A new proof of a theorem of Littlewood

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Abstract

In this paper we give a new combinatorial proof of a result of Littlewood [3]: $S_\mu(1, q, q^2, \ldots) = \frac{q^{n(\mu)}}{\prod_{s\in\mu}(1-q^{h_\mu(s)})}$, where S_μ denotes the Schur function of the partition μ, $n(\mu)$ is the sum of the legs of the cells of μ and $h_{(\mu)}(s)$ is the hook number of the cell $s \in \mu$.

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Proposed running head: A proof of a theorem of Littlewood
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1 Introduction

A classical result in the theory of symmetric functions concerns the so-called "principal specialization" of the Schur function $S_\mu(x_1, x_2, \dots)$ to $S_\mu(1, q, q^2, \dots)$. An explicit formula for this specialization was given by Littlewood [3], and can be expressed as

Theorem 1 (Littlewood)**.**

$$S_\mu(1, q, q^2, \dots) = \frac{q^{n(\mu)}}{\prod_{s\in\mu}(1 - q^{h_\mu(s)})}. \tag{1}$$

Here $n(\mu)$ is the value $\sum_i (i-1)\mu_i$, and $h_\mu(s)$ is the hook-length of the cell s in the diagram of μ. This formula is treated in many modern expositions of symmetric functions. See, for example, the books by Macdonald [4] (Section I.3, Example 2) and Stanley [5] (Corollary 7.21.3). Our contribution is a new proof of (1), which, in contrast to those cited above, is essentially combinatorial.

The proof proceeds by showing that the right hand side of (1) satisfies the following recursion:

$$\frac{q^{n(\mu)}}{\prod_{s\in\mu}(1 - q^{h_\mu(s)})} B_\mu(q, q^{-1}) = \frac{1}{1-q} \sum_{\nu\to\mu} \frac{q^{n(\nu)}}{\prod_{s\in\nu}(1 - q^{h_\nu(s)})}. \tag{2}$$

Here $\nu \to \mu$ means that ν covers μ in the usual Bruhat order on partitions, and $B_\mu(q, t)$ is the 'biexponent generator' of μ:

$$B_\mu(q, t) = \frac{1}{qt} \sum_{s\in\mu} q^{(\text{row index of } s)} t^{(\text{column index of } s)}$$

We then give a bijective proof that the left side of (1) satisfies the same recursion. To accomplish this, we interpret the left hand side of the identity

$$S_\mu(1, q, q^2, \dots) B_\mu(q, q^{-1}) = \frac{1}{1-q} \sum_{\nu\to\mu} S_\nu(1, q, q^2, \dots) \tag{3}$$

as

$$S_\mu(1, q, q^2, \dots) B_\mu(q, q^{-1}) = \sum_{n\geq 0} |A_n| q^n, \tag{4}$$

where A_n is the set of pairs (T, s) where T is a semistandard tableau of shape μ on the alphabet $\mathbb{Z}_{\geq 0}$, s is a cell in T, and the pair (T, s) must have the sum of the entries in T plus the content of the cell s equal to n.

We interpret the right hand side of (3) by

$$\frac{1}{1-q} \sum_{\nu\to\mu} S_\nu(1, q, q^2, \dots) = \sum_{n\geq 0} |B_n| q^n \tag{5}$$

where B_n is the set of semistandard tableaux T on the alphabet $\mathbb{Z}_{\geq 0}$, where the shape of T is μ with a single corner cell removed, and the sum of the entries in T is less than or equal to n.

With these interpretations, we have the following simple map from A_n to B_n: remove the cell containing s from T, and use *jeu de taquin* to slide the empty cell to the boundary of T. We show that this map is indeed a bijection to complete the proof of (3).

2 Definitions and Notation

Given a partition μ, we identify it with its Ferrers diagram in French notation. For every cell s in μ, we call the *arm* and the *leg* of s the parameters $a_\mu(s)$ and $l_\mu(s)$ giving the number of cells of μ that are respectively

East and North of s in μ. For every $s \in \mu$ we denote by $h_\mu(s) := a_\mu(s) + l_\mu(s) + 1$ the hook number of s in μ, and we set also

$$n(\mu) := \sum_{s\in\mu} l_\mu(s) = \sum_i (i-1)\mu_i,$$

and

$$B_\mu(q,t) := \sum_{(i,j)\in\mu} t^{i-1}q^{j-1}.$$

where by i, j we mean the row index and column index, respectively, of the cells in μ. Note that replacing t by q^{-1} gives

$$B_\mu(q,q^{-1}) = \sum_{(i,j)\in\mu} q^{j-i}.$$

The quantity $j - i$ is known as the *content* of the cell $s = (i, j) \in \mu$, and denote it by $|s|$.

3 Proof of the formula (2)

We will derive the formula (2) from the following formula in [2]

$$B_\mu(q,t) = \sum_{\nu\to\mu} c_{\mu\nu}(q,t), \tag{6}$$

where

$$c_{\mu\nu}(q,t) := \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}}.$$

Here $\mathcal{R}_{\mu/\nu}$ (resp. $\mathcal{C}_{\mu/\nu}$) denotes the set of the cells of ν that are in the same row (resp. same column) as the cell we must remove from μ to obtain ν. We follow the notes of Adriano Garsia (personal communication).

Remark 1. Notice that (6) is proved in a completely elementary way in ([2]).

We begin by replacing t with $1/q$ in (6). Observe now that this specialization ($t \mapsto 1/q$) in $c_{\mu\nu}(q,t)$ gives

$$\begin{aligned} c_{\mu\nu}(q,1/q) &= \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{q^{-l_\mu(s)} - q^{a_\mu(s)+1}}{q^{-l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{a_\mu(s)} - q^{-(l_\mu(s)+1)}}{q^{a_\nu(s)} - q^{-(l_\nu(s)+1)}} \\ &= \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{q^{l_\nu(s)}}{q^{l_\mu(s)}} \cdot \frac{1-q^{h_\mu(s)}}{1-q^{h_\nu(s)}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{l_\nu(s)+1}}{q^{l_\mu(s)+1}} \cdot \frac{q^{h_\mu(s)}-1}{q^{h_\nu(s)}-1} \\ &= \frac{\prod_{s\in\mu} 1-q^{h_\mu(s)}}{\prod_{s\in\nu} 1-q^{h_\nu(s)}} \cdot \frac{q^{n(\nu)}}{q^{n(\mu)}}, \end{aligned}$$

where the last equality follows from the definition of $h_\mu(s)$ and the fact that when $s \notin \mathcal{R}_{\mu/\nu} \cup \mathcal{C}_{\mu/\nu}$ then $l_\mu(s) = l_\nu(s)$ and $a_\mu(s) = a_\nu(s)$.

Hence replacing t by $1/q$ in (6) and multiplying both sides by $q^{n(\mu)}$ gives

$$q^{n(\mu)} B(q,1/q) = \sum_{\nu\to\mu} \frac{\prod_{s\in\mu} 1-q^{h_\mu(s)}}{\prod_{s\in\nu} 1-q^{h_\nu(s)}} q^{n(\nu)}.$$

This may also be written as

$$\frac{q^{n(\mu)}}{\prod_{s\in\mu}(1-q^{h_\mu(s)})} B_\mu(q,q^{-1}) = \frac{1}{1-q} \sum_{\nu\to\mu} \frac{q^{n(\nu)}}{\prod_{s\in\nu}(1-q^{h_\nu(s)})},$$

which we take as our starting point.

4 Proof of the theorem

As we already pointed out, we must show that the Schur functions satisfy the recursion (2).

The initial condition is obviously satisfied, because when μ is the partition with only one box, we have $n(\mu) = 0$ and the hook number of the cell is one. So in this case, the formula reduces to

$$\frac{q^{n(\mu)}}{\prod_{s\in\mu}(1-q^{h_\mu(s)})} = \frac{1}{1-q} = \sum_{n\geq 0} q^n,$$

which is clearly the Schur function $S_\mu(1, q, q^2, \dots)$.

Observe that by the definition of Schur functions as the generating function for semi-standard tableaux, we have

$$S_\mu(1, q, q^2, \dots) = \sum_{k\geq 0} c(k, \mu) q^k,$$

where $c(k, \mu)$ is the number of semi-standard tableaux T of shape μ such that the sum of the entries, each of them decreased by one, is k. So we can consider semi-standard tableaux of shape μ with entries in $\mathbb{Z}_{\geq 0}$ denote this set with $SST_0(\mu)$, and say that

$$c(k, \mu) = \#\{T \in SST_0(\mu) \mid wt(T) = k\},$$

where $wt(T)$ is the *weight* of the tableaux T, i.e. the sum of its entries.

We now express both sides of (2) in terms of combinatorial objects. We begin with the left hand side:

$$S_\mu(1, q, q^2, \dots) B_\mu(q, q^{-1}) = \sum_{(i,j)\in\mu} \sum_{k\geq 0} c(k, \mu) q^{k+j-i} = \sum_{n\geq 0} a_n q^n,$$

where a_n is the cardinality of the set

$$A_n := \{(T, s) \mid T \in SST_0(\mu),\ s \in \mu \ \text{ and } \ wt(T) + |s| = n\}.$$

For the right hand side, we have instead

$$\begin{aligned} \frac{1}{1-q} \sum_{\nu\to\mu} S_\nu(1, q, q^2, \dots) &= \left(\sum_{k\geq 0} q^k\right) \left(\sum_{n\geq 0} \left(\sum_{\nu\to\mu} c(n, \nu)\right) q^n\right) \\ &= \sum_{n\geq 0} \left(\sum_{j=0}^{n} \sum_{\nu\to\mu} c(j, \nu)\right) q^n \\ &= \sum_{n\geq 0} b_n q^n \end{aligned}$$

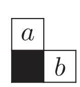
where $b_n := \sum_{j=0}^{n} \sum_{\nu\to\mu} c(j, \nu)$ is the cardinality of the set

$$B_n := \{T \in SST_0(\nu) \mid \nu \to \mu \ \text{ and } \ wt(T) \leq n\}.$$

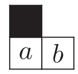
So we prove the theorem if we show that $a_n = b_n$ for all n. To do that, we will find bijections between the sets A_n and B_n.

We now describe a map from A_n to B_n. We shall see that this map will use nothing more than the well-known operation of *jeu de taquin*, or *sliding* (see [1]). The majority of the rest of the article will be devoted to showing that this map is in fact invertible.

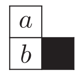
First, recall the definition of sliding: Give a tableau with a "hole", we consider the cells above and immediately to the right of the hole. We then "slide" a cell according to the rule below.



goes to

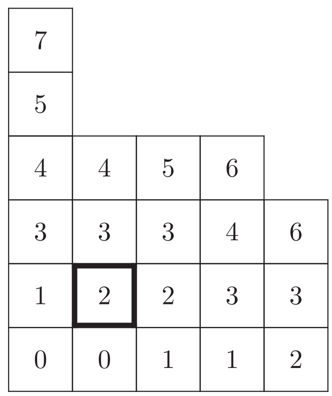


if $a \leq b$ (or if b is not part of the diagram) and to

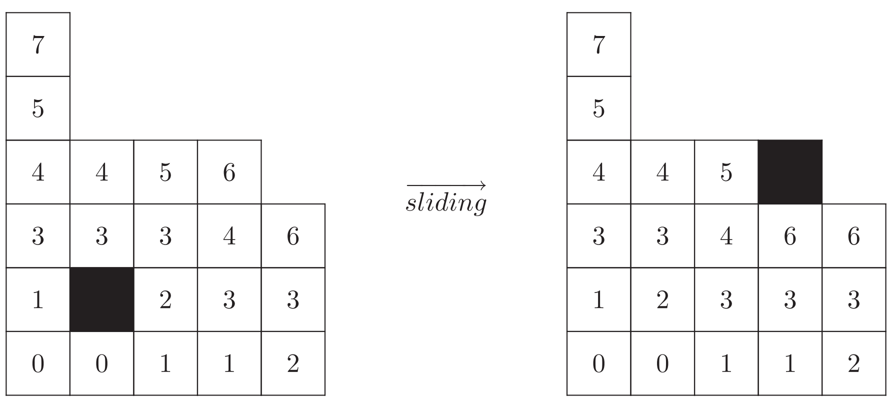


otherwise.

Now, given $(T, s) \in A_n$ we remove the cell s, leaving a hole in its place. We then repeat the sliding operation until we arrive at a new tableau. Here is an example:



Here the tableau T of shape μ is pictured and s is the cell marked in bold. We have $wt(T) = 65$ and the content of s is $|s| = 2 - 2 = 0$, so $n = 65$. Replacing s with a hole and sliding (jeu de taquin) gives



In the picture we have denoted with a black box the final position of the hole after sliding. This new tableau T' is of shape ν, where clearly $\nu \to \mu$, and the weight is $wt(T') = 65 - 2 = 63$, with $63 \leq 65 = n$. So T' is in B_n.

In general, from the definition of semi-standard tableau it follows that the sum of the entry in a cell s and its content $|s|$ is greater then or equal to 0. (To see this, consider the tableau with minimal weight). Hence if (T, s) is in A_n, throwing out the cell s from T and sliding we will get an element in B_n.

We want to prove that this map is invertible. We start with the following

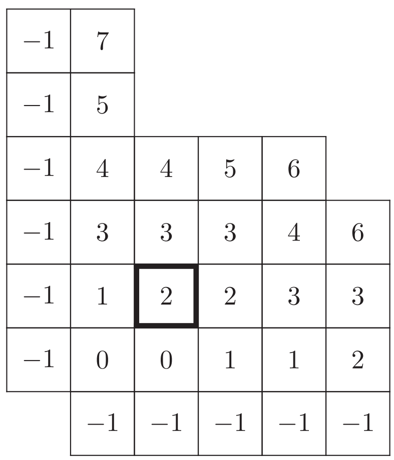
Definition 1. Given a pair of a filling of a semi-standard tableau and a cell in it, we call the sum of the weight of the tableau and the content of the cell the *index* of the pair.

Now, given T in B_n we are going to describe an entry to put in the distinguished cell $\mu \setminus \nu$. The index of this pair will be $\geq n$. We will then *slide* this cell *back* (a procedure we define below). At each step of this procedure, we will either move the distinguished cell, or change the value of its entry. The object we are considering at each step will always be a filling of the shape μ; sometimes it will be a semi-standard tableau and sometimes not. At the times when we do have a semi-standard tableau, we will record the index. The algorithm will terminate when we record an index of n (we will demonstrate that this will always happen). This will describe a map from $B_n \to A_n$, and we shall see that it is indeed the inverse of the map described above.

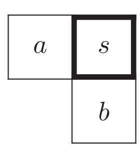
In what follows we will sometimes describe the cells by the values of their entries; we will say that a cell is bigger than, smaller than or equal to another cell if these relations hold between their entries.

At each step of the algorithm we assume that we have a filling of the diagram of μ with a distinguished cell with entry s. The algorithm will begin with the distinguished cell $\mu \setminus \nu$ and the entry will be the smallest value for which the resulting filling is a semi-standard tableau with index $\geq n$. Each step of the algorithm will depend only on the values of the cells immediately South and West of s.

To manage the situations which involve the first (left) column and the bottom row of the diagram, we assume that there is one more border column on the left of the diagram and one more border row on the bottom, both consisting of cells filled with the entry -1 (see the picture for an example).



Thus, the general situation for s in the diagram (we don't consider s in these new border cells) is illustrated as follows:



Now, the algorithm proceeds according to the following rules:

1. If $s > a$ and $s > b$, we *record the index*, then decrease s by 1.

2. If $s = a$ and $s > b$, we *record the index*, then swap the cells s and a. (Of course, since $s = a$, this has no affect on the entries of the tableau, but it does change the distinguished cell.)

3. If $s < a$ and $a > b$, we swap a and s.

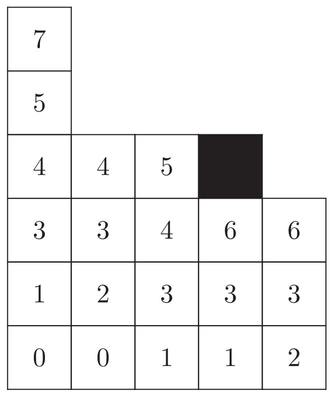
4. If $s < b$ and $a \leq b$, we swap s and b.

5. If $s = b$ and $a \leq b$, we decrease s by 1.

Remark 2. Notice that these cases are exclusive and cover all the possible situations. Also, cases (1) and (2) are the only cases which satisfy the condition of being semi-standard.

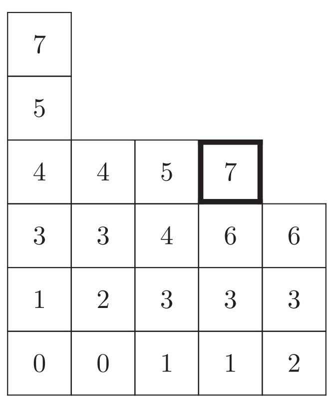
We iterate this algorithm, and terminate when we record an index of n. We claim that each recorded index is one less than the previous, and that the algorithm always terminates with the distinguished cell still part of the diagram.

Before showing this, we present an example to show how the algorithm works. Consider the tableau



This is the tableau we got from the first example. So in this case $n = 63$.

In this case, we can start with $s = 7$. We have



We now have to apply rule (1), so we record the index, which is 70 (the content of $s = 7$ is $4 - 4 = 0$), and decrease $s = 7$ by 1. We have $s = 6$:

7				
5				
4	4	5	**6**	
3	3	4	6	6
1	2	3	3	3
0	0	1	1	2

and we must apply rule (5) and decrease s by 1. Now we have to apply rule (4) and hence swap $s = 5$ and $b = 6$:

7				
5				
4	4	5	**5**	
3	3	4	6	6
1	2	3	3	3
0	0	1	1	2

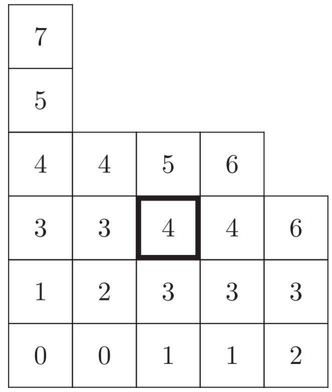
$\longrightarrow$

7				
5				
4	4	5	6	
3	3	4	**5**	6
1	2	3	3	3
0	0	1	1	2

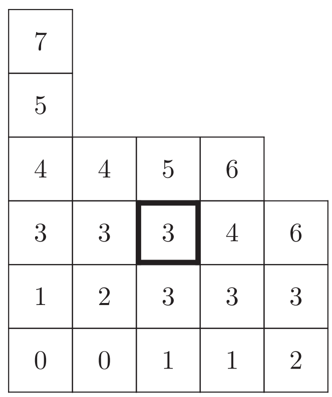
Now the weight of the tableau is 68, but the content of $s = 5$ is 1, so the index is 69. We record this index, and decrease s by 1 again (rule (1)):

7				
5				
4	4	5	6	
3	3	4	**4**	6
1	2	3	3	3
0	0	1	1	2

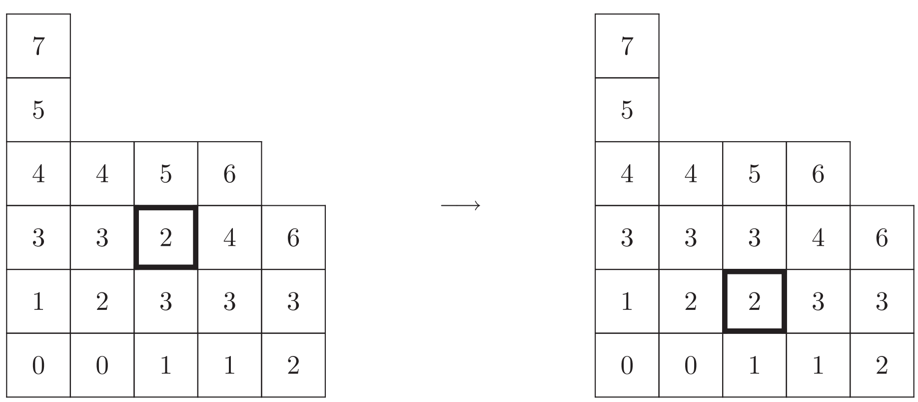
This is still semi-standard, so we record the index of 68, and, since the cell left to $s = 4$ has the same value, we swap them according to rule (2), getting



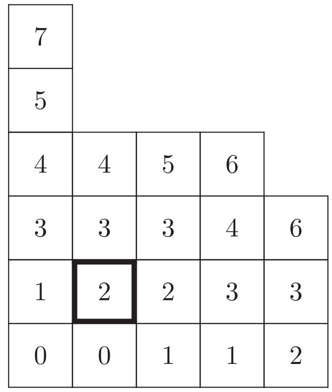
This way we decreased the content of $s = 4$ by 1. Now rule (1) applies, so we record the index, which is now 67, and decrease the value of s by 1



which does not give a semi-standard tableau. So, applying rule(5) we decrease $s = 3$ by 1 again, and we apply rule (4):



This is semi-standard, so we record the index of 66. Now we swap $s = 2$ with the cell left as before (rule (2))

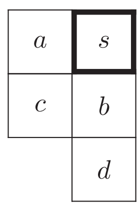


This gives a semi-standard tableau with index 65, so the algorithm terminates. Observe that this is exactly the pair of A_n we started with.

Remark 3. In this example we didn't use rule (3). Actually, in this way we'll never need rule (3), since whenever $a > b$, decreasing each time s by 1, we will get first the situation $s = a > b$, where we apply rule (2), swapping s and a. We defined the rule (3) anyway, for making the crucial observation in *Remark* 4.

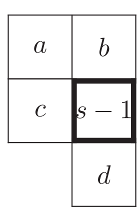
We now prove the algorithm has the desired properties.

We first show that each time we record the index, it goes down by one. Consider the following general situation, where we have just recorded the index of a semi-standard tableau with distinguished cell s:



Since we have a semi-standard tableau, we must have $s \geq a$ and $s > b$. Consider first the case $s = a$. Here the algorithm proceeds by swapping a and s (rule (2)). This clearly results in a semi-standard tableau (the same one) so we again record the index: it has dropped by one, since the content of the distinguished cell decreases by one with a step to the West.

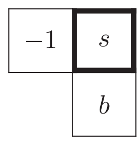
The other case is $s > a$, $s > b$. Here the algorithm proceeds by replacing s by $s' = s - 1$ (rule(1)). If we still have a semi-standard tableau, we again record the index and it is clear that it has decreased by one. If we do not, we must have $s' = b$, and $b = s' \geq a$. The algorithm proceeds by decreasing s' by 1 to get $s'' = s' - 1 = s - 2$ (rule (5)). We then have $s'' < b$, hence applying rule (4) we swap s'' and b getting the following:



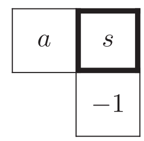
By semi-standardness we have that $d \leq b-1 = s-2$ and $c \leq a-1 \leq s-2$, so this is a semi-standard tableau. Now the weight of the tableau has decreased by two, but the content of the distinguished cell has increased by one with a move South, so again the index has decreased by one.

We now show that the algorithm must terminate before the distinguished cell can leave the diagram.

First note that we only swap the distinguished cell s with another of greater or equal value. Hence in case s reaches the most left column or the bottom row of the diagram, but not the South-West corner, we have one of the following:



$(b \geq 0)$ or



$(a \geq 0)$.

In the first case we swap s with b, in the second case we swap s with a. Hence in both cases we swap s with a cell of the diagram. Finally, if s reaches the South-West corner, since we decrease each time s by 1, before leaving the corner it must have had value 0. But in that case we had necessary a semi-standard tableau, whose index was the weight of the original tableau (since the content of the South-West corner is 0 and $s = 0$), which is, by definition of B_n, less or equal then n. This shows that the procedure terminates successfully in this case too.

We now have constructed a map from B_n to A_n. To complete the proof, we must show that this is the inverse of the map $A_n \to B_n$ described above.

Remark 4. In the algorithm going from $B_n \to A_n$, we start with some value s in the distinguished cell, and end with some weakly smaller value s'. We claim that starting the process with any value between s' and s will result in exactly the same outcome. The only potential difference is when rule (3) applies to the smaller value. But here, it will do exactly what rule (2) did to the larger.

Now it is easy to see that the "sliding" path of an element s as we go from $A_n \to B_n$ is exactly the reverse of the "sliding back" path if we start the algorithm from $B_n \to A_n$ with s. Remark 4 shows that we are able to assume without loss that we start the $B_n \to A_n$ algorithm with s.

5 Acknowledgements

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