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Research Article

Numerical Range of Two Operators in Semi-Inner Product Spaces

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In this paper, the numerical range for two operators (both linear and nonlinear) have been studied in semi-inner product spaces. The inclusion relations between numerical range, approximate point spectrum, compression spectrum, eigenspectrum, and spectrum have been established for two linear operators. We also show the inclusion relation between approximate point spectrum and closure of the numerical range for two nonlinear operators. An approximation method for solving the operator equation involving two nonlinear operators is also established.

1. Introduction

Lumer [1] introduced the concept of semi-inner product. He defined semi-inner product as follows.

Semi-Inner Product

Let X be a vector space over the field F of real or complex numbers. A functional $[\ ] : X\times X \to F$ is called a semi-inner product if it satisfies the following conditions:

(i) $[x + y, z] = [x, z] + [y, z]$, for all $x, y, z \in X$;
(ii) $[\lambda x, y] = \lambda[x, y]$, for all $\lambda \in F$ and $x, y \in X$;
(iii) $[x, x] > 0$, for $x \neq 0$;
(iv) $|[x, y]|^2 \leq [x, x][y, y]$, for all $x, y \in X$.

The pair $(X, [\ ,\ ])$ is called a semi-inner product space.

A semi-inner product space is a normed linear space with the norm $\|x\| = [x, x]^{1/2}$. Every normed linear space can be made into a semi-inner product space in infinitely many different ways. Giles [2] had shown that if the underlying space is a uniformly convex smooth Banach space, then it is possible to define a semi-inner product uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x, \lambda y] = \overline{\lambda}[x, y]$, for all scalars λ;

(ii) $[x, y] = 0$ if and only if y is orthogonal to x, that is, if and only if $\|y\| \leq \|y + \lambda x\|$, for all scalars λ;

(iii) generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y]$, for all $x \in X$;

(iv) the semi-inner product is continuous.

The sequence space $l^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces uniquely. Giles [2] had shown that the functions space $L^p, p > 1$ are semi-inner product spaces with the semi-inner product defined by

$$[f, g] := \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1} \operatorname{sgn}(g(x)) d\mu, \quad \forall f, g \in L^P(X, \mu). \tag{1.1}$$

To study the generalized eigenvalue problem $Tx = \lambda Ax$, Amelin [3] introduced the concept of numerical range for two linear operators in a Hilbert space. His purpose was to obtain some new results on the stability of index of a Fredholm operator perturbed by a bounded operator. Zarantonello [4] had introduced the concept of numerical range for a nonlinear operator in a Hilbert space. He proved that the numerical range contains the spectrum. He used this concept to solve the nonlinear functional equations. A great deal of literature on numerical range in unital normed algebras, numerical radius theorems, spatial numerical ranges, algebra numerical ranges, essential numerical ranges, joint numerical ranges, and matrix ranges are available in Bonsall [5, 6]. For recent work on numerical range, one may refer Chien and Nakazato [7, 8], Chien at al. [9], Gustafson and Rao [10], and Li and Tam [11].

In [1], Lumer discussed the numerical range for a linear operator in a Banach space. Williams [12] studied the spectra of products of two linear operators and their numerical ranges. The numerical range of two nonlinear operators in a semi-inner product space was defined by Nanda [13]. For two nonlinear operators T and A, he defined the numerical range $W_{nl}(T, A)$, as

$$W_{nl}(T, A) := \left\{ \frac{[Tx - Ty, Ax - Ay]}{\|Ax - Ay\|^2} : x \neq y, x, y \in D(T) \cap D(A) \right\}, \tag{1.2}$$

where $D(T)$ and $D(A)$ denote the domains of the operators T and A, respectively. The numerical radius $w_{nl}(T, A)$ is defined as $w_{nl}(T, A) = \sup\{|\lambda| : \lambda \in W_{nl}(T, A)\}$. $W_{nl}(T, A)$ may not be convex. If T and A are continuous and $D(T), D(A)$ are connected, then $W_{nl}(T, A)$ is connected. The numerical range of a nonlinear operator using the generalized Lipschitz norm was studied by Verma [14]. He defined the numerical range $V_L(T)$ of a nonlinear operator T, as

$$V_L(T) := \left\{ \frac{[Tx, x] + [Tx - Ty, x - y]}{\|x\|^2 + \|x - y\|^2} : x, y \in D(T), x \neq y \right\}. \tag{1.3}$$

He used this concept to solve the operator equation $Tx - \lambda x = y$, where T is a nonlinear operator.

This paper is concerned with the numerical range in a Banach space. Nanda [15] studied the numerical range for two linear operators and the coupled numerical range in a Hilbert space which was initially introduced by Amelin [3]. He also introduced the concepts of spectrum, point spectrum, approximated point spectrum, and compression spectrum for two linear operators. In Section 2, we generalize the results of Nanda [15] to semi-inner product space. Verma [14] introduced the numerical range of a nonlinear operator in a Banach space using the generalized Lipschitz norm. In Section 3, we generalize the numerical range of Verma [14] for two nonlinear operators using the generalized Lipschitz norm. We also give examples of operators in semi-inner product spaces and compute their numerical range and numerical radius.

2. Numerical Range of Two Linear Operators

Let T and A be two linear operators on a uniformly convex smooth Banach space X. To study the properties of the numerical range, coupled numerical range for the two operators T and A, and to discuss the results of the classical spectral theory associated with the numerical range, we need the following definitions in the sequel.

Numerical Range $W(T, A)$

The numerical range $W(T, A)$ of the two linear operators T and A is defined as $W(T, A) := \{[Tx, Ax] : \|Ax\| = 1, x \in D(T) \cap D(A)\}$, where $D(T)$ and $D(A)$ are denoted as the domain of T and the domain of A, respectively. The numerical radius $w(T, A)$ is defined as $w(T, A) = \sup\{|\lambda| : \lambda \in W(T, A)\}$.

Spectrum $\sigma(T, A)$

The spectrum $\sigma(T, A)$ of the two linear operators T and A is defined as

$$\sigma(T, A) := \{\lambda \in \mathbb{C} : (T - \lambda A) \text{ is not invertible}\}. \tag{2.1}$$

The spectral radius $r(T, A)$ is defined as $r(T, A) = \sup\{|\lambda| : \lambda \in \sigma(T, A)\}$.

Eigenspectrum $e(T, A)$

The eigenspectrum or point spectrum $e(T, A)$ of two linear operators T and A is defined as

$$e(T, A) := \{\lambda \in \mathbb{C} : Tx = \lambda Ax, x \neq 0\}. \tag{2.2}$$

Approximate Point Spectrum $\pi(T, A)$

The approximate point spectrum $\pi(T, A)$ of two linear operators T and A is defined as $\pi(T, A) := \{\lambda \in \mathbb{C}$ such that there exists a sequence x_n in X with $\|Ax_n\| = 1$ and $\|Tx_n - \lambda Ax_n\| \to 0$ as $n \to \infty\}$.

Compression Spectrum $\sigma_0(T, A)$

The compression spectrum $\sigma_0(T, A)$ of two linear operators T and A is defined as

$$\sigma_0(T, A) := \left\{\lambda \in \mathbb{C} : \text{Range } (T - \lambda A) \text{ is not dense in } X\right\}. \tag{2.3}$$

Coupled Numerical Range $W_A(T)$

The coupled numerical range $W_A(T)$ of T with respect to A is defined as

$$W_A(T) := \left\{\frac{[ATx, x]}{[Ax, x]} : \|x\| = 1, [Ax, x] \neq 0\right\}. \tag{2.4}$$

We can easily prove the following properties of the numerical range of two linear operators.

Theorem 2.1. *Let T_1, T_2, T, and A be linear operators and α, μ, and λ be scalars. Then*

(i) $W(T_1 + T_2, A) \subseteq W(T_1, A) + W(T_2, A)$;
(ii) $W(\alpha T, A) = \alpha W(T, A)$;
(iii) $W(T, \mu A) = \overline{\mu} W(T, A)$;
(iv) $W(T - \lambda A, A) = W(T, A) - \{\lambda\}$;
(v) $w(T_1 + T_2, A) \leq w(T_1, A) + w(T_2, A)$;
(vi) $w(\lambda T, A) = |\lambda| w(T, A)$.

Theorem 2.2. *For the coupled numerical range we have the following properties:*

(i) $W_A(T_1 + T_2) \subseteq W_A(T_1) + W_A(T_2)$;
(ii) $W_A(\alpha T) = \alpha W_A(T)$;
(iii) $W_{\alpha A}(T) = W_A(T)$.

We establish the following theorems which generalize the classical spectral theory results.

Theorem 2.3. *The approximate point spectrum $\pi(T, A)$ is contained in the closure of the numerical range $W(T, A)$.*

Proof. Let $\lambda \in \pi(T, A)$. Then there exists a sequence x_n in X such that $[Ax_n, Ax_n] = 1$ and $\|(T - \lambda A)x_n\| \to 0$ as $n \to \infty$.

Now

$$|[Tx_n, Ax_n] - \lambda| = |[(T - \lambda A)x_n, Ax_n]| \leq \|(T - \lambda A)x_n\| \|Ax_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

This implies that $[Tx_n, Ax_n] \to \lambda$ as $n \to \infty$.

Hence, $\lambda \in \overline{W(T, A)}$, and consequently $\pi(T, A) \subset \overline{W(T, A)}$. □

Theorem 2.4. *Eigenspectrum $e(T, A)$ is contained in the spectrum $\sigma(T, A)$.*

Proof. Let $\lambda \in e(T, A)$. Then there exists $x_0 \neq 0$ such that $(T - \lambda A)x_0 = 0$. Thus $(T - \lambda A)^{-1}$ does not exist otherwise $(T - \lambda A)^{-1}(T - \lambda A)x_0 = (T - \lambda A)^{-1}0 = 0$.

That is $Ix_0 = x_0 = 0$, which is a contradiction to the fact that $x_0 \neq 0$. Hence $\lambda \in \sigma(T, A)$ and consequently, $e(T, A) \subset \sigma(T, A)$. □

In the following theorems, we assume that the linear operator A is invertible.

Theorem 2.5. *Compression spectrum $\sigma_0(T, A)$ is contained in the numerical range $W(T, A)$.*

Proof. Let $\lambda \in \sigma_0(T, A)$, then range$(T - \lambda A)$ is not dense in X.

So we can find a y in X with $\|Ay\| = 1$ such that Ay is orthogonal to the range of $(T - \lambda A)$.

This implies $0 = [(T - \lambda A)y, Ay] = [Ty, Ay] - \lambda$. So $\lambda = [Ty, Ay] \in W(T, A)$, and consequently $\sigma_0(T, A) \subset W(T, A)$. □

The generalized Riesz representation theorem asserts that one can define semi-inner product using bounded linear functionals. In the following theorem, we denote that $(x, \phi(y)) = [x, y]$ for all $x, y \in$ X and $\phi \in$ X*.

Theorem 2.6. *Spectrum $\sigma(T, A)$ is contained in the closure of the numerical range $W(T, A)$.*

Proof. Let $\lambda \in \sigma(T, A)$. To show that $\lambda \in \overline{W(T, A)}$.

Suppose that $\lambda \notin \overline{W(T, A)}$, then $d(\lambda, \overline{W(T, A)}) = \delta > 0$.

For $\|Ax\| = 1$, we have

$$\|(T - \lambda A)x\| \geq |[(T - \lambda A)x, Ax]| = |[Tx, Ax] - \lambda| \geq \delta > 0. \tag{2.6}$$

Hence $(T - \lambda A)$ is one-to-one with a closed range. Again for $\phi \in X^*$, X^* being the dual space of X, we have

$$\left\|A^{-1}\right\| \|(T - \lambda A)^* \phi(Ax)\| \geq \left|(x, (T - \lambda A)^* \phi(Ax))\right| = |[(T - \lambda A)x, Ax]| \geq \delta. \tag{2.7}$$

Hence $(T - \lambda A)^*$ is bounded below on the range of ϕ and since this is dense in X^*, $(T - \lambda A)^*$ is bounded below, and it is one-to-one. This implies that $(T - \lambda A)$ has a dense range. By open mapping theorem $(T - \lambda A)$ has a bounded inverse, which is a contradiction to the fact that $\lambda \in \sigma(T, A)$.

Therefore, $\lambda \in \overline{W(T, A)}$, and consequently $\sigma(T, A) \subset \overline{W(T, A)}$. □

Remark 2.7. Theorem 2.6 is a generalization of a known result for Hilbert space operators to Banach space operators. Here, T and A are bounded linear operators on a Banach space X. If A is invertible, then the spectrum $\sigma(T, A)$ coincides with the classical spectrum $\sigma(TA^{-1})$ of TA^{-1}. The numerical range $W(T, A)$ coincides with the classical numerical range $W(TA^{-1})$ of TA^{-1}. So the assertion of Theorem 2.6 can also be deduced from a classical result on Banach space.

Theorem 2.8. *Let T and A be two linear operators on a semi-inner product space X, so that $w(T,A) < 1$. If A is invertible, then $A-T$ is invertible, and $\|A(A-T)^{-1}\| \le 1/(1-w(T,A))$.*

Proof. We have $r(T,A) \le w(T,A) < 1$. For $\|Ax\| = 1$, we have

$$\begin{aligned}\|(A-T)x\| &= \left\|A\left(I-A^{-1}T\right)x\right\| \\ &= \left\|A\left(I-A^{-1}T\right)x\right\|\|Ax\| \\ &\ge \left|\left[A\left(I-A^{-1}T\right)x, Ax\right]\right| \\ &\ge [Ax,Ax] - |[Tx,Ax]| \\ &\ge 1-w(T,A) > 0.\end{aligned} \tag{2.8}$$

This implies that $(A-T)$ is invertible in its range.

Again $\|(A-T)x\| \ge (1-w(T,A))\|Ax\|$. Setting $x = A^{-1}(I-A^{-1}T)^{-1}y$ with $\|y\| = 1$, we get

$$\begin{aligned}1 &\ge (1-w(T,A))\left\|A(A-T)^{-1}y\right\| \\ &\Rightarrow \left\|A(A-T)^{-1}y\right\| \le (1-w(T,A))^{-1}\|y\| \\ &\Rightarrow \left\|A(A-T)^{-1}\right\| \le \frac{1}{1-w(T,A)}.\end{aligned} \tag{2.9}$$

□

3. Numerical Range of Two Nonlinear Operators

Let $\mathrm{Lip}(X)$ denote the set of all Lipschitz operators on X. Suppose that $T \in \mathrm{Lip}(X)$, and $x, y \in \mathrm{Dom}(T)$ with $x \ne y$. The generalized Lipschitz norm $\|T\|_L$ of a nonlinear operator T on a Banach space X is defined as $\|T\|_L = \|T\| + \|T\|_l$, where $\|T\| = \sup_x \|Tx\|/\|x\|$ and $\|T\|_l = \sup_{x\ne y}\|Tx-Ty\|/\|x-y\|$. If there exists a finite constant M such that $\|T\|_L < M$, then the operator T is called the generalized Lipschitz operator (Verma [14]). Let $G_L(X)$ be the class of all generalized Lipschitz operators.

Now we define the concepts of resolvent set, spectrum, eigenspectrum, and point spectrum for a nonlinear operator with respect to another nonlinear operator, which generalize the concepts of the classical spectral theory.

A-Resolvent Set

A-resolvent set $\rho_A(T)$ of a nonlinear operator T with respect to another operator A is defined as

$$\rho_A(T) := \left\{\lambda \in \mathbb{C} : (T-\lambda A)^{-1} \text{ exists and is generalized Lipschitzian}\right\}. \tag{3.1}$$

A-Spectrum of T

A-spectrum of T, $\sigma_A(T)$ is the complement of the A-resolvent set of T.

Numerical Range of Two Nonlinear Operators

The numerical range $V_L(T,A)$ of two nonlinear operators T and A is defined as

$$V_L(T,A) := \left\{ \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} : x, y \in D(T) \cap D(A),\ x \neq y \right\}, \tag{3.2}$$

where $D(T)$ and $D(A)$ are the domains of the operators T and A, respectively. The numerical radius $w_L(T,A)$ is defined as $w_L(T,A) = \{\sup|\lambda| : \lambda \in V_L(T,A)\}$.

We give examples of two nonlinear operators in a semi-inner product space and compute their numerical range and numerical radius.

Example 3.1. Consider the real sequence space $l^p, p > 1$. Let $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in l^p$. Consider the two nonlinear operators $T, A : l^p \to l^p$ defined by $Tx = (\|x\|, x_1, x_2, \ldots)$ and $Ax = (\|x\|, 0, 0, \ldots)$. The semi-inner product on the real sequence space l^p is defined as $[x, y] = (1/\|y\|^{p-2}) \sum_{n=1}^{\infty} |y_n|^{p-2} y_n x_n, \forall x = \{x_n\}, y = \{y_n\} \in l^p$. One can easily compute that $\|Ax\| = \|x\|$, $\|Ax - Ay\| = |\|x\| - \|y\||$, $[Tx, Ax] = \|x\|^2$ and $[Tx - Ty, Ax - Ay] = (1/\|Ax - Ay\|^{p-2})\{|(\|x\| - \|y\|)|^{p-2}(\|x\| - \|y\|)^2\} = 1/|(\|x\| - \|y\|)|^{p-2}|(\|x\| - \|y\|)|^p = |(\|x\| - \|y\|)|^2$.

We can calculate

$$\frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} = \frac{\|x\|^2 + |(\|x\| - \|y\|)|^2}{\|x\|^2 + |(\|x\| - \|y\|)|^2} = 1, \quad \forall x, y \in l^p. \tag{3.3}$$

Therefore, $V_L(T,A) = \{1\}$, and $w_L(T,A) = 1$.

Example 3.2. Consider the real sequence space $l^p, p > 1$. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in l^p$. Consider the two nonlinear operators $T, A : l^p \to l^p$ defined by $Tx = (\|x\|, x_1, x_2, \ldots)$ and $Ax = (\alpha, x_1, x_2, \ldots)$, where $\alpha \in \mathbb{R}$ is any constant.

One can easily compute $\|Ax - Ay\|^p = \sum_{n=1}^{\infty} |x_n - y_n|^p$ and

$$[Tx - Ty, Ax - Ay] = \frac{1}{\|Ax - Ay\|^{p-2}} \sum_{n=1}^{\infty} |x_n - y_n|^p = \frac{\|Ax - Ay\|^p}{\|Ax - Ay\|^{p-2}} = \|Ax - Ay\|^2. \tag{3.4}$$

For any $x, y \in l^p$, we have

$$\frac{[Tx - Ty, Ax - Ay]}{\|Ax - Ay\|^2} = \frac{\|Ax - Ay\|^2}{\|Ax - Ay\|^2} = 1. \tag{3.5}$$

Therefore, the numerical range of two nonlinear operators T and A in the sense of Nanda [13] is $W_{nl}(T,A) = \{1\}$ and the numerical radius $w_{nl}(T,A) = 1$.

We have the following elementary properties for the numerical range of two nonlinear operators.

Theorem 3.3. *Let X be a Banach space over $\mathbb{C}$. If T, A, T_1, and T_2 are nonlinear operators defined on X, and λ and μ are scalars, then*

(i) $V_L(\lambda T, A) = \lambda V_L(T, A)$;

(ii) $V_L(T, \mu A) = (1/\mu) V_L(T, A)$;

(iii) $V_L(T_1 + T_2, A) \subseteq V_L(T_1, A) + V_L(T_2, A)$;

(iv) $V_L(T - \lambda A, A) = V_L(T, A) - \{\lambda\}$.

Proof. To prove (i):

$$\frac{[\lambda Tx, Ax] + [\lambda Tx - \lambda Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} = \lambda \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2}. \tag{3.6}$$

Hence, $V_L(\lambda T, A) = \lambda V_L(T, A)$.

To show (ii):

$$\begin{aligned} \frac{[Tx, \mu Ax] + [Tx - Ty, \mu Ax - \mu Ay]}{\|\mu Ax\|^2 + \|\mu Ax - \mu Ay\|^2} &= \frac{\overline{\mu}[Tx, Ax] + \overline{\mu}[Tx - Ty, Ax - Ay]}{|\mu|^2\left(\|Ax\|^2 + \|Ax - Ay\|^2\right)} \\ &= \frac{\overline{\mu}}{|\mu|^2} \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\ &= \frac{1}{\mu} \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2}. \end{aligned} \tag{3.7}$$

Hence $V_L(T, \mu A) = (1/\mu) V_L(T, A)$.

Let $x, y \in D(T_1) \cap D(T_2)$.

Then,

$$\begin{aligned} &\frac{[(T_1 + T_2)x, Ax] + [(T_1 + T_2)x - (T_1 + T_2)y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\ &\quad = \frac{[T_1x, Ax] + [T_2x, Ax] + [T_1x - T_1y, Ax - Ay] + [T_2x - T_2y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\ &\quad = \frac{[T_1x, Ax] + [T_1x - T_1y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} + \frac{[T_2x, Ax] + [T_2x - T_2y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2}. \end{aligned} \tag{3.8}$$

Therefore, $V_L(T_1 + T_2, A) \subseteq V_L(T_1, A) + V_L(T_2, A)$. Thus, (iii) is proved.

Finally, to prove (iv):

$$\begin{aligned} &\frac{[(T-\lambda A)x, Ax] + [(T-\lambda A)x - (T-\lambda A)y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\ &\quad = \frac{[Tx, Ax] - \lambda\|Ax\|^2 + [Tx - Ty, Ax - Ay] - \lambda\|Ax - Ay\|^2}{\|Ax\|^2 + \|Ax - Ay\|^2} \\ &\quad = \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} - \lambda. \end{aligned} \tag{3.9}$$

This implies that $V_L(T - \lambda A, A) = V_L(T, A) - \{\lambda\}$. □

Approximate Point Spectrum of Two Nonlinear Operators

Approximate point spectrum $\pi(T, A)$ of two nonlinear operators T and A is defined as $\pi(T, A) := \{\lambda \in \mathbb{C}$: there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\|Ax_n\| = 1, \|Ax_n - Ay_n\| = 1$, $\|(T - \lambda A)x_n\| \to 0$ and $\|(T - \lambda A)x_n - (T - \lambda A)y_n\| \to 0$ as $n \to \infty\}$.

Strongly Generalized A-Monotone Operator

A nonlinear operator $T : D(T) \subset X \to X$ is called strongly generalized A-monotone operator if there is a constant $C > 0$, such that $|[Tx, Ax] + [Tx - Ty, Ax - Ay]| \geq C(\|Ax\|^2 + \|Ax - Ay\|^2)$.

Theorem 3.4. *Approximate point spectrum $\pi(T, A)$ of two nonlinear operators T and A is contained in the closure of their numerical range $V_L(T, A)$.*

Proof. Let $\lambda \in \pi(T, A)$. Now

$$\begin{aligned} &\left| \frac{[Tx_n, Ax_n] + [Tx_n - Ty_n, Ax_n - Ay_n]}{\|Ax_n\|^2 + \|Ax_n - Ay_n\|^2} - \lambda \right| \\ &\quad = \frac{|[(T-\lambda A)x_n, Ax_n] + [(T-\lambda A)x_n - (T-\lambda A)y_n, Ax_n - Ay_n]|}{\|Ax_n\|^2 + \|Ax_n - Ay_n\|^2} \\ &\quad \leq \frac{\|(T-\lambda A)x_n\|\|Ax_n\| + \|(T-\lambda A)x_n - (T-\lambda A)y_n\|\|Ax_n - Ay_n\|}{\|Ax_n\|^2 + \|Ax_n - Ay_n\|^2}. \end{aligned} \tag{3.10}$$

The right-hand side goes to 0 as $n \to \infty$. This implies that $\lambda \in \overline{V_L(T, A)}$, and hence $\pi(T, A) \subset \overline{V_L(T, A)}$. □

Theorem 3.5. *Let μ be a complex number. Then μ is at a distance $d > 0$ from $\overline{V_L(T, A)}$ if and only if $(T - \mu A)$ is strongly generalized A-monotone.*

Proof. We have

$$0 < d \leq \left| \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} - \mu \right| = \frac{\left|[(T - \mu A)x, Ax] + [(T - \mu A)x - (T - \mu A)y, Ax - Ay]\right|}{\|Ax\|^2 + \|Ax - Ay\|^2}. \tag{3.11}$$

This implies that

$$\left|[(T - \mu A)x, Ax] + [(T - \mu A)x - (T - \mu A)y, Ax - Ay]\right| \geq d\left(\|Ax\|^2 + \|Ax - Ay\|^2\right). \tag{3.12}$$

Hence, $(T - \mu A)$ is a strongly generalized A-monotone operator. The converse part follows easily and hence omitted. □

The following theorem is an approximation method for solving an operator equation involving two nonlinear operators.

Theorem 3.6. *Let X be a complex Banach space, $T \in G_L(X)$, and $\|T\|_L < 1$. Also let A be another generalized Lipschitzian and invertible operator on X. If T is A-Lipschitz with constant $K \neq 1$, then $A - T$ is invertible in $G_L(X)$ and $\|(A-T)^{-1}\| \leq (\|A^{-1}\|_L(2 - \|A^{-1}T\|_L))/(1 - \|A^{-1}T\|)(1 - \|A^{-1}T\|_l)$. Again if $B_0 = I$ and $B_n = I + (A^{-1}T)B_{n-1}$ for $n = 1, 2, 3, \ldots$, and $\|A^{-1}T\|_l < 1$, then $\lim_{n\to\infty} B_n x = (I - A^{-1}T)^{-1}x$ for every $x \in X$, as $n \to \infty$ and $\|(I - A^{-1}T)^{-1}x - B_n x\| \leq \|A^{-1}T\|_l^n \|A^{-1}Tx\|(1 - \|A^{-1}T\|_l)^{-1}$, for $x \in X$, $n = 0, 1, 2, \ldots$.*

Proof. For each $x, y \in X$ with $Ax \neq Ay$, we have

$$\begin{aligned} \|(A - T)x - (A - T)y\| &\geq \|Ax - Ay\| - \|Tx - Ty\| \\ &\geq \|Ax - Ay\| - K\|Ax - Ay\| \\ &= (1 - K)\|Ax - Ay\| > 0, \end{aligned} \tag{3.13}$$

since $K \neq 1$, and $Ax \neq Ay$. This implies that $(A - T)$ is injective.

Next if $u, v \in R(A - T)$, then

$$\begin{aligned} \left\|(A - T)^{-1}\right\| &= \left\|A^{-1}\left(I - A^{-1}T\right)^{-1}\right\| \\ &\leq \left\|A^{-1}\right\|\left\|\left(I - A^{-1}T\right)^{-1}\right\| \leq \left\|A^{-1}\right\|\left(1 - \left\|A^{-1}T\right\|\right)^{-1}. \end{aligned} \tag{3.14}$$

Similarly, we have $\|(A - T)^{-1}\|_l \leq \|A^{-1}\|_l (1 - \|A^{-1}T\|_l)^{-1}$.

Now

$$\begin{aligned} \left\|(A - T)^{-1}\right\| + \left\|(A - T)^{-1}\right\|_l &\leq \frac{\left\|A^{-1}\right\|}{1 - \left\|A^{-1}T\right\|} + \frac{\left\|A^{-1}\right\|_l}{1 - \left\|A^{-1}T\right\|_l} \\ &= \frac{\left\|A^{-1}\right\|(1 - \left\|A^{-1}T\right\|_l) + \left\|A^{-1}\right\|(1 - \left\|A^{-1}T\right\|_l)}{(1 - \left\|A^{-1}T\right\|)(1 - \left\|A^{-1}T\right\|_l)} \end{aligned}$$

$$\begin{aligned}
&\le \frac{\|A^{-1}\|_L(1-\|A^{-1}T\|_l)+\|A^{-1}\|_L(1-\|A^{-1}T\|_l)}{(1-\|A^{-1}T\|)(1-\|A^{-1}T\|_l)} \\
&= \frac{2\|A^{-1}\|_L-\|A^{-1}\|_L\|A^{-1}T\|_l-\|A^{-1}\|_L\|A^{-1}T\|}{(1-\|A^{-1}T\|)(1-\|A^{-1}T\|_l)} \\
&= \frac{2\|A^{-1}\|_L-\|A^{-1}\|_L\|A^{-1}T\|_L}{(1-\|A^{-1}T\|)(1-\|A^{-1}T\|_l)}, \\
\Longrightarrow \left\|(A-T)^{-1}\right\|_L &\le \frac{\|A^{-1}\|_L(2-\|A^{-1}T\|_L)}{(1-\|A^{-1}T\|)(1-\|A^{-1}T\|_l)}.
\end{aligned} \tag{3.15}$$

To prove the second part, consider the sequence of approximating operators

$$B_0 = I, \quad B_n = I + \left(A^{-1}T\right)B_{n-1}, \quad \text{for } n = 1,2,3,\dots. \tag{3.16}$$

We claim that

$$\|B_{n+1}x - B_nx\| \le \left\|A^{-1}T\right\|_l^n \left\|A^{-1}Tx\right\|. \tag{3.17}$$

For $n = 0$, $\|B_1x - B_0x\| = \|(I + A^{-1}T)Ix - Ix\| = \|A^{-1}Tx\|$.
For $n = 1$, $\|B_2x - B_1x\| = \|(I + (A^{-1}T)B_1)x - (I + (A^{-1}T)B_0)x\| \le \|A^{-1}T\|_l^1\|A^{-1}Tx\|$.
Assume that (3.17) is true for $n = k-1$, that is $\|B_kx - B_{k-1}x\| \le \|A^{-1}T\|_l^{k-1}\|A^{-1}Tx\|$.
Now for $n = k$,

$$\begin{aligned}
\|B_{k+1}x - B_kx\| &= \left\|\left(I + \left(A^{-1}T\right)B_k\right)x - \left(I + \left(A^{-1}T\right)B_{k-1}\right)x\right\| \\
&= \left\|\left(A^{-1}T\right)B_kx - \left(A^{-1}T\right)B_{k-1}x\right\| \\
&\le \left\|A^{-1}T\right\|_l \|B_kx - B_{k-1}x\| \\
&\le \left\|A^{-1}T\right\|_l^k \left\|A^{-1}Tx\right\|.
\end{aligned} \tag{3.18}$$

Now for a positive integer p,

$$\begin{aligned}
\left\|B_{n+p}x - B_nx\right\| &= \left\|\sum_{k=0}^{p-1} B_{n+k+1}x - B_{n+k}x\right\| \\
&\le \sum_{k=0}^{p-1} \|B_{n+k+1}x - B_{n+k}x\| \\
&\le \sum_{k=0}^{p-1} \left\|A^{-1}T\right\|_l^{n+k} \left\|A^{-1}Tx\right\|
\end{aligned}$$

$$= \left\|A^{-1}T\right\|_l^n \left\|A^{-1}Tx\right\| \left\{1 + \left\|A^{-1}T\right\|_l + \left\|A^{-1}T\right\|_l^2 + \cdots + \left\|A^{-1}T\right\|_l^{p-1}\right\}$$
$$\leq \left\|A^{-1}T\right\|_l^n \left\|A^{-1}Tx\right\| \left(1 - \left\|A^{-1}T\right\|_l\right)^{-1}. \tag{3.19}$$

Since $\|A^{-1}T\|_l < 1$, the sequence $\{B_n x\}$ is a Cauchy sequence. Again since X is complete, we have $\lim_{m\to\infty} B_m(x) = Ex$ exists for all $x \in X$.

For $m = n + p$,

$$\|Ex - B_n x\| = \lim_{p\to\infty} \left\|B_{n+p}x - B_n x\right\| \leq \left\|A^{-1}T\right\|_l^n \left\|A^{-1}Tx\right\| \left(1 - \left\|A^{-1}T\right\|_l\right)^{-1}. \tag{3.20}$$

As $A^{-1}T$ is continuous, we have

$$Ex = \lim_{n\to\infty} B_n(x) = \lim_{n\to\infty}\left(I + \left(A^{-1}T\right)B_{n-1}\right)x = \left(I + \left(A^{-1}T\right)E\right)x$$
$$\Longrightarrow Ex - \left(A^{-1}T\right)Ex = Ix \tag{3.21}$$
$$\Longrightarrow E = \left(I - A^{-1}T\right)^{-1}.$$

Now multiplying A^{-1} we get $(A - T)^{-1} = A^{-1}E$. □

Using this technique, one can solve an operator equation involving two nonlinear operators under the condition that one of the operator is invertible.

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