COLLOCATION METHOD FOR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

NEHZAT EBRAHIMI[a1] AND JALIL RASHIDINIA[b]

[ab]Department of Mathematics, Islamic Azad University, Central Tehran Branch, Iran

ABSTRACT

This paper introduces an approach for obtaining the numerical solution of the linear and nonlinear Volterra-Fredholm integro-differential equations based on quintic B-spline functions.The solution is collocated by quintic B-spline and then the integrand is approximated by 5-points Gauss-Tur´an quadrature formula with respect to the Legendre weight function.The main characteristic of this approach is that it reduces linear and nonlinear Volterra -Fredholm integro-differential equations to a system of algebraic equations, which greatly simplifying the problem. The error analysis of proposed numerical method is studied theoretically. Numerical examples illustrate the validity and applicability of the proposed method.

KEYWORDS: Volterra-Fredholm, integro-differential equations, quintic B-spline, Gauss-Tur´an quadrature formula, Error analysis

AMS Subject Classifications 41A15, 65R20

Consider the nonlinear Volterra-Fredholm integro-differential equation of the form

$$\sum_{r=0}^{m} p_r(t)\, y^{(r)}(t) = g(t) + \int_a^t k_1\big(t, x, y(x)\big)dx + \int_a^b k_2\big(t, x, y(x)\big)dx\,, 1 \le m \le 4, t \in [a, b], \quad (1)$$

with the boundary conditions,

$$\sum_{r=0}^{m-1} \left[\alpha_{i.r} y^{(r)}(a) + \beta_{i,r} y^{(r)}(b)\right] = \gamma_i\,, \qquad 0 \le i \le m-1, \qquad (2)$$

where $\alpha_{i.r}, \beta_{i,r}$ and γ_i are given real constants. The given kernels k_1, k_2 are continuous on [a, b] and satisfie a uniform Lipschitz, and $g(t)$ and $p_r(t)$ are the known functions and y is unknown function. The boundary value problems in terms of integro-differential equations have many practical applications. A physical event can be modelled by the differential equation, an integral equation or an integro-differential equation or a system of these equations. Some of the phenomena in physics, electronics, biology, and other applied sciences lead to nonlinear Volterra- Fredholm integro-differential equations. Of course, these equations can also appear when transforming a differential equation into an integral equation [1-7].

In general, nonlinear Volterra-Fredholm integro-differential equations do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered are almost impossible to solve by this technique. Due to this, some authors have proposed numerical methods to approximate the solutions of nonlinear Fredholm-Volterra integro-differential equations. To mention a few, in [8] the authors have discussed the Taylor polynomial method for solving integro-differential equations (1).The triangular functions method has been applied to solve the same equations in [9].Furthermore,

The operational matrix with block-pulse functions method is carried out in [10] for the aforementioned integro-differential equations. The Hybrid Legendre polynomials and Block- Pulse functions approach for solving integro-differential equations (1) are proposed in [11]. Yalcinbas in [12] developed the Taylor polynomial solutions for the nonlinear Volterra-Fredholm integral equations and in [13] considered the high-order linear Volterra-Fredholm integro-differential equations. The numerical solvability of Fredholm and Volterra integro-differential equations and other related equations can be found in [14-28].Using a global approximation to the solution of Fredholm and Volterra integral equation of

---

[1]**Corresponding author**

the second kind is constructed by means of the spline quadrature in [32-37].

In this paper we will develop a collocation method based on quintic B-spline to approximate the unknown function in equation (1) then the 5-points Gauss-Tur´an quadrature formula with respect to the Legendre weight function is used to approximate the nonlinear Volterra-Fredholm integro-differential equations of second kind.

The organization of the paper is as follows. In Section 2, we describe the basic formulation of the quintic B-spline collocation method then the Gauss-Tur´an quadrature formula is discussed in Section 3.Section 4

devoted to the solution of Eq. (1) by using collocation method for Volterra-Fredholm integro-differential equation. In section 5, description of new approaches for the error analysis of the schemes is given. Finally numerical examples are given in section 6 to illustrate the efficiency of the presented method.

Quintic B-spline

We introduce the quintic B-spline space and basis functions to construct an interpolation s to be used in the formulation of the quintic B-spline collocation method. Let $\pi: \{a = t_0 < t_1 < \cdots < t_N = b\}$, be a uniform partition of the interval $[a, b]$ with step size $h = \frac{b-a}{N}$. The quintic spline space is denoted by

$$S_5(\pi) = \{s \in C^4[a, b]; s|[t_i, t_{i+1}] \in P_5 \quad , \quad i = 0, 1, \ldots, N\},$$

where P_5 is the class of quintic polynomials. The construction of the quintic B-spline interpolate s to the analytical solution y for (1) can be performed with the help of the ten additional knots such that

$$t_{-5} < t_{-4} < t_{-3} < t_{-2} < t_{-1} \text{ and } t_{N+1} < t_{N+2} < t_{N+3} < t_{N+4} < t_{N+5}.$$

Following [33] we can define a quintic B-spline s(t) of the form

$$s(t) = \sum_{i=-2}^{N+2} c_i B_i^5(t), \tag{3}$$

where

$$B_i(t) = \frac{1}{120h^5} \begin{cases} (t - t_{i-3})^5 & , t \in [t_{i-3}, t_{i-2}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 & , t \in [t_{i-2}, t_{i-1}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 & , t \in [t_{i-1}, t_i] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5 & , t \in [t_i, t_{i+1}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5 + 15(t - t_{i+1})^5 & , t \in [t_{i+1}, t_{i+2}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5 + 15(t - t_{i+1})^5 - 6(t - t_{i+2})^5 & , t \in [t_{i+2}, t_{i+3}] \\ 0 & , \text{ otherwise,} \end{cases}$$

Satisfying the following interpolator conditions:

$$s(t_i) = y(t_i), \qquad 0 \leq i \leq N,$$

and the end conditions

$$(i) s^j(t_0) = y^j(t_0), \qquad s^j(t_N) = y^j(t_N), \quad j = 1,2,$$

$$\text{or}$$

$$(ii) D^j s(t_0) = D^j s(t_N), \quad j = 1,2,3,4, \tag{4}$$

$$\text{or}$$

$$(iii) s^j(t_0) = 0 \quad , s^j(t_N) = 0, \qquad j = 3,4.$$

On quadrature formulae of Gauss–Tura´n

Let P_m be the set of all algebraic polynomials of degree at most m. The Gauss-Tur´an quadrature formula in [29,31] is

$$\int_a^b f(x)d\lambda(x) = \sum_{i=0}^{2s}\sum_{v=1}^{n} A_{i,v}f^{(i)}(\tau_v) + R_{n,2s}(f), \qquad (5)$$

where $n \in N, s \in N_0$ and $d\lambda(x)$ is a nonnegative measure on the interval (a, b) which can be the real axis R, with compact or infinite support for which all moments:

$$\mu_k = \int_a^b x^k d\lambda(x) \quad, \qquad k = 0,1,\dots$$

exist and are finite , moreover $\mu_0 > 0, and\ A_{i,v} = \int_a^b l_{v,i}(x)d\lambda(x),\ (i = 0,\dots,2s, v = 1,\dots,n)$ and $l_{v,i}(x)$ are the fundamental polynomials of Hermite interpolation. The nodes $\tau_v(v = 1,\dots,n)$ in (5) are the zeros of polynomial $\pi_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ which minimizes the integral

$$F(a_0, a_1, \dots, a_{n-1}) = \int_a^b [\pi_n(x)]^{2s+2}d\lambda(x) \quad, \qquad (6)$$

then the formula (5) is exact for all polynomials of degree at most $2(s + 1)n - 1$, that is, $R_{n,2s}(f) = 0, \forall f \in P_{2(s+1)n-1}$. The condition (6) is equivalent with the following conditions:

$$\int_a^b [\pi_n(x)]^{2s+1}x^k d\lambda(x) = 0 \quad, \qquad (k = 0,\dots,n-1)$$

let $\pi_n^s(x)$ denoted by $P_{n,s}$ and $d\lambda(x) = w(x)$ on $[a, b]$. In this article, we use 5-points Gauss–Tura´n quadrature formula with respect to the weight function Legendre $w(x) = 1$ and $[-1, 1]$ with $n = 5, s = 3$, therefore we can approximate integral as

$$\int_{-1}^1 f(x)d(x) = \sum_{i=0}^{6}\sum_{v=1}^{5} A_{i,v}f^{(i)}(\tau_v) + R_{5,6}(f) \quad, \qquad (7)$$

$$\int_{-1}^1 [\pi_5(x)]^7 x^k d(x) = 0 \quad, \qquad (k = 0,1,2,3,4)\,, \qquad (8)$$

where $\pi_5(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$ by solving system (8) we can obtain $a_i\ (i = 0, 1, 2, 3, 4)$ coefficients, on the other hand we have

$$\pi_{v+1}(x) = (x - \alpha_v)\pi_v(x) - \beta_v\pi_{v-1}(x) \,, \qquad v = 0,1,2,3,4,$$

$$\pi_{-1}(x) = 0\,, \qquad \pi_0(x) = 1\,,$$

Where

$$\alpha_v = \alpha_v(3,5) = \frac{(x\pi_v, \pi_v)}{(\pi_v, \pi_v)} = \frac{\int_{-1}^1 x\pi_v^2(x)\pi_n^{2s}dx}{\int_{-1}^1 \pi_v^2(x)\pi_n^{2s}dx} = \frac{\int_{-1}^1 x\pi_v^2(x)\pi_5^6dx}{\int_{-1}^1 \pi_v^2(x)\pi_5^6dx}\,,$$

$$\beta_v = \beta_v(3,5) = \frac{(\pi_v, \pi_v)}{(\pi_{v-1}, \pi_{v-1})} = \frac{\int_{-1}^1 \pi_v^2(x)\pi_n^{2s}dx}{\int_{-1}^1 \pi_{v-1}^2(x)\pi_n^{2s}dx} = \frac{\int_{-1}^1 \pi_v^2(x)\pi_5^6dx}{\int_{-1}^1 \pi_{v-1}^2(x)\pi_5^6dx}\,,$$

$$\beta_0 = \int_{-1}^1 \pi_5^6dx,$$

so that we can obtain the zeros of polynomial $\pi_5^{3,5}(x)$ of eigenvalue Jacobian matrix

$$J_5 = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & 0 & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & 0 & 0 \\ 0 & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & 0 \\ 0 & 0 & \sqrt{\beta_3} & \alpha_3 & \sqrt{\beta_4} \\ 0 & 0 & 0 & \sqrt{\beta_4} & \alpha_4 \end{bmatrix},$$

and the values of $\tau_\upsilon, \alpha_\upsilon, \beta_\upsilon$ which are tabulated in Table 1.

Table 1: Determined values of $\tau_\upsilon, \alpha_\upsilon, \beta_\upsilon$

ν	τ_υ	α_υ	β_υ	
0	-0.938106999475975	- 0.204189952	3.83357763e(-7)	
1	-0.573307914524401	-0.0013538804	0.2697108295	
2	-2.00573118159208e(-16)	0.00105893788	0.3046524435	
3	0.573307914524401	0.2173728849	0.2789709710	
4	0.938106999475975	-0.4744354850	0.2554389765	
Note: -2.00573118159208e(-16)= -2.00573118159208× 10^{-16}				

Finally to determine $A_{i,\nu}$, we use the following polynomial for approximation of function f(x)

$$f_{k,\upsilon}(x) = (x - \tau_\nu)^k \Omega_\nu(x) = (x - \tau_\upsilon)^k \prod_{i \neq \upsilon} (x - \tau_i)^{2s+1}, \qquad (9)$$

where $0 \leq k \leq 2s, \ 1 \leq \nu \leq n$ and

$$\Omega_\nu(x) = \left(\frac{\pi_\nu(x)}{x - \tau_\nu}\right)^{2s+1} = \prod_{i \neq \upsilon} (x - \tau_i)^{2s+1} \qquad , \qquad \nu = 1, \dots, n,$$

since (5) is exact for all polynomials of degree at most $2(s + 1)n − 1$ then accuracy degree $f_{k,\upsilon}$ is

$$\deg f_{k,\upsilon} = (n − 1)(2s + 1) + k \leq (2s + 1)n − 1.$$

Then (5) is exact for polynomials (9), that is, $R(f_{k,\nu}) = 0, (0 \leq k \leq 2s, 1 \leq \nu \leq n)$ then by replacing $f_{k,\upsilon}(x)$ instead of f(x) in (5) we have

$$\sum_{i=0}^{2s} \sum_{j=1}^{n} A_{i,j} f_{k,\nu}^{(i)}(\tau_j) = \int_a^b f_{k,\upsilon}(x) d\lambda x = \mu_{k,\nu} \ , \qquad (10)$$

therefore for each $\nu = j$, we get the linear system $(2s + 1) \times (2s + 1)$, where $A_{i,\nu}$ are unknowns $i = 0, \dots, 2s, \upsilon = 1, \dots, n$.

The approximate solution of nonlinear Volterra- Fredholm integro-differential equation

In the given nonlinear Volterra- Fredholm integro-differential Eq. (1), we can approximate the unknown function by quintic B-spline (3), then we obtain:

$$\sum_{r=0}^{m} p_r(t) s^{(r)}(t) = g(t) + \int_a^t k_1(t, x, s(x)) dx + \int_a^b k_2(t, x, s(x)) dx , 1 \leq m \leq 4, t \in [a, b], (11)$$

with the boundary conditions,

$$\sum_{r=0}^{m-1} \left[\alpha_{i,r} s^r(a) + \beta_{i,r} s^r(b) \right] = \gamma_i , \qquad 0 \le i \le m-1.$$

We now collocate Eq. (11) at collocation points $t_j = a + jh, h = \frac{b-a}{N}$, $j = 0,1,\dots,N$, and we obtain

$$\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \int_a^{t_j} k_1\left(t_j, x, s(x)\right) dx + \int_a^b k_2\left(t_j, x, s(x)\right) dx , 1 \le m \le 4, j = 1,\dots,N . (12)$$

By partitioning the interval [a, b] to N equal subintervals we obtain

$$\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \sum_{p=0}^{j-1} \int_{t_p}^{t_{p+1}} k_1\left(t_j, x, s(x)\right) dx + \sum_{p=0}^{N-1} \int_{t_p}^{t_{p+1}} k_2\left(t_j, x, s(x)\right) dx , j = 1,\dots,N . (13)$$

For using the Gauss–Tura´n formula we need to change each subinterval $[t_p, t_{p+1}]$ to the interval $[-1, 1]$. Then by the following change of variable, we have

$$x = \frac{1}{2}\left[(t_{p+1} - t_p)u + (t_{p+1} + t_p) \right], \qquad dx = \frac{t_{p+1} - t_p}{2} du = \frac{h}{2} du ,$$

and then

$$\int_{t_p}^{t_{p+1}} k\left(t, x, s(x)\right) dx = \frac{h}{2} \int_{-1}^{1} k\left(t, \frac{1}{2}[(t_{p+1} - t_p)u + (t_{p+1} + t_p)], s\left(\frac{1}{2}[(t_{p+1} - t_p)u + (t_{p+1} + t_p)]\right)\right) du.$$

To approximate the integral Equation (13), we can use the 5-points Gauss–Tura´n quadrature formula in the case $n = 5$ and $s = 3$, then we get the following nonlinear system:

$$\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \frac{h}{2}\sum_{p=0}^{j-1}\sum_{v=1}^{5}\sum_{l=0}^{6} A_{l,v} k_1^{(l)}\left(t_j, \xi_{pv}, s(\xi_{pv})\right) + \frac{h}{2}\sum_{p=0}^{N-1}\sum_{v=1}^{5}\sum_{l=0}^{6} A_{l,v} k_2^{(l)}\left(t_j, \xi_{pv}, s(\xi_{pv})\right) \qquad , j$$
$$= 1,\dots,N, \quad (14)$$

where $\xi_{pv} = \frac{1}{2}[(t_{p+1} - t_p)\tau_v + (t_{p+1} + t_p)]$ and we have the nodes τ_v and coefficients $A_{l,v}$ of previous section.

We need four more equations to obtain the unique solution for Equation (14), we impose the end conditions (4). Hence by associating equation (14) with (4) we have the following nonlinear system $(N + 5) \times (N + 5)$:

$$\begin{cases} \sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \frac{h}{2}\sum_{p=0}^{j-1}\sum_{v=1}^{5}\sum_{l=0}^{6} A_{l,v} k_1^{(l)}\left(t_j, \xi_{pv}, s(\xi_{pv})\right) + \frac{h}{2}\sum_{p=0}^{N-1}\sum_{v=1}^{5}\sum_{l=0}^{6} A_{l,v} k_2^{(l)}\left(t_j, \xi_{pv}, s(\xi_{pv})\right), \\ \sum_{r=0}^{m-1}\left[\alpha_{i,r} s^{(r)}(a) + \beta_{i,r} s^{(r)}(b)\right] = \gamma_i , \qquad 0 \le i \le m-1, j = 1,\dots,N, \qquad (15) \\ s^{(r+1)}(t_0) = s^{(r+1)}(t_N), \quad 0 \le r \le 4 - m. \end{cases}$$

By solving the above nonlinear system ,we determine the coefficients $c_i , i = -2,\dots,N + 2$,by setting c_i in (3), we obtain the approximate solution for Equation (11).

Error Analysis

In this section, we consider the error analysis of the Volterra- Fredholm integro-differential equation of the second kind .To obtain the error estimation of our approximation, first we recall the following definitions in [29-31,33].

Definition 5.1 The Gauss-Tur´an quadrature formula with multiple nodes,

$$\int_a^b f(x)d\lambda(x) = \sum_{i=0}^{2s}\sum_{\upsilon=1}^{n} A_{i,\upsilon}f^{(i)}(\tau_\upsilon) + R_{n,2s}(f),$$

is exact for all polynomials of degree at most $2(s+1)n-1$, that is, $R_{n,2s}(f) = 0 \; \forall f \in P_{2(s+1)n-1}$.

Definition 5.2 The most immediate error analysis for spline approximates S to a given function f defined on an interval $[a, b]$ follows from the second integral relations.

If $f \in C_6[a, b]$, then $\left\| D^j(f-S) \right\| \le \gamma h^{6-j}, j = 0, \dots, 6$. Where $\|f\|_\infty = \max_{0 \le i \le N} \sup_{t_{i-1} \le t \le t_i} |f(t)|$,

and D^j the j-th derivative.

Theorem 5.1   The approximate method

$$\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \frac{h}{2}\sum_{p=0}^{j-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}k_1^{(l)}\left(t_j, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right) + \frac{h}{2}\sum_{p=0}^{N-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}k_2^{(l)}\left(t_j, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right),$$

$$\sum_{r=0}^{m-1}\left[\alpha_{i,r}s^{(r)}(a) + \beta_{i,r}s^{(r)}(b)\right] = \gamma_i, \qquad\qquad 0 \le i \le m-1, j = 1, \dots, N, \qquad\qquad (16)$$

for solution of the Volterra- Fredholm integro-differential Eq .(11) is converge and the error bounded is

$$\left| e_j^{(m)} \right| \le \frac{hL}{2|p_{mj}|}\sum_{p=0}^{j-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6}|A_{l,\upsilon}||e_j^{(m)}| + \frac{hL^*}{2|p_{mj}|}\sum_{p=0}^{N-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6}|A_{l,\upsilon}||e_{1p\upsilon}| + \frac{1}{|p_{mj}|}\sum_{r=0}^{m-1}|p_{rj}||e_{2p\upsilon}| .$$

Proof :   We know that at $t_j = a + jh, h = \frac{t-a}{N}$, $j = 0,1, \dots, N$,the corresponding approximation method for nonlinear Volterra- Fredholm integro-differential Eq. (11) is

$$\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \frac{h}{2}\sum_{p=0}^{j-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}k_1^{(l)}\left(t_j, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right) + \frac{h}{2}\sum_{p=0}^{N-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}k_2^{(l)}\left(t_j, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right),$$

$$j = 1, \dots, N, \qquad 1 \le m \le 4. \qquad\qquad (17)$$

By discrediting (1) and approximating the integral by the 5-points Gauss–Tura´n rules, we obtain

$$\sum_{r=0}^{m} p_r(t_j) y^{(r)}(t_j) = g(t_j) + \frac{h}{2}\sum_{p=0}^{j-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}k_1^{(l)}\left(t_j, \xi_{p\upsilon}, y(\xi_{p\upsilon})\right) + \frac{h}{2}\sum_{p=0}^{N-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}k_2^{(l)}\left(t_j, \xi_{p\upsilon}, y(\xi_{p\upsilon})\right),$$

$$j = 1, \dots, N, \qquad 1 \le m \le 4. \qquad\qquad (18)$$

By subtracting (18) from (17) and using interpolatory conditions of quintic B-spline, we get

$$\sum_{r=0}^{m} p_r(t_j) [s^{(r)}(t_j) - y^{(r)}(t_j)]$$

$$= \frac{h}{2}\sum_{p=0}^{j-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}[k_1^{(l)}\left(t_i, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right) - k_1^{(l)}\left(t_i, \xi_{p\upsilon}, y(\xi_{p\upsilon})\right)] + \frac{h}{2}\sum_{p=0}^{N-1}\sum_{\upsilon=1}^{5}\sum_{l=0}^{6} A_{l,\upsilon}[k_2^{(l)}\left(t_i, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right)$$

$$- k_2^{(l)}\left(t_i, \xi_{p\upsilon}, y(\xi_{p\upsilon})\right)],$$

So

$$\Rightarrow |p_m(t_j)| |s^{(m)}(t_j) - y^{(m)}(t_j)|$$

$$\leq \frac{h}{2} \sum_{p=0}^{j-1} \sum_{\upsilon=1}^{5} \sum_{l=0}^{6} |A_{l,\upsilon}| |k_1^{(l)}\left(t_j, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right)$$

$$- k_1^{(l)}\left(t_j, \xi_{p\upsilon}, y(\xi_{p\upsilon})\right)| + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{\upsilon=1}^{5} \sum_{l=0}^{6} |A_{l,\upsilon}| |k_2^{(l)}\left(t_j, \xi_{p\upsilon}, s(\xi_{p\upsilon})\right) - k_2^{(l)}\left(t_j, \xi_{p\upsilon}, y(\xi_{p\upsilon})\right)|$$

$$+ \sum_{r=0}^{m-1} |p_r(t_j)| |s^{(r)}(t_j) - y^{(r)}(t_j)| \quad , j = 1, \dots, N.$$

We suppose that $s^{(m)}(t_j) = s_j^{(m)}, y^{(m)}(t_j) = y_j^{(m)}$, $j = 1, \dots, N, m = 1,2,3,4$ and kernels $k_1^{(l)}, k_2^{(l)}, l = 0, \dots, 6$ satisfy a Lipschitz condition in its third argument of the form

$$\left|k_1^{(l)}(t, \xi, s) - k_1^{(l)}(t, \xi, y)\right| \leq L|s - y|, \left|k_2^{(l)}(t, \xi, s) - k_2^{(l)}(t, \xi, y)\right| \leq L^*|s - y|,$$

where L, L^* is independent of t, ξ, s and y. We get

$$|p_{mj}| \left|s_j^{(m)} - y_j^{(m)}\right| \leq \frac{hL}{2} \sum_{p=0}^{j-1} \sum_{\upsilon=1}^{5} \sum_{l=0}^{6} |A_{l,\upsilon}| |s_{p\upsilon} - y_{p\upsilon}| + \frac{hL^*}{2} \sum_{p=0}^{N-1} \sum_{\upsilon=1}^{5} \sum_{l=0}^{6} |A_{l,\upsilon}| |s_{p\upsilon} - y_{p\upsilon}| + \sum_{r=0}^{m-1} |p_{rj}| \left|s_j^{(r)} - y_j^{(r)}\right| \quad , j$$

$$= 1, \dots, N.$$

Since that $|p_{mj}| \neq 0$, then we have

$$\left|e_j^{(m)}\right| \leq \frac{hL}{2|p_{mj}|} \sum_{p=0}^{j-1} \sum_{\upsilon=1}^{5} \sum_{l=0}^{6} |A_{l,\upsilon}| |e_{1p\upsilon}e_j^{(m)}| + \frac{hL^*}{2|p_{mj}|} \sum_{p=0}^{N-1} \sum_{\upsilon=1}^{5} \sum_{l=0}^{6} |A_{l,\upsilon}| |e_{2p\upsilon}| + \frac{1}{|p_{mj}|} \sum_{r=0}^{m-1} |p_{rj}| |e_j^{(r)}| \quad .$$

Where $e_j^{(m)} = s_j^{(m)} - y_j^{(m)}$, $j = 1, \dots, N, r = 0, \dots, m.$

When $h \to 0$ then the above first and second term are zero and the third term in the above tends to zero because this term is due to interpolating of y(t) by quintic B-spline. We get for a fixed j,

$$\left|e_j^{(m)}\right| \to 0 \text{ as } h \to 0, m = 0, \dots, 4.$$

Numerical Examples

In order to test the applicability of the presented method, we consider two examples of the nonlinear Volterra - Fredholm integro-differential equations, these examples have been solved with various values of N. The absolute errors in the solution for various values of N are tabulated in Tables. The tables verified that our approach is more accurate. Programs are preformed by Mathematica for all the examples.

Example 6.1 Consider the following nonlinear Volterra-Fredholm integro-differential equation with the exact solution $y(t) = t^2$,

$$y'(t) + y(t) = -\frac{1}{2} \int_0^t t y^2(x) dx + \frac{1}{4} \int_0^1 x y^3(x) dx + g(t), \qquad 0 \leq x, t \leq 1,$$

where $g(t) = \frac{1}{10} t^6 + t^2 + 2t - \frac{1}{32}$, with boundary conditions: $y(0) = 0$.

This equation has been solved by our method with N = 6, 16, 46, the absolute error at the particular grid points is tabulated in table 2 .Table 3 shows, a comparison between the absolute errors of our method together with triangular functions method [9], operational matrix with block-pulse functions method [10], and Hybrid Legendre polynomials and block-pulse functions

method [11] and reproducing kernel Hilbert space method [38]. As it is evident from the comparison results, it was found that our method in comparison with the mentioned methods is better with a view to accuracy and utilization.

Example 6.2 Cosider the following nonlinear Volterra-Fredholm integro-differential equation with the exact solution $y(t) = -1 + t^2$,

$$y'''(t) + y(t) = \int_0^t y^2(x)\,dx + \int_0^1 (t^2x + tx^2)\,y^2(x)dx + g(t), \qquad 0 \le x, t \le 1,$$

where $g(t) = \frac{-1}{5}t^5 + \frac{2}{3}t^3 + \frac{5}{6}t^2 - \frac{113}{105}t - 1$, with boundary conditions: $y(0) = -1, y'(0) = 0, y''(0) = 2$.

The approximate solutions are calculated for different values of N = 5, 15, 35, the absolute error at the particular grid points is tabulated in table 4 . While in [39], the adopted method yields no solution for N = 3, takes time to give the approximate answer for N = 5, and is proper only for N = 4, our method is significant not only because it yeilds solutions for any N, but also because the approximate solutions it gives are very close to exact ones. This table verified that our results are more accurate.

Table 2: The error ||E|| in solution of example 6.1 at particular points

t	N=6	N=16	N=46
0	8.67E − 19	1.08E − 19	1.35E − 20
0.1	1.48E − 09	2.01E − 10	2.42E − 11
0.2	2.74E − 09	3.83E − 10	4.62E − 11
0.3	3.85E − 09	5.45E − 10	6.60E − 11
0.4	4.87E − 09	6.89E − 10	8.35E − 11
0.5	5.75E − 09	8.14E − 10	9.86E − 11
0.6	6.43E − 09	9.14E − 10	1.11E − 10
0.7	6.90E − 09	9.84E − 10	1.19E − 10
0.8	7.11E − 09	1.01E − 09	1.22E − 10
0.9	6.91E − 09	9.87E − 10	1.19E − 10
1	6.20E − 09	8.91E − 10	1.08E − 10

Table 3: Numerical comparison of absolute error for Example 6.1

t	our method	method in [9]	method in [10]	method in [11]	method in [38]
	N = 26	m = 32	m = 32	n = 8, m = 8	N = 26, n = 5
0	0	0	0	0	0
0.1	7.60E − 11	1.66E − 04	2.18E − 03	3.10E − 05	6.14E − 07
0.2	1.45E − 10	2.54E − 04	1.46E − 03	7.50E − 05	1.23E − 06
0.3	2.06E − 10	2.62E − 04	1.67E − 03	1.71E − 04	1.86E − 06
0.4	2.61E − 10	1.91E − 04	7.24E − 03	9.40E − 05	2.52E − 06
0.5	3.08E − 10	4.10E − 05	1.60E − 02	2.28E − 04	3.19E − 06
0.6	3.46E − 10	2.02E − 04	1.15E − 02	5.02E − 04	3.90E − 06
0.7	3.73E − 10	2.83E − 04	4.51E − 03	5.83E − 04	4.65E − 06
0.8	3.84E − 10	2.83E − 04	4.87E − 03	3.74E − 04	5.44E − 06
0.9	3.74E − 10	2.02E − 04	1.67E − 02	4.70E − 05	6.29E − 06
1	3.38E − 10	3.70E − 05	3.09E − 02	1.40E − 05	7.20E − 06

Table 4: The error ||E|| in solution of example 6.2 at particular points

t	N = 5	N = 15	N = 35
0	1.11E − 16	2.21E − 16	0
0.1	3.38E − 13	4.34E − 14	2.28E − 14
0.2	1.68E − 12	2.78E − 13	5.62E − 14
0.3	1.01E − 12	1.54E − 12	2.59E − 13
0.4	2.15E − 11	5.05E − 12	1.09E − 12
0.5	7.78E − 11	1.28E − 11	2.53E − 12
0.6	1.87E − 10	2.66E − 11	5.14E − 12
0.7	3.70E − 10	4.88E − 11	9.42E − 12
0.8	6.51E − 10	8.20E − 11	1.55E − 11
0.9	1.05E − 09	1.28E − 10	2.42E − 11
1	1.59E − 09	1.90E − 10	3.57E − 11

CONCLUSION

In the present work, a technique has been developed for solving the linear and nonlinear Volterra-Fredholm integro-differential equations by using the 5-points Gauss-Turan quadrature formula with respect to the Legendre weight function and collocating by quintic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions. This method tested on 2 examples. The absolute errors in the solutions of these examples show that our

approach is more accurate in comparison with the methods given in [9,10,11,38,39] and our results verified the accurate nature of our method.

REFERENCES

M. A. Abdou, On Asymptotic Methods for Fredholm-Volterra Integral Equation of the Second Kind in Contact Problems, Journal of Computational and AppliedMa- thematics, Vol. 154, No. 2(2003), 431-446.

F. Bloom, Asymptotic Bounds for Solutions to a System of Damped Integro-Differential Equations of Electro-magnetic Theory, Journal of Mathematical Analysis and Applications, Vol. 73(1980), 524-542.

M. A. Jaswon and G. T. Symm, Integral Equation Me thods in Potential Theory and Elastostatics, Academic Press, London, (1977).

S. Jiang , V. Rokhlin, Second Kind Integral Equations for the Classical Potential Theory on OpenSurface II, Journal of Computational Physics, Vol. 195, No. 3 (2004), 1-16.

P. Schiavane and C. Constanda and A. Mioduchowski, Integral Methods in Science and Engineering, Birk- hauser, Boston, (2002).

B. I. Smetanin, On an Integral Equation for Axially-Symmetric Problems in the Case of an Elastic Body Con- taining an Inclusion, Journal of Applied Mathematics and Mechanics, Vol. 55, No. 3 (1991), 371-375.

C. Minggen , D. Hong, Representation of exact solution for the nonlinear Volterra-Fredholm integralequations, Appl. Math. Comput., 182 (2006), 1795- 1802.

K. Maleknejad and Y. Mahmoudi, Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations,Applied Mathematics and Computation, vol. 145, No. 2-3(2003), 641-653.

E. Babolian, Z. Masouri, and S. Hatamzadeh-Varmazyar, Numerical solution of nonlinear Volterra-Fredholm integro-differential equations via direct method using triangular functions,Computers and Mathematics with Applications, vol. 58, No. 2 (2009), 239-247.

E. Babolian, Z. Masouri, and S. Hatamzadeh-Varmazyar, New direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equations using operational matrix with blockpulse functions,Progress In Electromagnetics Research B, vol. 8 (2008), 59-76.

K. Maleknejad, B. Basirat, and E. Hashemizadeh, Hybrid Legendre polynomials and block-pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations, Computers and Mathematics with Applications, vol. 61, No. 9(2011),2821-2828.

S. Yalcinbas, Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equtions, Appl. Math. Comput. 127 (2002), 195-206.

S. Yalcinbas, M. Sezer, The approximate solution of high-order linear Volterra-Fredholm integrodifferential equtions in terms of Taylor polynomials, Appl. Math. Comput. 112 (2000), 291-308.

E. Babolian and Z. Masouri, Numerical solution of Volterra integro-differential equation by direct method via block-pulse functions, J. Sci. Tarbiat Moallem Univ. 9(1) (2010), 1-9.

J. Rashidinia and M. Zarebnia, The numerical solution integro-differential equation by the sinc method, Appl. Math. Comput. 188 (2007), 1124-1130.

A. Mohsen and M. El-Gamel, A sinc-collocation method for the linear Fredholm integrodifferential equations, ZAMP. Z. Angew. Math. Phys. 58 (2007), 380-390.

M. Zarebnia, Sinc numerical solution for the Volterra integro-differential equation, Commun. Nonlinear Sci. Numer. Simul. 15(3) (2010), 700-706.

A. Golbabai and M. Javidi, Application of Hes homotopy perturbation method for nthorder integrodifferential equations, Appl. Math. Comput. 190 (2007), 1409-1416.

J. Biazar and M. Eslami, Exact solutions for non-linear Volterra-Fredholm integrodifferential equations by Hes homotopy perturbation method, Int. J. Nonlinear Sci. 9(3) (2010), 285-289.

S.R. Seyed Alizadeh, G.G. Domairry, and S. Karimpour, An Approximation of the analytical solution of the linear and nonlinear integro- differential equations by homotopy perturbation method, Acta Appl. Math. 104 (2008), 355-366.

M. Ghasemi, M. Tavassoli Kajani, and E. Babolian, Application of Hes homotopy perturbation method to nonlinear integro-differential equations, Appl. Math. Comput. 188 (2007), 538-548.

N.H. Sweilam, Fourth order integro-differential equations using variational iteration method, Comput. Math. Appl. 54 (2007), 1086-1091.

Y. Ordokhani, M. Razzaghi , Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions, Applied Mathematics Letters ,21 (2008), 4-9.

E. Babolian, F. Fattahzadeh, E. Golpar Raboky, A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type, Appl. Math. Comput. 189 (2007), 641-646.

Farshid Mirzaee , Ali Akbar Hoseini, Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series ,Alexandria Engineering Journal, 52(2013) , 551-555.

M. Zarebnia , A Numerical solution of nonlinear Volterra-Fredholm integral equations , Journal of Applied Analysis and Computation , Vol. 3, No. 1, February 2013.

F.A. Hendi , A.M. Albugami , Numerical solution for Fredholm-Volterra integral equation of the second kind by using collocation and Galerkin methods, Journal of King Saud University (Science),22(2010) , 37-40.

E. Hashemizadeh, K.Maleknejad, B. Basirat, Hybrid functions approach for the nonlinear Volterra-Fredholm integral equations, Proc. Comput. Sci. 3 (2011), 1189-1194.

M.M. Jamei , S.Fakhravar, Gauss-Turan numerical integration method. Master of Science thesis, Khaje Nassir Al-Deen Toosi University of Technology (2011).

G.V. Milovanovici , M.M. Spalevici, M.S.Pranici , Error bounds of some Gauss-Turan-Kronrod quadratures with Gori-Micchelli weights for analytic functions .J. Math. 30(2007),221-234.

E. D.Danciu , On quadrature formulas of Gauss-Turan and Gauss-Turan-Stancu type. General Mathematics, 17(3)(2009) ,203-220.

Z. Mahmoodi , J. Rashidinia , E. Babolian, B-Spline collocation method for linear and nonlinear Fredholm and Volterra integro-differential equations. Applicable Analysis,(2012),1-16.

P.M.Prenter, Spline and Variational Methods. Wiley & Sons, New-York(1975).

A.N.Netravali, R.J.P.Figueiredo , Spline approximation to the solution of the linear Fredholm integral equation of the second kind .SIAM. J. Numer. Anal.11(1974), 538-549.

M.T.Rashed, An expansion method to treat integral equations. Appl. Math. Comput.135 (2003),65-72.

J.Rashidinia , E.Babolian , Z.Mahmoodi , Spline Collocation for nonlinear Fredholm Integral Equations. International Journal of Mathematical Modelling and Computations, 1 (1) (2011), 69-75.

J.Rashidinia , E.Babolian , Z.Mahmoodi , Spline Collocation for Fredholm Integral Equations. Mathematical Sciences,5 (2) (2011),147-158.

O. A. Arqub, M.Al-Smadi, and Sh. Momani,Application of Reproducing Kernel Method for Solving Nonlinear Fredholm-Volterra Integro-differential Equations,Hindawi Publishing Corporation Abstract and Applied Analysis. Vol. 16 (839-836),2012.

W. Wang, An Algorithm for Solving the High-Order Nonlinear Volterra-Fredholm Integro-Differential Equation with Mechanization, Applied Mathematics and Computation, Vol. 172, No. 1(2006), 1-23.