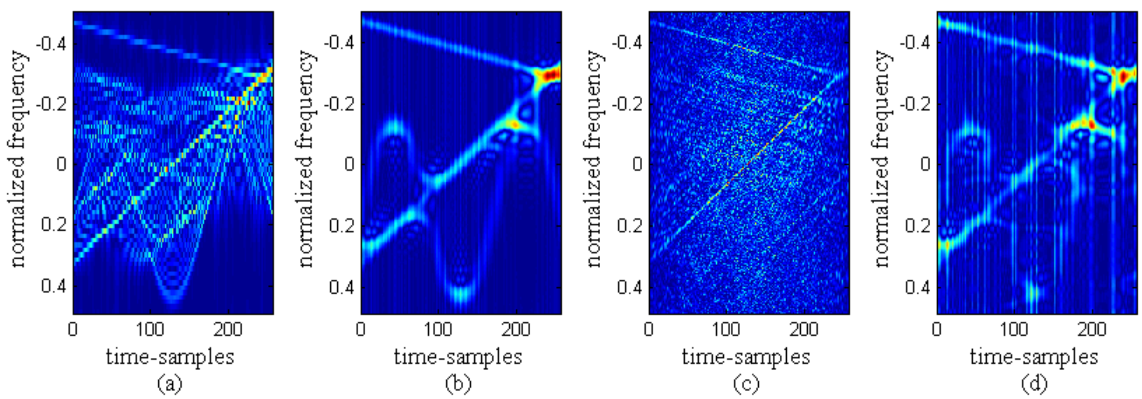

Fig. 2. Time-frequency representations of multicomponent signal: (a) WVD obtained by using full data; (b) Choi-Williams distribution of full data; (c) WVD obtained by using incomplete data; (d) Choi-Williams distribution obtained by using incomplete data.

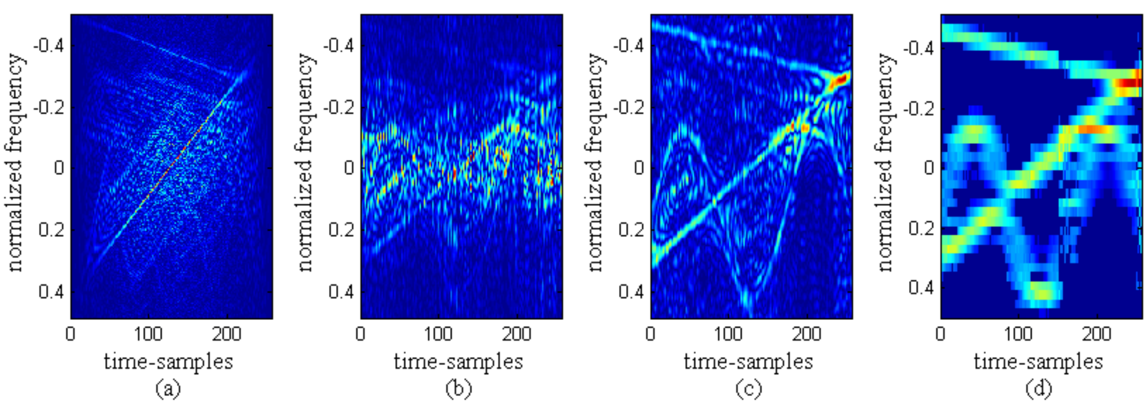

Fig. 3. Time-frequency representations of randomly undersampled multicomponent signal: (a) WVD obtained after using linear interpolation in time domain; (b) WVD obtained after using linear interpolation along lag axis in the IAF; (c) WVD obtained after using linear interpolation along time axis in the IAF; (d) Time-frequency representation obtained by using sparse reconstruction over overlapping windows.

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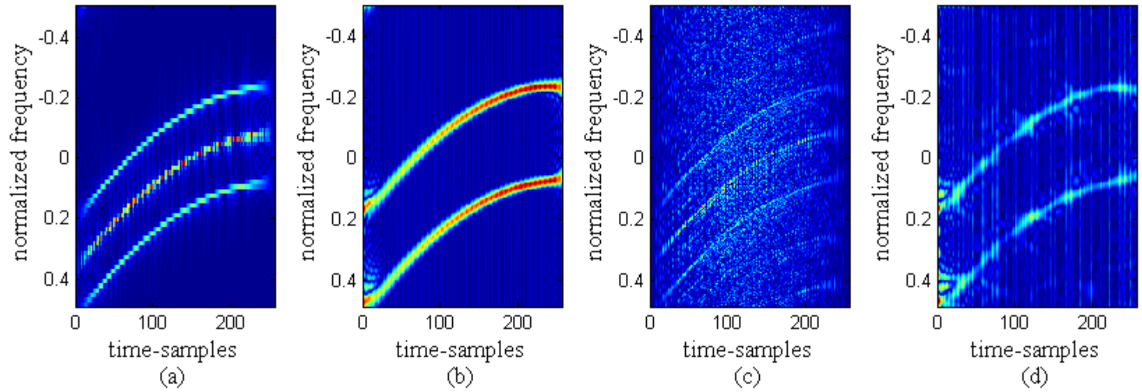

Fig. 4. Time-frequency representations of multicomponent signal: (a) WVD obtained by using full data; (b) Choi-Williams distribution of full data; (c) WVD obtained by using incomplete data; (d) Choi-Williams distribution obtained by using incomplete data.

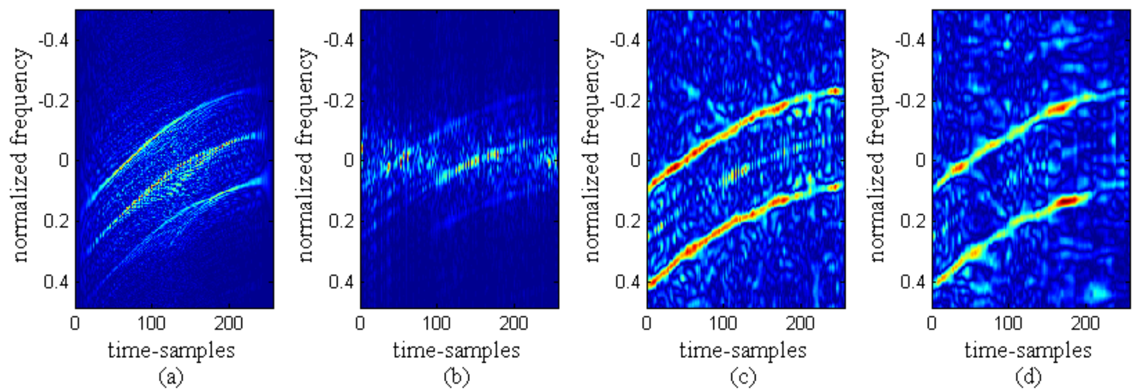

Fig. 5. Time-frequency representations of randomly undersampled multicomponent signal: (a) WVD obtained after using linear interpolation in time domain; (b) WVD obtained after using linear interpolation along lag axis in the IAF; (c) WVD obtained after using linear interpolation along time axis in the IAF; (d) WVD obtained after using linear interpolation along time axis in the IAF - 35% of data samples are present.

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