An Exact Energy for TRM Theory

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ABSTRACT

A time residual mean (TRM) energy is obtained by averaging a transformation of the energy of the Boussinesq hydrostatic incompressible equations of motion. The transformation is the fundamental TRM transformation between level Cartesian coordinates and coordinates that are the mean positions of density surfaces. The TRM energy consists of a sum of mean kinetic, mean potential, wave kinetic, and wave potential energies. It is shown that the interaction between the mean kinetic energy and mean potential energy can be expressed entirely in terms of mean fields. The wave forcing of the mean TRM momentum equations is expressed as a divergence. An explicit and exact form of the TRM equations, with the transformed pressure term expressed in terms of the mean and wave fields, is also noted. It is suggested that the mean domain for the TRM equations and the Cartesian domain may not be the same, which would have consequences for the TRM boundary conditions.

1. Introduction

Most ocean models incorporate some kind of average. Accordingly, a variety of theories have been developed that put the appropriate equations of fluid motion in averaged form. Invariably, nonlinearities couple the resulting equations for the mean fields with correlations of the perturbation (wave) fields so that there is some kind of wave–mean flow interaction. To assess the possible uses of the averaged equations, it is necessary to state precisely the interaction terms as they appear in the mean equations as well as the equations for conserved quantities such as energy. The purpose of this paper is, first, to find an energy equation under time residual mean (TRM) averaging and, second, to clearly formulate the interaction terms. TRM theory was introduced by McDougall and McIntosh (1996, 2001) and generalized by Greatbatch and McDougall (2003).

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TRM theory puts the density equation in a convenient form, a form which contains no correlations of perturbation quantities. Other theories are also capable of doing this: an example is the generalized Lagrangian mean (GLM) theory of Andrews and McIntyre (1978). However, the TRM theory has some other useful and special properties. In particular, the TRM velocity field is nondivergent if the unaveraged velocity field is. It has been suggested by McDougall and McIntosh (2001) that these two properties make the TRM equations a more suitable description of the behavior of z-coordinate non-eddy-resolving climate models than the Eulerian-averaged Cartesian equations.

Although it will not be discussed at length in this paper, a second advantage of TRM theory is that its three-dimensional extra advective velocity can be derived from a streamfunction (the so-called quasi-Stokes streamfunction). This streamfunction satisfies simple boundary conditions, which makes it an attractive tool for eddy parameterization. Some work in this direction has been done by Aiki et al. (2004). The authors recently learned that similar work was undertaken by Killworth (2003, personal communication).

On the other hand, TRM theory has some potential drawbacks. Wave effects on the mean field must appear somewhere. In TRM theory they appear in the mean momentum equations as a Reynolds stress and as additional terms associated with the transformed pressure gradient. These effects may ultimately prove difficult to study—just as they are for direct Eulerian averages—because the TRM average does not incorporate information about horizontal particle motion or momentum transfer (whereas its ability to put the averaged density equation in a special form is partly based on following the vertical motion of density surfaces).

Another concern about TRM appears in section 2, in which it is suggested that the domain of the mean coordinates for the TRM equations and the Cartesian domain may not be the same. If this is so, a likely result would be that the simple boundary conditions previously suggested for the quasi-Stokes streamfunction would be replaced with more complex boundary conditions, perhaps parameterized ones.

In section 2, the fundamentals required for TRM theory are stated, the equations of motion are transformed into a form suitable for averaging, and properties of the TRM average are discussed. The exact TRM equations are found. The connection between the derivation of TRM given here and that given by McDougall and McIntosh (2001) is noted in section 2a. The formulation given here differs from theirs in that explicit use of density coordinates is avoided. However, the necessary tools for the representation of a thickness-weighted isopycnal average in mean z coordinates have been developed by Andrews and McIntyre (1978), De Szoeke and Bennett (1993), McDougall and McIntosh (1996, 2001), Greatbatch (1998), Iwasaki (2001), Greatbatch and McDougall (2003), and others.

In section 3 the TRM energy and mean energy interaction terms are presented. The energy is found from the mean of the transformation of the standard Eulerian energy. Some similar work has been done in an atmospheric context by Iwasaki (2001) and his predecessors. An equation is given for the rate of change of wave potential energy.

2. Fundamentals

The equations for an unforced, rotating, incompressible, hydrostatic, Boussinesq fluid are

$$\gamma_0 u_{t^c} + \gamma_0 \boldsymbol{\nabla}_c \cdot (\mathbf{U}u) - \gamma_0 f v + p_{x^c} = 0, \tag{1}$$

$$\gamma_0 v_{t^c} + \gamma_0 \boldsymbol{\nabla}_c \cdot (\mathbf{U}v) + \gamma_0 f u + p_{y^c} = 0, \tag{2}$$

$$p_{z^c} + g\gamma = 0, \tag{3}$$

$$\boldsymbol{\nabla}_c \cdot \mathbf{U} = 0, \quad \text{and} \tag{4}$$

$$\gamma_{t^c} + \boldsymbol{\nabla}_c \cdot (\mathbf{U}\gamma) = 0. \tag{5}$$

Here (x^c, y^c, z^c, t^c) are rectangular Cartesian coordinates. The dependent variables u, v, w, p, and γ are evaluated at (x^c, y^c, z^c, t^c). Also, $\boldsymbol{\nabla}_c = (\partial_{x^c}, \partial_{y^c}, \partial_{z^c})$ and $\mathbf{U} = (u, v, w)$. We will assume that all boundaries are solid and that $\mathbf{U} \cdot \mathbf{n} = 0$ there. To be consistent with prior work on TRM, γ is the density.

There is an associated energy equation. It is obtained by taking the inner product of the horizontal momentum equations with (u, v) and using the other equations to bring the pressure terms into a conservation form:

$$\left[\frac{\gamma_0}{2}(u^2 + v^2) + g\gamma z^c\right]_{t^c} + \boldsymbol{\nabla}_c \cdot \left\{\mathbf{U}\left[\frac{\gamma_0}{2}(u^2 + v^2) + g\gamma z^c + p\right]\right\} = 0. \tag{6}$$

a. TRM transformation

The exact TRM theory is based on a transformation from mean z coordinates to the Cartesian coordinates. The form of the transformation may be written

$$\begin{aligned}(x^c, y^c, z^c, t^c, \tau^c) &= \mathcal{T}[(x, y, z, t, \tau)] \\ &= [x, y, z + z'(x, y, z, t, \tau), t, \tau],\end{aligned} \tag{7}$$

where $\overline{z'} = 0$, and this average is taken over τ at constant x, y, z, and t. The mean coordinates are given without superscripts. As in the case of transformation to density coordinates, only the height coordinate is transformed nonidentically. Earlier studies have focused on the case that τ represents a shift in time so that the average represents a low-pass time filter. In that case, functional dependence is on the combination $t + \tau$. However, expressing the coordinates separately allows general ensemble averaging as well as low-pass time averaging. It is assumed that, with little or no error, the average commutes with partial differentiation, that $\overline{\phi\overline{\psi}} = \overline{\phi}\,\overline{\psi}$, and that $\overline{\overline{\phi}} - \overline{\phi} = 0$, for any ϕ and ψ.

The Jacobian determinant of the transformation is

$$J = z^c_z = 1 + z'_z. \tag{8}$$

It follows immediately that $\overline{J} = 1 + \overline{z'}_z = 1$, a result which will be useful in deriving the mean TRM equations in section 2d.

The following notation is adopted:

$$u^c(x, y, z, t, \tau) = u(x^c, y^c, z^c, t^c, \tau^c) = u(x, y, z^c, t, \tau),$$

and likewise for other functions of the coordinates. This notation is useful when referring to the flow fields in the mean coordinates.

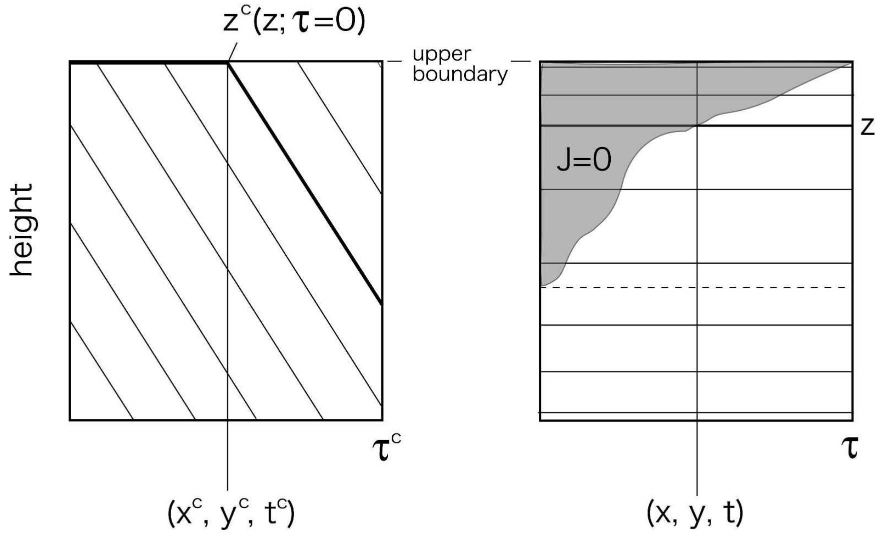

FIG. 1. (left) A height–τ slice of the Cartesian coordinates and (right) a height–τ slice of the mean coordinates. Mapping between the two systems is accomplished by associating with each density surface in Cartesian coordinates (contour lines on left) its mean height in mean coordinates (contours on right). It is possible for a range of mean coordinates in each column to map to the upper surface in Cartesian coordinates, with the result that the Jacobian of the transformation vanishes in those locations. More details are given in section 2a.

Up to this point, the framework of the TRM transformation has been given, but the specific transformation has yet to be defined. We now fix the TRM transformation, making reference to Fig. 1. On the left is shown the Cartesian coordinate system, with the x^c, y^c, and t^c axes suppressed and with their extension in height represented only by the centered vertical line. On the right is shown the mean coordinate system, again with x, y, and t represented by the centered vertical line. The contour lines show constant density surfaces. If a density surface outcrops at the sea surface, it is considered to be continued at the sea surface, as is shown for the thick contour line on the left. Consider the τ average of the Cartesian height of the density surface shown by this thick contour line. This average determines a mean height z, indicated by the heavy horizontal line on the right. The transformation at mean height z is now fixed (for the particular choice of x, y, t that generated this diagram) by relating to z, for each τ, the Cartesian height corresponding to the physical location of the density surface associated with the mean height z. In other words, the thick horizontal line on the right maps to the heights of the thick density surface on the left under the TRM transformation. The result of carrying out this process for several other density contours can be seen by relating the thinner mean-height contours on the right to the heights of the thin density contours on the left. Carrying out the process for every density contour determines a map between every mean coordinate on the right and a Cartesian coordinate on the left, which is the TRM transformation.

Stability of each fluid column will guarantee $J \geq 0$ for the TRM transformation. Stability demands that the density surfaces be monotonically ordered in the Cartesian height coordinate and, hence, so must be the corresponding mean heights.

By the construction shown in the figure, the τ mean of z' is zero, as desired. Moreover, if x, y, z, and t are fixed, the TRM density $\overline{\gamma}(x, y, z, t, \tau) \equiv \gamma^c(x, y, z, t, \tau) = \gamma[x^c, y^c, z^c(x, y, z, t, \tau), t^c, \tau^c]$ is constant. Graphically, the line of constant z and varying τ maps to a single density surface. As a result, $\tilde{\gamma}$ is a mean quantity (it is unchanged by the mean operator).

For readers familiar with GLM theory, it may be helpful to note that the foundations of TRM theory resemble the foundations of GLM theory except that the TRM perturbation position z' is one-dimensional and follows density surfaces. The perturbation ξ in GLM is three-dimensional and follows particles (Andrews and McIntyre 1978).

As the upper boundary is approached on the right-hand side of Fig. 1, the corresponding density surfaces must approach the upper boundary on the left and spend more and more of the averaging period as outcropping surfaces. (If they did not, their mean positions could not approach the upper boundary on the right.) In fact, the mean upper boundary must map entirely onto the Caretsian upper boundary, which establishes the boundary condition $z' = 0$ on the mean upper boundary. This is essentially the same argument as that which was given in Fig. 3 and the surrounding discussion in McDougall and McIntosh (2001).

Indeed, most of the above was detailed by McDougall and McIntosh (2001). The difference between the derivation given there and that given here is that in this paper the equations of motion are considered in the two coordinate systems (x, y, z, t, τ) and $(x^c, y^c, z^c, t^c, \tau^c)$, but not in the system (x, y, γ, t, τ), nor do we express z' approximately in terms of γ and γ'. We feel that avoiding density coordinates makes the exact derivation more straightforward (particularly when it comes to the energy equation) and keeps the focus on the fundamental TRM quantity z'. However, the reader who wishes to explore the connection may note that $z' = -\gamma'/\overline{\gamma}_z + O(\alpha^2)$, where α measures perturbation amplitude; also $J = \tilde{\gamma}_z(1/\gamma_z)|_{z+z'}$ [see McDougall and McIntosh (2001), their (11) and (28)]. Substitution of these relationships above into some of the equations of this paper will reproduce the formulas of McDougall and McIntosh (2001).

A caveat, which does not appear to have been noted in earlier work on TRM, arises in connection with the domain of the transformation. It is clear from considering the process that defines the TRM transformation that there are mean heights for which the Jacobian determinant of the transformation will be zero. This arises because outcropping limits the positions of density surfaces to a maximum height at the top (and, although it is not pictured in the figure, a minimum height at the bottom). For example, at centerline $\tau = 0$ shown in Fig. 1, there is a range of mean heights (all those above z) that map to the Cartesian upper boundary under the TRM transformation. The Jacobian determinant must be zero at these points. In Fig. 1, a possible $J = 0$ region is shaded. Examination of the transformation shows that J goes discontinuously to zero across the edge of this region.

Despite the presence of a $J = 0$ region, it is still possible to transform all unaveraged flow variables to be functions of the mean coordinates over the entire range of mean heights (up to the upper boundary pictured in Fig. 1), although they will be constant in each vertical column in the $J = 0$ region. However, since J is not differentiable everywhere, it may not be legitimate to transform (over the entire range of mean heights) differential equations in which J appears. Since we will shortly be concerned with such equations as part of the process of deriving the TRM equations [e.g., below in (9)–(13)], we suspect that it is necessary when considering the TRM equations to limit the mean domain to exclude the $J = 0$ region. The same issue should occur in density-coordinate-based derivations of TRM [since γ_z is effectively infinite in magnitude at the boundary where outcropping occurs and hence $J = \tilde{\gamma}_z(1/\gamma_z)|_{z+z'} = 0$].

Moreover, the edge of the $J = 0$ region is the edge of the inverse image of the Cartesian upper boundary under the TRM transformation. As a result, it is the location at which the transformed version of the velocity boundary condition $\mathbf{U} \cdot \mathbf{n} = 0$ should be applied. Averaging over τ in the mean domain to produce the TRM equations (see section 2d) would tend to destroy this boundary condition if the discontinuity in J is not taken into account. On the other hand, if the discontinuity in J is accounted for, averages over derivatives of J (delta functions) will pick up information at the location of the discontinuity, which will require some kind of boundary tracking or parameterized boundary tracking. This leads us to suspect that the mean domain should be limited to the levels below the dotted line in the right-hand panel when considering the TRM equations. Unfortunately, doing so would remove the natural boundary conditions along with information about the near-boundary flow field.

In this paper, we will generally point out the simple consequences of the possibility that the TRM equations can be applied over a mean domain that is the same as the Cartesian domain and that $z' = 0$ on the boundary of that mean domain. It is less clear what should be done about boundary conditions if the mean domain is to be limited. We note that such a limitation would have serious consequences for aspects of TRM theory not considered elsewhere in this paper. In particular, it would probably not be correct to assert that the quasi-Stokes streamfunction (which is an averaged variable) vanishes at the *limited* mean boundaries. Also, if limited, the mean domain would in most cases be time dependent.

b. Transformed equations

Consider (1)–(5). If one wishes to express these equations in the coordinates (x, y, z, t), the mean positions associated with (x^c, y^c, z^c, t^c) at any given instant, the equations must be transformed. (We may ignore τ for the moment, as the equations have no explicit dependence on the averaging parameter.) This is analogous to the common transformation to density coordinates. Expression of the partial derivatives ∂_{x^c}, and so on, in terms of ∂_x, and so on, is required. From the differential of the TRM transformation (7), these are found to be

$$\partial_{t^c} = \partial_t - \frac{\partial z^c/\partial t}{J}\partial_z,\ \partial_{x^c} = \partial_x - \frac{\partial z^c/\partial x}{J}\partial_z,$$

$$\partial_{y^c} = \partial_y - \frac{\partial z^c/\partial y}{J}\partial_z,$$

along with $\partial_{z^c} = (1/J)\partial_z$. The equations of motion in mean coordinates are obtained by substituting these differential relationships in all terms of the original equations and manipulating:

$$\gamma_0(Ju^c)_t + \gamma_0\nabla\cdot(J\mathbf{U}^a u^c) - \gamma_0 fJv^c - (z^c_x p^c)_z + (Jp^c)_x = 0, \tag{9}$$

$$\gamma_0(Jv^c)_t + \gamma_0\nabla\cdot(J\mathbf{U}^a v^c) + \gamma_0 fJu^c - (z^c_y p^c)_z + (Jp^c)_y = 0, \tag{10}$$

$$(p^c)_z + gJ\tilde{\gamma} = 0, \tag{11}$$

$$J_t + \nabla\cdot(J\mathbf{U}^a) = 0, \tag{12}$$

and

$$(J\tilde{\gamma})_t + \nabla\cdot(J\mathbf{U}^a\tilde{\gamma}) = 0. \tag{13}$$

Here, $\nabla = (\partial_x, \partial_y, \partial_z)$ and $\mathbf{U}^a = (u^c, v^c, w^*)$. The superscript a and the asterisk are chosen to be consistent with previous papers on TRM. In the course of carrying out the manipulations, one finds that w^* is defined in terms of w^c by

$$\frac{D^a}{Dt}z^c \equiv (\partial_t + u^c\partial_x + v^c\partial_y + w^*\partial_z)z^c = w^c, \tag{14}$$

where the first equivalence defines in turn the operator D^a/Dt. Of course, if the equations are to be solved without reference to a known solution in Cartesian coordinates, then w^* may simply be treated as one of the dependent variables. In either case, one of J or $z^c(x, y, z, t)$ must be known. In section 2d, averaging over τ will remove the requirement that J be known. One may note that (14) is a one-dimensional analog of the vector equation in (2.6a) of Andrews and McIntyre (1978):

$$(\partial_t + \mathbf{v}\cdot\nabla)\Xi = \mathbf{u}^\xi.$$

That equation was used to define $\overline{\mathbf{u}}^L$ $(=\mathbf{v})$, the Lagrangian mean velocity.

If the mean domain is the same as the Cartesian domain and if $z' = 0$ on the boundary, the boundary conditions on velocity are easily found. Since $z^c = z$ on the boundary in that case, (14) simplifies to $w^* = w^c$. As a result, the transformed boundary condition $\mathbf{U}^c\cdot\mathbf{n} = 0$ implies $\mathbf{U}^a\cdot\mathbf{n} = 0$.

Equations (9)–(13) have been written without any forcing for simplicity. However, forcing is easily handled. Let the forcing X be added to the right-hand side of one of (1)–(5). The forcing would appear in the corresponding equations of (9)–(13) as JX^c, and could be carried though the subsequent calculations without difficulty.

For the energy, let $E^c \equiv \gamma_0[(u^c)^2 + (v^c)^2]/2 + g\tilde{\gamma}z^c$. Then the transformation of (6) is

$$(JE^c)_t + \nabla\cdot[J\mathbf{U}^a(E^c + p^c)] + (z^c_t p^c)_z = 0. \tag{15}$$

In the case that the mean domain is the same as the Cartesian domain and $z' = 0$ on the boundary of the mean domain, one finds that $z^c_t = z_t + 0_t = 0$ on the mean domain boundary. This guarantees the conservation of energy on such a fixed domain.

c. TRM averaging and TRM quantities

Almost every term in (9)–(13) is multiplied by J. This suggests thickness-weighted (J weighted) averaging. Let the TRM mean quantities be defined by

$$\mathbf{U}^a = \overline{J\mathbf{U}^a} + \mathbf{U}'' = \hat{\mathbf{U}} + \mathbf{U}'', \tag{16}$$

$$\gamma^c = \overline{J\gamma^c} + \gamma'' = \tilde{\gamma} + 0, \quad \text{and} \tag{17}$$

$$(p^c)_z = \overline{(p^c)_z} + p'_z = \hat{p}_z + p'_z = g\tilde{\gamma} + gz'_z\tilde{\gamma}. \tag{18}$$

Here, $\hat{\mathbf{U}} = (\hat{u}, \hat{v}, \hat{w})$ and the carets, tildes, and double primes are used for consistency with earlier works. A property of the averaging for the perturbation velocities is

$$\overline{J\mathbf{U}''} = 0. \tag{19}$$

As a result of (19) one has, for example,

$$\overline{Ju^c u^c} = \overline{\hat{u}^2 + 2J\hat{u}u'' + Ju''u''} = \hat{u}^2 + \overline{Ju''u''}. \tag{20}$$

This kind of identity is useful when considering the Reynolds stresses in the momentum equations and the energies in the energy equation.

The exact fields can be reconstructed from the TRM mean and perturbation quantities and a knowledge of z', which is a perturbation field.

d. Mean TRM equations

With (16) and (19) in mind, it is easy to obtain the mean equations: simply take averages of the transformed (9)–(13). Doing this and manipulating yields

$$\gamma_0\hat{u}_t + \gamma_0\nabla\cdot(\hat{\mathbf{U}}\hat{u}) - \gamma_0 f\hat{v} + \hat{p}_x = -\nabla\cdot[\overline{\gamma_0 J\mathbf{U}''u''} + (-\overline{z'p'_z}, 0, \overline{z'p'_x})], \tag{21}$$

$$\gamma_0\hat{v}_t + \gamma_0\nabla\cdot(\hat{\mathbf{U}}\hat{v}) + \gamma_0 f\hat{u} + \hat{p}_y = -\nabla\cdot[\overline{\gamma_0 J\mathbf{U}''v''} + (0, -\overline{z'p'_z}, \overline{z'p'_y})], \tag{22}$$

$$\hat{p}_z + g\tilde{\gamma} = 0, \tag{23}$$

$$\nabla\cdot\hat{\mathbf{U}} = 0, \quad \text{and} \tag{24}$$

$$\tilde{\gamma}_t + \nabla\cdot(\hat{\mathbf{U}}\tilde{\gamma}) = 0. \tag{25}$$

The wave interaction terms are in divergence form. In the case that the mean and Cartesian domains are the same and $z' = 0$ on the mean boundary, the wave terms do not contribute to the time rate of change of the domain integral of the TRM mean momenta. Also, by applying the operator $(-\partial_y, \partial_x)$ to the momentum equations, it is seen that the only pressure wave term (i.e., excluding the Reynolds stresses) that can affect the "TRM vorticity" $\hat{\upsilon}_x - \hat{u}_y$ is

$$\frac{\partial}{\partial z}(\overline{p'_y z'_x - p'_x z'_y}).$$

3. TRM energy

The energy is found by averaging the transformed energy [see (15)]:

$$\overline{JE^c} = \frac{\gamma_0}{2}(\hat{u}^2 + \hat{\upsilon}^2) + \frac{\gamma_0}{2}(\overline{Ju''u'' + J\upsilon''\upsilon''}) + g\tilde{\gamma}z + g\overline{z'z'_z}\tilde{\gamma}. \tag{26}$$

This shows that it is possible to define kinetic and potential energies associated with the mean field, as well as wave kinetic and wave potential energy. Since $J \geq 0$, the wave kinetic energy is nonnegative. Except for a difference of boundary terms, the same can be shown for the *domain-integrated* wave potential energy by an integration by parts, provided that $\tilde{\gamma}_z < 0$ (stable stratification). If z' is zero on the mean boundary, the difference of boundary terms drops out of the domain-integrated wave potential energy.

a. Mean energy interactions

It is straightforward to show that the energy interaction between mean field potential energy and mean field kinetic energy can be expressed in terms of the mean fields. The argument is the same as in the familiar Eulerian case:

$$\begin{aligned}(g\tilde{\gamma}z)_t &= -gz\nabla\cdot(\hat{\mathbf{U}}\tilde{\gamma})\\ &= -\nabla\cdot(gz\hat{\mathbf{U}}\tilde{\gamma}) + g\tilde{\gamma}\hat{w}\\ &= -\nabla\cdot(gz\hat{\mathbf{U}}\tilde{\gamma}) - \hat{p}_z\hat{w}\\ &= -\nabla\cdot(gz\hat{\mathbf{U}}\tilde{\gamma}) - (\hat{p}\hat{w})_z + \hat{p}\hat{w}_z\\ &= -\nabla\cdot(gz\hat{\mathbf{U}}\tilde{\gamma}) - (\hat{p}\hat{w})_z - \hat{p}(\hat{u}_x + \hat{\upsilon}_y)\\ &= -\nabla\cdot[\hat{\mathbf{U}}(gz\tilde{\gamma} + \hat{p})] + \hat{u}\hat{p}_x + \hat{\upsilon}\hat{p}_y.\end{aligned}$$

The last term of the last equation on the right-hand side is precisely the energy transfer term that arises from multiplying the mean pressure term in TRM horizontal momentum equations by $(\hat{u}, \hat{\upsilon})$. The remaining transfers of mean kinetic energy must arise from multiplication of the wave terms in the TRM horizontal momentum equations by $(\hat{u}, \hat{\upsilon})$ and must represent transfers to wave kinetic and potential energy.

b. Rate of change of wave potential energy

The rate of change of the wave energies can be found from the definition (26) and manipulation of the TRM equations in (21)–(25) and transformed equations in (9)–(13). An interesting case is the rate of change of the wave potential energy, which is analogous to the density variance equation in McDougall and McIntosh (2001). One has

$$(g\tilde{\gamma}\overline{z'z'_z})_t = (g\tilde{\gamma}\overline{Jz'})_t = -\overline{gz'\nabla\cdot(JU^a\tilde{\gamma})} + g\tilde{\gamma}\overline{Jz'_t}.$$

Using (14), the definition of w^*, one may put this equation in the form

$$(g\tilde{\gamma}\overline{z'z'_z})_t + \nabla\cdot(g\tilde{\gamma}\overline{J\mathbf{U}^a z'}) = g\tilde{\gamma}\overline{J(w^c - w^*)}, \tag{27}$$

which shows that the wave potential energy is forced by the weighted mean difference of w^c and w^*.

4. Summary

We have defined a TRM energy, which is obtained as an average of a transformation of the Eulerian energy of the Boussinesq hydrostatic incompressible equations of motion. The energy consists of a sum of mean kinetic, mean potential, wave kinetic, and wave potential energy. It has been shown that interactions between the mean field energies take place entirely in terms of mean field quantities.

At present, the main uses of a TRM energy appear to be as a diagnostic quantity and in the design of mesoscale parameterizations based on an energy principle, perhaps one analogous to the principle of mean potential energy decrease suggested by Gent et al. (1995).

A by-product of the derivation of the TRM energy was an explicit and exact representation of the TRM equations in terms of the mean fields and wave fields. In particular, the transformed pressure term has been expressed in terms of correlations of a perturbation pressure and the density surface height perturbation z'. Notwithstanding their frequent use of $O(\alpha)$ notation, the exactness of the TRM equations was mentioned by McDougall and McIntosh (2001) and further demonstrated in Greatbatch and McDougall (2003). However, the explicit representation in this paper serves to emphasize the point. In particular, error-free diagnosis of TRM quantities in eddy-resolving models or observations can only be done on the basis of the exact definitions and equations.

Last, we have suggested that, for the TRM equations, the mean domain and the Cartesian domain may not be the same. If this is the case, modification of the TRM boundary conditions may be necessary.

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