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Ranking Intervals and Dominance Relations for Ratio-Based Efficiency Analysis

We develop comparative results for ratio-based efficiency analysis, based on the decision making units' (DMUs) relative efficiencies over sets of feasible weight that characterize preferences for input and output variables. Specifically, we determine (i) *ranking intervals* which indicate the best and worst efficiency rankings that a DMU can attain relative to other DMUs, (ii) *dominance structures* which convey what other DMUs a given DMU dominates in one-on-one efficiency comparisons, and (iii) *efficiency bounds* which show how much more efficient a DMU can be relative to a given DMU or a subset of other DMUs. These efficiency results–which reflect the full range of feasible input and output weights–are robust in the sense that they are insensitive to possible outliers and do not necessitate particular returns-to-scale assumptions. We also report a real case study where these results supported the efficiency analysis of the twelve departments at a large technical university.

Key words: performance measurement, data efficiency analysis, preference modeling

1. Introduction

Inspired by the seminal paper of Charnes et al. (1978), the Data Envelopment Analysis (DEA) literature offers numerous methods for examining the efficiency of decision making units (DMUs) (see, e.g., Cooper et al., 2007). These methods are often employed in settings where information about the values of input and output variables is not readily available, but where subjective preference information about these values can nevertheless be elicited (Thompson et al., 1986; Allen et al., 1997). As an application domain, higher education has these characteristics, which is one of the reasons for why the DEA and its variants have been employed extensively in education (see, e.g., Sarrico and Dyson, 2000).

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Technically, the efficiency score for a DMU (CCR-DEA; Charnes et al., 1978) is computed relative to an efficient frontier, characterized by efficient DMUs whose efficiency ratio (defined as the ratio between the aggregate value of their outputs and that of their inputs) is highest among all DMUs for some input/output weights. By definition, efficient DMUs have an efficiency score of one. For inefficient DMUs, the efficiency score will be less than one, indicating how 'close' to the efficient frontier the DMU can be when its inputs and outputs are aggregated using weights that are *most favorable* to it. In both cases, however, the resulting efficiency score conveys no information about how high the DMU's efficiency ratio can be relative to those of other DMUs for *other* input/output weights: for example, even if a DMU is efficient, there may exist some feasible weights for which some other DMUs have strictly higher efficiency ratios. This suggests that it is instructive to examine how the DMUs' efficiency ratios change *relative to each other* as a function of input/output weights.

Because the efficiency scores are computed relative to the efficiency frontier, these scores are potentially sensitive to what DMUs are included in or excluded from the analysis: specifically, the introduction/removal of a single outlier (e.g., an exceptionally efficient DMU that produces more outputs per inputs than the other DMUs) may shift the efficient frontier considerably, which may disrupt the reported efficiency scores for other DMUs and hence perplex the users of efficiency results (see, e.g., Seiford and Zhu, 1998ab; Zhu, 1996). Such disruptions do not take place in direct one-on-one comparison of the DMUs' efficiency ratios which are not affected by outliers. Nor do such pairwise comparisons necessitate specific returns-to-scale assumptions which may be difficult to validate *ex ante* when defining the efficient frontier (see, e.g., Galagedera and Silvapulle, 2003; Dyson et al., 2001).

Motivated by the above considerations, we develop efficiency results which allow us to answer questions such as:

- What are the best/worst rankings that DMU A can attain in comparison with other DMUs, based on their efficiency ratios?

- Does DMU A dominate DMU B in terms of its efficiency? (in the sense that the efficiency ratio of DMU A is at least as high as that of DMU B for all feasible weights and strictly higher for some)
- How much more/less efficient can DMU A be relative to DMU B? Or, more generally, relative to some selected subset of other DMUs?

The first question is partly motivated by the popularity of ranking lists, as exemplified by the ranking of 'best' universities by the Shanghai Jiao Tong University (cf. Liu and Cheng, 2005; see also Köksalan et al., 2009). It results in *ranking intervals* which are defined by the DMUs' best/worst rankings over the sets of feasible weights and which are robust because they will exhibit only small changes due to the introduction/removal of DMUs. The second question helps establish and communicate *dominance structures* based on one-on-one comparisons between the DMUs. The third question (which is related to super-efficiency; see, e.g., Andersen and Petersen, 1993) yields *efficiency bounds* that provide information about relative efficiency *differences* among the DMUs. All these efficiency results can be employed for the identification of most/least efficient DMUs and the specification of performance targets.

We also describe a case study where these efficiency results were employed to analyze the departments at a large technical university. The results of this case study were communicated to and appreciated by the Board of the University. They catalyzed an informed debate on the possibilities and limitations of using efficiency analyses for the purpose of guiding resource allocation decisions. We present salient arguments from this case study and outline topics for future research as well.

The rest of this paper is organized as follows. Section 2 discusses earlier methods for ratio-based efficiency analysis and their uses in higher education. Section 3 formulates the methodological contributions. Section 4 describes the case study, and Section 5 concludes.

2. DEA Methods and Their Uses in Higher Education

In the DEA literature, there are numerous methods for analyzing the relative efficiencies of DMUs that transform multiple inputs into multiple outputs (see, e.g., Cooper et al., 2007). These methods have been extensively applied in contexts where subjective preference information about the relative values of these input/output variables can be elicited (cf. Thanassoulis et al., 2004). Early approaches for incorporating preference information include, among others, the specification of assurance regions (Thompson et al., 1990) and cone ratios (Charnes et al., 1990). Subsequently, relationships between DEA models and multi-criteria decision making (MCDM) methods have been explored extensively (Stewart, 1996; Joro et al., 1998; Bouyssou, 1999). These relationships also underpin the Value Efficiency Analysis method (Halme et al., 1999; Halme and Korhonen; 2000; Korhonen et al., 2002) which makes inferences about the DMUs' value efficiencies with the help of an implicit value function. Recent advances at the juncture of DEA and preference modeling include models based on the explicit construction of the decision maker's (DM) value function (Gouveia et al., 2008) and the specification of context-sensitive assurance regions for input/output weights (Cook and Zhu, 2008).

Higher education is an attractive domain for DEA because universities consume several inputs and produce multiple outputs (e.g., degrees, research articles) to which prices may be difficult to attach. In consequence, DEA has been employed extensively at various levels in higher education by treating universities, departments, research units or even students as 'units of analysis'.

For example, Ahn et al. (1988) analyze the production behavior of higher education institutions and compare the relative efficiencies of public and private doctoral-granting universities in the US. Athanassopoulos and Shale (1997) compare the relative efficiencies of 45 higher education institutions in the United Kingdom (UK). Johnes (2006a) considers the possibilities and limitations of DEA models in higher education and analyzes more than 100 higher educational institutions in the UK. Based on her work with multi-level modeling–where efficiency scores are established for individual students and at the higher departmental level–she reports that the efficiency scores at the different levels are not necessarily closely correlated (Johnes, 2006bc).

Abbott and Doucouliagos (2003) consider the efficiencies of Australian public universities using input/output measures from 1995, and conclude that the universities operate at a fairly high level of efficiency. Avrikan (2001) analyzes the relative efficiencies of Australian universities using three models that focus on overall performance, delivery of educational performance, and performance on fee-paying enrollments. McMillan and Chan (2006), too, consider Australian universities and determine efficiency scores with DEA and stochastic frontier analysis: specifically, they compare the efficiency *rankings* from these two methods, and observe that there is a relatively high degree of consistency in these rankings. Räty (2002) employs Finnish university data and demonstrates that the use of efficient facets can help identify variables for efficiency measurement.

Focusing on business education, Colbert et al. (2000) determine the relative efficiencies of 24 leading MBA programs in the US. Tauer et al. (2007) examine the efficiencies of the 26 academic departments at Cornell University as a step towards establishing performance targets. Korhonen et al. (2001) apply VEA to determine efficiency scores for research units at the Helsinki School of Economics, and develop an approach for allocating resources to support the attainment of higher aggregate efficiency.

Sarrico and Dyson (2000) illustrate the use of DEA at the University of Warwick as a planning tool which recognizes multiple perspectives into departmental activities and which is also explicitly linked to other techniques such as strategic options formulation. Feng et al. (2004) describe a multi-method approach where the Analytic Hierarchy Process (AHP) is first employed to evaluate the research activities at 26 Chinese universities, whereafter they use DEA to assess the relative efficiencies of the R&D strength of these universities. Kao and Hung (2008) assess the relative efficiencies of the departments at the National Cheng Kung University by defining an assurance region and by categorizing four groups of similar departments through efficiency decomposition and cluster analysis.

Building on the data in *Times Higher Educational Supplement* (THES), Tulkens (2007) evaluates the attractiveness of universities by determining how many other universities a university either dominates or is dominated by using a conservative approach that does not develop an explicit value representation using criterion weights (i.e., university A dominates B if A outperforms B on all the six evaluation criteria in the THES listing). Köksalan et al. (2009), on the other hand, admit weight information and provide optimization formulations for exploring how sensitive the rankings of MBA programmes are to different assumptions about criterion weights. But their approach, too, is formulated only in terms of output variables and is therefore not applicable to efficiency analyses.

3. Comparative Results for Ratio-Based Efficiency Analysis

3.1. Efficiency Ratios

There are K DMUs which consume M inputs and produce N outputs. The k-th DMU (DMU$_k$ for short) consumes $x_{mk} \geq 0$ units of the m-th input and produces $y_{nk} \geq 0$ units of the n-th output. The input consumption and output production vectors are $x_k = (x_{1k}, \ldots, x_{Mk})^T$ and $y_k = (y_{1k}, \ldots, y_{Nk})^T$, respectively.

Preference information about the relative values of inputs and outputs is captured by non-negative weights $v = (v_1, \ldots, v_M)^T$ and $u = (u_1, \ldots, u_N)^T$, respectively. These weights are assumed to satisfy homogeneous linear constraints (cf. Podinovski, 2001, 2005)

$$S_u = \{u = (u_1, \ldots, u_N) \neq 0 \mid A_u u \leq 0\} \tag{1}$$

$$S_v = \{v = (v_1, \ldots, v_M) \neq 0 \mid A_v v \leq 0\}, \tag{2}$$

where A_u, A_v are coefficient matrices derived from the DM's preference statements about the relative values of inputs and outputs.

For any feasible input weights $v \in S_v$, the virtual input of DMU$_k$ is $v^T x_k = \sum_{m=1}^{M} v_m x_{mk}$. Similarly, the virtual output for $u \in S_u$ is $u^T y_k = \sum_{n=1}^{N} u_n y_{nk}$. We assume that the virtual inputs and the virtual outputs are strictly positive for all feasible weights (i.e., $\sum_m v_m x_{mk} > 0 \ \forall \ v \in S_v$ and $\sum_n u_n y_{nk} > 0 \ \forall \ u \in S_u$ for all $k = 1, \ldots, K$). This assumption holds, for example, if inputs and outputs have strictly positive weights, and if there is at least one input (output) that is consumed (produced) by every DMU. It also holds if all DMUs consume/produce some positive amounts of all inputs/outputs. The assumption of positive virtual inputs/outputs ensures that the (absolute) efficiency ratio (cf. Podinovski, 2001) of DMU$_k$, defined as

$$E_k(u, v) = \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}}, \tag{3}$$

is well-defined for any $u \in S_u, v \in S_v$ (see also Dyson et al., 2001).

3.2. Efficiency Rankings

The DMUs' efficiency ratios (3) depend on input and output weights. For any feasible weights, the DMUs can be ranked based on their efficiency ratios. As the weights assume different values in their respective feasible sets, the resulting rankings may change relative to each other.

In this setting, we first determine the best (=smallest) efficiency ranking that a DMU can attain relative to the other DMUs over the set of input/output weights. Similarly, we also compute the worst (=largest) ranking. These two bounds establish a *ranking interval* which conveys information about the relative efficiencies of the DMUs.

Towards this end, we define the sets

$$R_k^{>}(u,v) = \{l \in \{1\ldots,K\} \mid E_l(u,v) > E_k(u,v)\}$$

$$R_k^{\geq}(u,v) = \{l \in \{1\ldots,K\} \mid E_l(u,v) \geq E_k(u,v),\ l \neq k\},$$

which contain the indexes of DMUs whose efficiency ratios are either strictly higher than that of DMU$_k$ (for $R_k^{>}(u,v)$), or at least as high as that of DMU$_k$ (for $R_k^{\geq}(u,v)$). By construction, $R_k^{>}(u,v) \subseteq R_k^{\geq}(u,v)$.

The corresponding efficiency rankings are defined as $r_k^{>}(u,v) = 1 + |R_k^{>}(u,v)|$ and $r_k^{\geq}(u,v) = 1 + |R_k^{\geq}(u,v)|$ (here, $|R|$ denotes the cardinality of the set R). For example, if the efficiency ratio of DMU$_k$ is strictly higher than the efficiency ratios of other DMUs at $(u,v) \in (S_u, S_v)$, both efficiency rankings are equal one, because $R_k^{>}(u,v) = R_k^{\geq}(u,v) = \emptyset$. Yet these rankings treat ties differently: for if exactly two DMUs have same highest efficiency ratio at $(u',v') \in (S_u, S_v)$, then $r^{>}(u',v')$ ranks them both as first, but $r^{\geq}(u',v')$ ranks them as second.

Formally, the *ranking interval* for DMU$_k$ is defined as $[r_k^{min}, r_k^{max}]$ where the minimum and maximum rankings for DMU$_k$ are given by

$$r_k^{min} = \min_{u,v} r_k^{>}(u,v)$$

$$r_k^{max} = \max_{u,v} r_k^{\geq}(u,v),$$

where the optimization problems are solved over $(u,v) \in (S_u, S_v)$. Both optimum solutions exist, because $r_k^{>}(u,v)$ and $r_k^{\geq}(u,v)$ assume values in the set $\{1,\ldots,K\}$.

From Theorems 1 and 2, the ranking interval $[r_k^{min}, r_k^{max}]$ can be determined from linear programming problems where the feasible weight sets are closed and bounded due to constraints (6) and (9). All proofs are in the Appendix.

Theorem 1. *Consider the minimization problem*

$$\min_{u,v,z} \quad 1 + \sum_{l \neq k} z_l \tag{4}$$

$$\text{subject to} \quad \sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} + C z_l, \qquad l \neq k \tag{5}$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk} = 1 \tag{6}$$

$$z_l \in \{0,1\}, \quad l \neq k$$

$$(u,v) \in (S_u, S_v),$$

where C is a large positive constant. Then the solution to (4) is r_k^{min}, the best (smallest) efficiency ranking of DMU_k.

Theorem 2. *Consider the maximization problem*

$$\max_{u,v,z} \quad 1 + \sum_{l \neq k} z_l \tag{7}$$

$$\text{subject to} \quad \sum_m v_m x_{ml} \leq \sum_n u_n y_{nl} + C(1 - z_l), \qquad l \neq k \tag{8}$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk} = 1 \tag{9}$$

$$z_l \in \{0,1\}, \quad l \neq k$$

$$(u,v) \in (S_u, S_v),$$

where C is a large positive constant. Then the solution to (7) is equal to r_k^{max}, the worst (largest) efficiency ranking of DMU_k.

In general, those DMUs whose input/output profiles differ considerably from what is consumed/produced by all DMUs on the average are more likely to have large ranking intervals, because these DMUs may achieve very high or low rankings for feasible weights that correspond to the extreme points of S_u and S_v. Conversely, the DMUs which are close to the average input/outputs may have narrower ranking intervals.

3.3. Efficiency Dominance

Although ranking intervals provide information about the relative efficiencies of the DMUs, they are not very well suited for the comparison of *pairs* of DMUs. For instance, even if two DMUs have overlapping ranking intervals, it is possible that one of them has a higher efficiency ratio (3) for all feasible input/output weights.

To compare the efficiency ratios of DMUs on a one-on-one basis, we build on concepts from preference programming (see, e.g., Salo and Hämäläinen, 1992, 2001) and define *efficiency dominance* between DMUs as follows.

Definition 1. DMU_k dominates DMU_l (denoted by $\text{DMU}_k \succ \text{DMU}_l$) if and only if

$$E_k(u,v) \geq E_l(u,v) \quad \text{for all } (u,v) \in (S_u, S_v) \tag{10}$$

$$E_k(u,v) > E_l(u,v) \quad \text{for some } (u,v) \in (S_u, S_v). \tag{11}$$

If $\text{DMU}_k \succ \text{DMU}_l$, the efficiency ratio of DMU_k is at least as high as that of DMU_l for all feasible weights and, moreover, there exist some weights for which its efficiency is strictly higher. Thus, if the inequalities (10)–(11) hold, DMU_k can be incontestably regarded as more efficient than DMU_l.

By construction, Definition 1 establishes an irreflexive, asymmetric, antisymmetric and transitive binary relation among the DMUs. This relation, however, is not necessarily total (i.e., it may be that neither $\text{DMU}_k \succ \text{DMU}_l$ nor $\text{DMU}_l \succ \text{DMU}_k$).

The dominance relation in Definition 1 can be determined by examining the efficiency ratio

$$D_{k,l}(u,v) = \frac{E_k(u,v)}{E_l(u,v)}. \tag{12}$$

By the following Lemma 1, this ratio (12) is invariant subject to multiplication of input/output weights by positive constants.

Lemma 1. *Take any $(u,v) \in (S_u, S_v)$ and let (u',v') be vectors that are obtained from (u,v) by multiplying them componentwise so that $u' = c_u v, v' = c_v v$ for some $c_u > 0,\ c_v > 0$. Then $(u',v') \in (S_u, S_v)$ and $D_{k,l}(u,v) = D_{k,l}(u',v')$.*

By Lemma 1, the ratio (12) remains invariant even if normalization constraints (such as $\sum_n u_n = 1$ and $\sum_m v_m = 1$) are imposed on input/output weights. After the introduction of these normalization constraints, the feasible sets S_u and S_v become closed and bounded. Because the ratio $D_{k,l}(u,v)$ is continuous in its arguments, the ratio (12) reaches its maximum and minimum.

A concern in the optimization of the relative efficiency ratio (12) is that this ratio is nonlinear in u and v. We thus establish Theorem 3 which allows the dominance structures to be computed using linear programming.

Theorem 3. *Consider the maximization/minimization problems*

$$\max/\min_{u,v} \quad \sum_n u_n y_{nk} \tag{13}$$

$$\text{subject to} \quad \sum_n u_n y_{nl} = \sum_m v_m x_{ml} \tag{14}$$

$$\sum_m v_m x_{mk} = 1 \tag{15}$$

$$(u,v) \in (S_u, S_v) \tag{16}$$

Then the maximum/minimum of $D_{k,l}(u,v) = E_k(u,v)/E_l(u,v)$ in (12) *over (S_u, S_v) is equal to the maximum/minimum of* (13) *subject to constraints* (14)-(16).

Specifically, if the minimum of (12), denoted by $\underline{D}_{k,l}$, is less than one, DMU$_k$ does not dominate DMU$_l$. If the minimum is greater than one, then dominance holds. Finally, if the minimum is one, the sufficiency condition (11) can be checked by computing the maximum of (13) subject to (14)–(16). If the resulting maximum, denoted by $\overline{D}_{k,l}$, is greater than one, dominance does hold; but if not, then DMU$_k$ and DMU$_l$ have the same efficiency ratio (3) for all feasible weights and dominance does not hold. In practice, the properties of $\succ$ (transitivity, asymmetricity) can be exploited to reduce the number of pairs for which this relation needs to be explicitly computed.

A DMU need not be dominated by a DMU which has a higher DEA efficiency score. For example, consider three DMUs A, B and C which all consume one unit of a single input and produce two outputs so that $A = (1,3), B = (2,1), C = (3,1)$. For the weight information $1/3u_1 \le u_2 \le 3u_1$, there are two efficient DMUs, A and C. Yet, for the weight vector $(u_1, u_2) = (0.75, 0.25)$, the virtual output of DMU$_B$ (i.e., $0.75 \times 2 + 0.25 \times 1 = 1.75$) is higher than that of DMU$_A$ (i.e., $0.75 \times 1 + 0.25 \times 3 = 1.50$) so that DMU$_B$ is not dominated by DMU$_A$.

3.4. Efficiency Differences among DMUs

The maximization and minimization problems in Theorem 3 provide information about much more (or less) efficient DMU$_k$ can be relative to DMU$_l$. For example, if $\overline{D}_{k,l} = 1.42$, the efficiency ratio of DMU$_k$ can be *at most* 42 % greater than that of DMU$_l$. Conversely, if $\underline{D}_{k,l} = 1.10$, then the efficiency ratio of DMU$_k$ will be *at least* 10 % higher than that of DMU$_l$; and if $\underline{D}_{k,l} = 0.80$, the efficiency ratio of DMU$_k$ will be *at least* 80 % of the efficiency ratio of DMU$_l$. In this way, the efficiency bounds $\overline{D}_{k,l}, \underline{D}_{k,l}$ provide information about the efficiency *differences* between the DMUs.

The consideration of pairwise ratios can be extended to situations where the efficiency DMU$_k$ is compared concurrently with *several* DMUs in $L \in \{1, \dots, K\}, k \notin L$. Specifically, the ratios

$$D_{k,\overline{L}}(u,v) = \frac{E_k(u,v)}{\max_{l \in L} E_l(u,v)} = \min_{l \in L} \frac{E_k(u,v)}{E_l(u,v)} \tag{17}$$

$$D_{k,\underline{L}}(u,v) = \frac{E_k(u,v)}{\min_{l \in L} E_l(u,v)} = \max_{l \in L} \frac{E_k(u,v)}{E_l(u,v)} \tag{18}$$

indicate how efficient DMU$_k$ is in comparison with the highest and lowest efficiency ratios of DMU$_l, l \in L$ for different input/output weights. By Theorems 4 and 5, the maximum of (17) and the minimum of (18) can be obtained from the following linear programs.

THEOREM 4. *The maximum of* (17) *over feasible input/output weights is the optimum value of*

$$\max_{u,v} \quad \sum_n u_n y_{nk} \tag{19}$$

$$\text{subject to} \quad \sum_n u_n y_{nl} \le \sum_m v_m x_{ml}, \qquad l \in L \tag{20}$$

$$\sum_n v_m x_{mk} = 1$$

$$(u,v) \in (S_u, S_v).$$

THEOREM 5. *The minimum of* (18) *over feasible input/output weights is the optimum value of*

$$\min_{u,v} \quad \sum_n u_n y_{nk} \tag{21}$$

$$\text{subject to} \quad \sum_n u_n y_{nl} \geq \sum_m v_m x_{ml}, \qquad l \in L \tag{22}$$

$$\sum_n v_m x_{mk} = 1$$

$$(u,v) \in (S_u, S_v).$$

In principle, one may also be interested in the *highest* efficiency ratio of DMU_k relative to the *smallest* of the efficiency ratios of $\text{DMU}_l, l \in L$, or the *smallest* efficiency ratio of DMU_k relative to the *highest* of the efficiency ratios of $\text{DMU}_l, l \in L$. These bounds can be determined by inspection and the repeated application of Theorem 3, because

$$\begin{aligned} \max_{u,v} D_{k,\underline{L}}(u,v) &= \max_{u,v} \frac{E_k(u,v)}{\min_{l \in L} E_l(u,v)} = \max_{u,v} \max_{l \in L} \frac{E_k(u,v)}{E_l(u,v)} \\ &= \max_{l \in L} \max_{u,v} D_{k,l}(u,v) = \max_{l \in L} \overline{D}_{k,l}(u,v) \end{aligned}$$

and

$$\begin{aligned} \min_{u,v} D_{k,\overline{L}}(u,v) &= \min_{u,v} \frac{E_k(u,v)}{\max_{l \in L} E_l(u,v)} = \min_{u,v} \min_{l \in L} \frac{E_k(u,v)}{E_l(u,v)} \\ &= \min_{l \in L} \min_{u,v} D_{k,l}(u,v) = \min_{l \in L} \underline{D}_{k,l}(u,v). \end{aligned}$$

3.5. Specification of Performance Targets

The above efficiency results can be employed to specify different kinds of performance targets. For example, based on efficiency rankings, one can introduce targets such that DMU_k will belong to (i) the $R_k^* \ (< r_k^{\min})$ best DMUs for *some* weights or (ii) the $R_k^\circ \ (< r_k^{\max})$ most efficient DMUs for *all* weights. The following Theorems address these two specific cases when these targets are to be attained through radial increases in the production of outputs.

THEOREM 6. *The minimization problem*

$$\min_{u,v,z} \quad \zeta \tag{23}$$

$$\text{subject to} \quad 1+\sum_{l\neq k} z_l \leq R_k^* \tag{24}$$

$$\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} + C z_l, \qquad l \neq k \tag{25}$$

$$\sum_n u_n y_{nk} = \zeta \tag{26}$$

$$\sum_m v_m x_{mk} = 1$$

$$z_l \in \{0,1\}, \quad l \neq k$$

$$(u,v) \in (S_u, S_v)$$

(where C is a large positive constant) has a solution $\zeta^* > 1$ which gives the least radial increase in outputs that improves the best possible ranking of DMU$_k$ from $r_k^{\min}$ to R_k^* $(< r_k^{\min})$.

THEOREM 7. *The minimization problem*

$$\min_{u,v,z} \quad \zeta \tag{27}$$

$$\text{subject to} \quad 1+\sum_{l\neq k} z_l \leq R_k^\circ$$

$$\sum_m v_m x_{ml} \leq \sum_n u_n y_{nl} + C(1-z_l), \qquad l \neq k,$$

$$\sum_n u_n y_{nk} = \zeta$$

$$\sum_m v_m x_{mk} = 1$$

$$z_l \in \{0,1\}, \quad l \neq k$$

$$(u,v) \in (S_u, S_v).$$

(where C is a large positive constant) has a solution $\zeta^* > 1$ which gives the least radial increase in outputs that improve the worst possible ranking of DMU$_k$ from $r_k^{\max}$ to R_k° $(< r_k^{\max})$.

These two kinds of performance targets can be imposed simultaneously by introducing relevant constraints from Theorems 6 and 7.

Even dominance structures can be employed in the specification of performance targets. First, we ask how much more a DMU$_k$ that *does* not dominate DMU$_l$ should produce in order to achieve a point where it starts to dominate DMU$_l$. In considering this question, we may assume that

$\underline{D}_{k,l} < 1$ in Theorem 3 (otherwise, we have $1 = \underline{D}_{k,l} = \overline{D}_{k,l}$ so that the efficiency ratios of both DMUs coincide for all feasible weights and thus any marginal improvement will suffice). When $\underline{D}_{k,l} < 1$, the efficiency ratio of the revised DMU$_k$ becomes at least as high as that of DMU$_l$ for all feasible weights when DMU$_k$ increases the production of its outputs by a factor of $\zeta_l^* = 1/\underline{D}_{k,l}$. If this necessary condition for dominance is required to hold for several DMUs, the required increase will be $\zeta^* = \max_{l \in L} \zeta_l^*$ where L contains the indexes of these other DMUs. Second, we may ask how much more a DMU$_k$ that *is* dominated by DMU$_l$ should produce to achieve the point where it is no longer necessarily dominated by DMU$_l$. In this case $1 \leq \underline{D}_{l,k} \leq \overline{D}_{l,k}$ with at least with one strict inequality. It follows that DMU$_k$ must increase the production of its outputs at least by a factor of $\zeta^* = \underline{D}_{l,k}$, or else it will continue to be dominated by DMU$_l$. If the existing dominance of DMU$_k$ by DMU$_l$, $l \in L$ is to be eliminated (so that no DMU$_l$ no longer dominates DMU$_k$), then the required increase will be at least $\zeta^* = \max_{l \in L} \underline{D}_{l,k}$.

The results in Section 3.4, too, can be used for setting performance targets. For instance, assume that the maximum of Theorem 4 is $\overline{D}_{k,\overline{L}}$ and the target is to increase the efficiency of DMU$_k$ so that for some weights its efficiency ratio becomes $\overline{\rho}$ times greater than the highest of the efficiency ratios of DMU$_l$, $l \in L$. Then $\overline{\zeta} = \overline{\rho}/\overline{D}_{k,\overline{L}}$ is the least radial increase in the production of outputs through which this target can be reached. Similarly, given the minimum $\underline{D}_{k,\underline{L}}$ in Theorem 5, the efficiency of DMU$_k$ can be improved so as to ensure that for all feasible weights its efficiency ratio will be at least $\underline{\rho}$ times greater than the smallest efficiency ratios among DMU$_l$, $l \in L$. This target can be attained by increasing the production of outputs a factor of $\underline{\zeta} = \underline{\rho}/\underline{D}_{k,\underline{L}}$.

In terms of efficiency implications, an increase in the production of outputs by a factor of $\zeta > 1$ corresponds to a decrease in the use of inputs by a factor of $1/\zeta < 1$, because

$$\frac{\sum_n u_n [\zeta y_{nk}]}{\sum_m v_m x_{mk}} = \frac{\sum_n u_n y_{nk}}{\sum_m v_m [1/\zeta] x_{mk}}.$$

As a result, radial targets on the output side can be easily mapped into corresponding requirements on the input side. Any such targets ζ^* can also be factored into radial targets ζ_u and ζ_v which are applied through $y'_{nk} = \zeta_u y_{nk}$ and $x'_{mk} = [1/\zeta_v] x_{mk}$ subject to the constraint $\zeta_u \zeta_v = \zeta^*$. Furthermore,

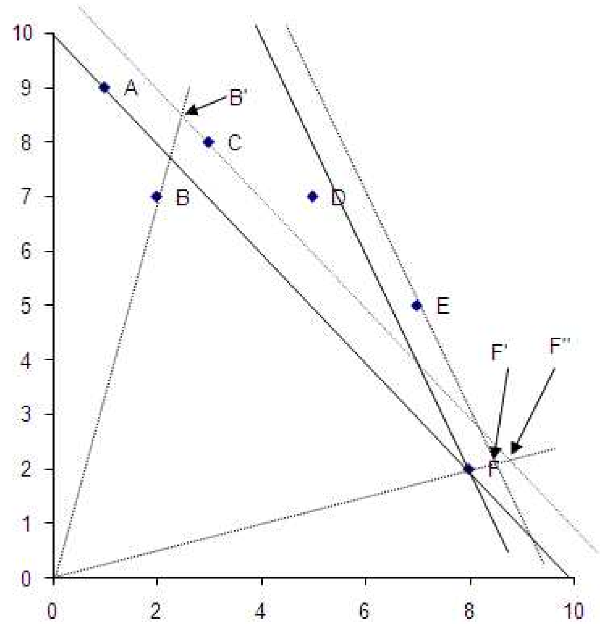

Figure 1 Output vectors of the six DMUs.

it is possible to introduce non-radial performance targets as well by constraining slack variables so that the targeted changes in ranking intervals, dominance relations or efficiency differences will be achieved.

3.6. Illustrative Examples

Consider the six DMUs in Figure 1 which consume one unit of a single input and produce two outputs such that $A = (1,9); B = (2,7); C = (3,8); D = (5,7); E = (7,5); F = (8,2)$. If the DM states that the (unit) value of the first output variable is greater than that of the second, but no more than two times as valuable, the set of feasible output weights becomes $S_u = \{(u_1, u_2) \neq (0,0) \mid u_2 \leq u_1 \leq 2u_2, u_i \geq 0, i = 1,2\}$ which is shown in Figure 1 and 2, together with the DMUs outputs. The DMUs' CCR-DEA efficiency scores are $(0.83, 0.75, 0.92, 1.00, 1.00, 0.95)$.

The DMUs' ranking intervals can be computed from Theorems 1 and 2 while the dominance structure for the DMUs can be established using Theorem 3. The results are shown in Figure 3 where the arrows indicate dominance relationships and the ranking intervals are shown after then DMUs' labels. Here, for example, the ranking interval [2,5] of DMU$_F$ shows that the relative

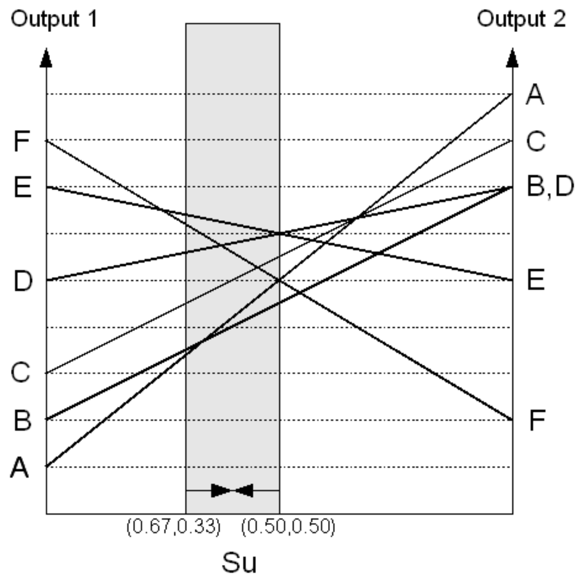

Figure 2 Feasible weights and DMUs' relative efficiencies.

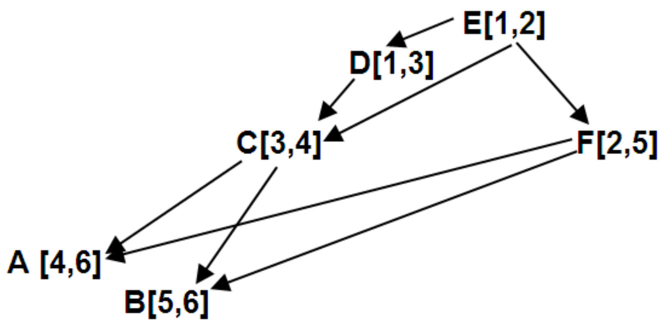

Figure 3 Figure 3. Ranking intervals and efficiency dominances among the DMUs.

efficiency of DMU_F is sensitive to weights. This can be contrasted with DMU_C whose ranking interval [3,4] shows that its efficiency is the third or fourth highest. These ranking intervals and dominance relations can also be read from Figure 2.

Figure 1 also illustrates the use of rankings in the specification of performance targets. For example, if DMU_F is required to become efficient through a proportional increase of all its outputs (which is equivalent to the requirement that it attains the highest ranking $r_F^{min} = 1$ for some feasible weights), it needs to move from F to $F' = (8\frac{4}{9}, 2\frac{1}{9}) \approx (8.44, 2.11)$ where its virtual value achieves that of DMU_E for output weights $(u_1, u_2) \propto (2, 1)$. However, if it is required to improve its performance so that it will be among the three most efficient DMUs for *all* feasible output weights, an even greater improvement to point $F'' = (8\frac{4}{5}, 2\frac{1}{5}) \approx (8.80, 2.20)$ is required. This is because for any radial increase of outputs that falls short of this, the use of equal output weights $(u_1, u_2) \propto (1, 1)$ will assign a higher virtual value to the three DMUs C, D and E, and hence DMU_F will not be among the three most efficient ones. Moreover, if DMU_B is required to improve its efficiency so that it *can* become one of three most efficient ones, it needs to produce more outputs until it reaches the point $B' = (2\frac{4}{9}, 8\frac{5}{9}) \approx (2.44, 8.56)$ where its virtual output achieves that of DMU_C (and remains less than that of DMU_D and DMU_E).

Dominance relations, too, can be employed in target specification. For example, DMU_A–which is dominated by DMU_C–should increase its output production by more than a factor of $(3+8)/(1+9) = 11/10 \approx 1.10$ in order *not* to be dominated DMU_C. On the other hand, the best possible ranking of DMU_F is two. This is better than three, the best possible ranking of DMU_C, yet neither one dominates the other. If DMU_C now seeks to dominate DMU_F, it needs to increase its output production by a factor of $(2\times 8 + 1\times 2)/(2\times 3 + 1\times 8) = 18/14 \approx 1.29$.

From the management perspective, an important benefit of these concepts is that at best they capture forceful verbal statements that correspond to realistic yet challenging targets. For example, the DMU could be required to it best possible ranking so that it achieves a position in the top 20% of most efficient DMU; or that it will no longer be dominated by some ‘rival’ DMUs. In contrast, the usual DEA projection of DMUs onto the efficient frontier may call for unrealistically large performance improvements, at least for the most inefficient DMUs. It may also be more difficult to communicate in non-technical simple terms what such projections mean.

4. A Case Study on the Comparison of University Departments

We illustrate the use of the preceding efficiency results by reporting a case study at a large Technical University with well over 10 000 students. At the time of the study, the University had twelve departments responsible for research activities and degree programmes of which most were in engineering and technical sciences, but also in fields such as architecture and production economics. The efficiency results were produced for the twelve departments. Administrative units, support functions, and separate research institutes without responsibilities for degree programmes were not considered.

A major impetus for the study came from the Board of the University which requested that the Resource Committee–which has the remit to develop principles for resource allocation within the University–consider alternative models for efficiency analysis and resource allocation. Further to this request, the pilot study was carried out in two phases. The initial results were first presented to the Board and, based on positive feedback, further analyses were conducted three months later using a more comprehensive set of preference statements supplied by the ten members of the Resource Committee.

The outputs were defined as three-year departmental averages in the University's reporting system which contained 44 outputs, structured under seven classes (Degrees and credits awarded/ International publications/ Domestic publications/ International mobility of staff/ Other international scientific activities/ Other domestic scientific activities/ Student exchanges). Statements about the relative values of these outputs were elicited from the Resource Committee members with a spreadsheet tool. For each of the seven output classes, the tool first assigned 10 points to the first (reference) output in a class (e.g., MSc degree), and then asked the respondents to give a number of points to the other outputs in the same class. Thus, for example, giving 80 points to a PhD degree would correspond to the preference statement that a PhD degree is eight times as valuable as an MSc degree. Second, the respondents were asked to compare these seven reference outputs through similar point allocations. Finally, the corresponding vector of relative weights was derived for each respondent. Based on this weight vector, the tool also showed for every output class its relative share of the University's total virtual output, computed by valuing all the outputs of the University using on this weight vector.

The two input variables were basic funding (which is provided by the Government and allocated to the Departments by the Rector) and external funding (which is acquired by the research groups

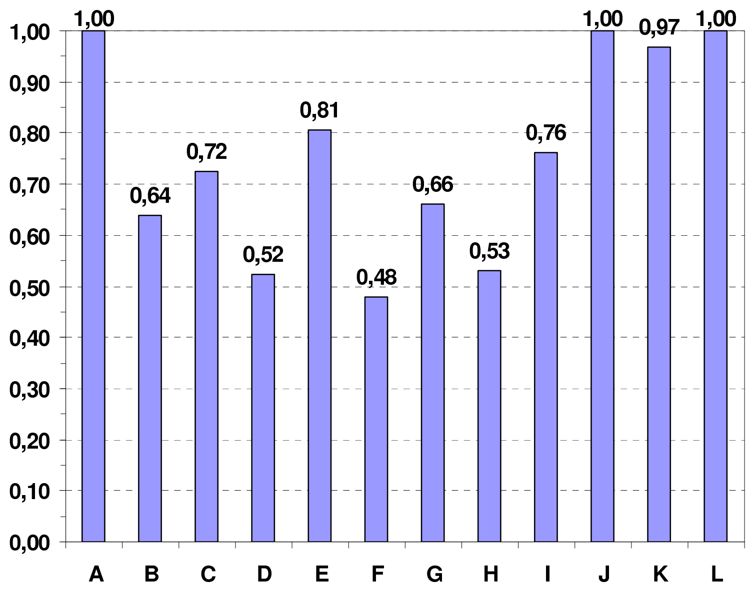

Figure 4 Efficiency scores for the twelve departments.

from external funding sources). The choice of these inputs was motivated by the recognition that (i) most other inputs (e.g., annual person-years, office space) are ultimately financed through these two sources of funding, and (ii) information about these two inputs was readily available. Because the management of external funding involves extra workload, and because this funding also puts additional constraints on how it is used, the respondents were asked to state how much more 'valuable' basic funding would be in comparison with external funding. Here, most respondents reported that basic funding would be 1.25-2.00 times as valuable as funding from external sources (e.g., a $100,000 of basic funding would have the same value of $125,000-200,000 of external funding).

Figure 4 shows the usual CCR-DEA scores using a feasible weight set that was formed from the respondents' weight vectors and their convex combinations. Among the twelve Departments, three are efficient (A, J and L), followed by the 'nearly' efficient Department K (with an efficiency score of 0.97), four Departments with efficiency scores in the range 0.60-0.90, and, finally, the three least efficient ones with efficiency scores less than 0.60.

The good performance of Departments J, K and L is confirmed by Figure 5 which shows the *ranking intervals* of the twelve Departments. For example, the efficiency ratio of L is among the three highest ones for *all* feasible weights while the efficiency ratios of J and K are among the top

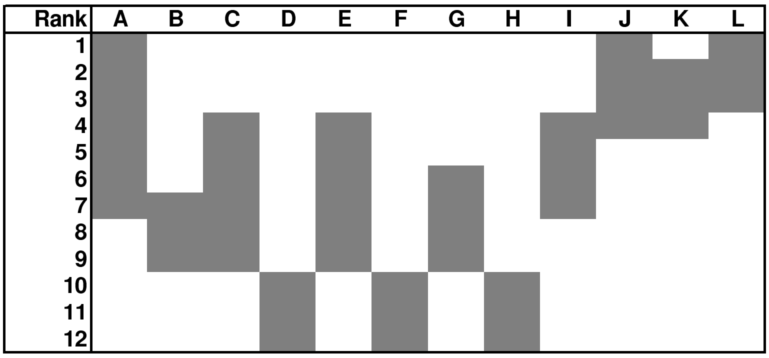

Figure 5 Best and worst efficiency rankings for the departments.

four. Although A is efficient, its efficiency ranking may be as low as 7 for some weights, indicating that its efficiency ratio is sensitive to the choice of weights. Figure 5 also shows that Departments D, F and H have the three smallest efficiency ratios regardless of weights.

Results about efficiency dominance relations can be inferred from Figure 6. Here, the cells contain minimum *lower* bounds on how much larger the efficiency ratios of the row-DMUs are in comparison with the column-DMUs, based on the minimization problem (13). For example, although Department K is not efficient (in the CCR-DEA sense), it is dominated by Department L only. An examination of the K-column also shows that K is *not* dominated by the efficient Departments A and J; thus, for example, there exist some weights for which the efficiency ratio of K is higher than that of A (and possibly some other weights for which the same is true when comparing K with J). Out of the three efficient Departments, A dominates the fewest number of other Departments (five, as opposed to eight for J and nine for L). Even the 'almost-efficient' Department K dominates more other Departments than A; these results could be readily explained by the untypical teaching and research profile of Department A. The indicated percentages, too, were instructive because they provided information about the relative efficiency differences among the Departments: for example, the 36,4 % for the pairwise comparison of A over D indicated that for all feasible weights, the efficiency ratio of DMU_A was *at least* this much higher than that of DMU_D.

During the examination of results, the question was raised if the definition of feasible weights as

	A	B	C	D	E	F	G	H	I	J	K	L
A		4,9 %	*	36,4 %	*	32,3 %	5,4 %	43,9 %	*	*	*	*
B	*		*	15,7 %	*	18,0 %	*	7,2 %	*	*	*	*
C	*	*		19,6 %	*	48,5 %	*	23,3 %	*	*	*	*
D	*	*	*		*	*	*	*	*	*	*	*
E	*	*	*	22,8 %		19,1 %	*	29,5 %	*	*	*	*
F	*	*	*	*	*		*	*	*	*	*	*
G	*	*	*	7,7 %	*	19,4 %		3,1 %	*	*	*	*
H	*	*	*	*	*	*	*		*	*	*	*
I	*	1,6 %	*	26,0 %	*	48,3 %	*	31,5 %		*	*	*
J	*	27,5 %	19,3 %	70,6 %	22,6 %	83,9 %	39,8 %	69,0 %	11,7 %		*	*
K	*	30,7 %	8,8 %	64,2 %	5,9 %	67,7 %	20,5 %	58,4 %	10,7 %	*		*
L	*	55,9 %	35,9 %	88,2 %	16,2 %	108,6 %	48,4 %	85,5 %	31,5 %	*	3,3 %	

Figure 6 Efficiency dominances among the departments.

convex combinations of the respondents' individual weights had permitted opportunistic behavior where some respondents–as representatives of their Departments–would give weights with the aim of maximizing the relative efficiencies of their own Departments. To discourage such behavior, the model was extended by putting upper bounds $\lambda_i \leq 0.5$ on the coefficients λ_i with which the respondents' normalized weights u^i were aggregated. This resulted in the reduced feasible weight set $S_u = \{u \mid u = \sum_i \lambda_i u^i, 0 \leq \lambda_i \leq 0.5, \sum_i \lambda_i = 1\}$ where the 'weight' of the i-th respondent's individual weights u^i could not exceed 50 % of the total; however, the efficiency results did not change noticeably. Yet, there may exist situations where such constraints can be used to ensure that the preferences of the key DMs are fully accounted for (for example, a minimum of 50 % could be ascribed to the coefficient of the Rector's weight vector).

The results catalyzed an informed discussion in the Committee and gave insights into why some departments were more efficient than others. In considering the limitations of the model, it was noted that the weighting of outputs can be problematic due to interdependencies between the different phases in educational processes. Such interdependencies arise, for example, when some departments produce 'final' outputs (e.g., PhD degrees) further down in the 'value chain' by building on intermediate outputs produced by other departments (e.g., introductory courses on mathematics). But because these intermediate outputs may not receive high weights, departments that produce them may appear less efficient than those that focus on the more highly valued final outputs–even if the efficiency of the University *as a whole* might decline unless sufficient resources are given to the production of intermediate outputs.

Second, the assignment of equal weights to the outputs of *different* departments assumes that these outputs are equally valuable. This assignment can be defended on the grounds that the departments are then treated in the same way; but this approach does not recognize that the production of some of the seemingly similar outputs may call for much more resources than others (e.g., articles in experimental physics vs. theoretical physics). Unless such factors are recognized in the interpretation of results, the straightforward use of efficiency analysis for resource allocation may promote the production of outputs that can be generated at a lower cost without ascribing due value to the *diversity* of different outputs (see, e.g., Stirling, 2007). To counter such tendencies, efficiency analyses *within* the university may need to be complemented by efficiency analyses *across* universities where the focus is on the comparison of similar enough 'units of analysis' (e.g., departments in the same scientific discipline at different universities).

Third, universities provide education, produce scientific knowledge and foster innovations through highly non-linear processes. These processes of knowledge production may be most efficient (say, in terms of articles published per person-year of effort) in fields where there already exist well-established scientific communities with specialized journals and a broad readership (which is not the case for new emerging fields that may be struggling to establish new paradigms; Kuhn, 1962). Seen from this perspective, excessive pursuit of demonstrated efficiency may foster "lock-ins" in existing scientific traditions, thus undermining the broader objectives of generating new knowledge.

5. Conclusion

We have developed efficiency results (ranking intervals, dominance structures, and efficiency bounds) which provide comparative information about the DMUs' relative efficiencies as a function of different input and output weights. In comparison with conventional DEA efficiency scores, these efficiency results are more robust, in the sense that they (i) reflect the ranges of DMUs' efficiency ratios for all feasible weights, (ii) tend to be insensitive to the introduction/removal of outlier DMUs, and (iii) do not call for particular returns-to-scale assumptions. These results can be employed for the specification of performance targets, too, for instance by requiring that a DMU will increase its output so as to become one of the three most efficient ones for *some* feasible weights; or that it will be among the five most efficient DMUs for *all* feasible weights.

We have also deployed these efficiency results in the comparison of the departments at a major technical university. The encouraging feedback from this case study suggest that the proposed efficiency results do provide important management insights and that they can meaningfully complement conventional efficiency scores. The usefulness of these efficiency results, however, is by no means limited to the context of higher education: rather, they can be deployed across the full range of decision contexts where DEA-like ratio-based efficiency analyses are being applied, particularly when it is of interest to explore the robustness and sensitivity of the efficiency results subject to alternative assumptions about input and output weights.

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Appendix

Proof of Theorem 1. Let the minimum rank of DMU_k be attained at $(u,v) \in (S_u, S_v)$. Then there exists $L = R_k^{>}(u,v) \subset \{1,\ldots,K\}$ so that $E_l(u,v) > E_k(u,v), l \in L$ and $E_k(u,v) \geq E_l(u,v),\ l \notin$

L. Let $v'_m = v_m/[\sum_m v_m x_{mk}]$ and $u'_n = u_n/[\sum_n u_n y_{nk}]$. Then $(u',v') \in (S_u, S_v)$ and $\sum_m v'_m x_{mk} = \sum_n u'_n y_{nk} = 1$.

For any $l \neq k$, let $z_l = 1$ if $l \in L$, and $z_l = 0$ if $l \notin L$. Then, for any $l \notin L$, we have

$$1 \leq \frac{E_k(u,v)}{E_l(u,v)} = \frac{E_k(u',v')}{E_l(u',v')} = \frac{\sum_m v'_m x_{mk}}{\sum_n u'_n y_{nk}} \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}} = \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}}$$

which gives $\sum_n u'_n y_{nl} \leq \sum_m v'_m x_{ml}$. For $l \in L$, multiplying $z_l = 1$ by the large positive constant C implies that the constraint (5) is satisfied for $l \in L$, too. Because $1 + \sum_{l \neq k} z_l = 1 + |L| = 1 + |R_k^{>}(u,v)| = r$, the solution to the minimization problem is not larger than the minimum rank.

Conversely, let (u,v,z) be a solution to the minimization problem. Let $L = \{l \mid l \neq k, z_l = 1\}$. Then introducing $z_l = 0, l \notin L$ into the first constraint in (5) gives $\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml}$ so that

$$\frac{E_k(u,v)}{E_l(u,v)} = \frac{\sum_m v_m x_{ml}}{\sum_n u_n y_{nl}} \geq 1,$$

because $E_k(u,v) = 1$ due to (6). Thus, any $l \notin L$ cannot belong to $R_k^{>}(u,v)$. For $l \in L$, the inequality $\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} \iff E_k(u,v) \geq E_l(u,v)$ cannot hold, because z is at optimum (otherwise, any such $z_l = 1$ could be changed to $z_l = 0$ without violating (5) while reducing the value of the objective function); hence $l \in L \subseteq R_k^{>}(u,v)$. It follows that $R_k^{>}(u,v) = L$ and $r_k^{\min} \leq 1 + |R_k^{>}(u,v)| = 1 + |L| = 1 + \sum_{l \neq k} z_l$.

Proof of Theorem 2. If the maximum rank of DMU$_k$ is attained at $(u,v) \in (S_u, S_v)$, there exists a subset $L = R_k^{\geq}(u,v) \subset \{1, \ldots, K\}, k \notin L$ such that $E_l(u,v) \geq E_k(u,v), l \in L$ and $E_k(u,v) > E_l(u,v), l \notin L$. If $\sum_j v_m x_{mk} \neq 1$, let $v'_m = v_m/[\sum_j v_m x_{mk}]$ so that $\sum_m v'_m x_{mk} = 1$; and if $\sum_n u_n y_{nk} \neq 1$, put $u'_n = u_n/[\sum_n u_n y_{nk}]$ so that $\sum_n u'_n y_{nk} = 1$.

For any $l \neq k$, let $z_l = 1$ if $l \in L$ and $z_l = 0$ if $l \notin L$. Then, for any $l \in L$,

$$1 \leq \frac{E_l(u,v)}{E_k(u,v)} = \frac{E_l(u',v')}{E_k(u',v')} = \frac{\sum_m u'_n y_{nl}}{\sum_m v'_m x_{ml}} \Rightarrow \sum_m v'_m x_{ml} \leq \sum_n u'_n y_{nl}$$

and thus (8) holds. For $l \notin L$, multiplying $(1 - z_l) = 1$ by the positive constant C implies that (8) is satisfied in this case too. Now, $1 + \sum_{l \neq k} z_l = 1 + |L| = 1 + |R_k^{\geq}(u,v)| = r_k^{\max}$. Thus, the solution to maximization problem is at least as large as the maximum rank.

Conversely, assume that (u,v,z) is a solution to the maximization problem and let $L=\{l \mid l\neq k, z_l=1\}$. For any $l\in L$ with $z_l=1$, the constraint $\sum_m v_m x_{ml}\leq \sum_n u_n y_{nl}$ implies

$$\frac{E_l(u,v)}{E_k(u,v)}=\frac{\sum_n u_n y_{nl}}{\sum_m v_m x_{ml}}\geq 1,$$

because u and v satisfy (8)–(9); thus, $L\subseteq R_k^{\geq}(u,v)$. Because z is at optimum, the inequality $E_k(u,v)\leq E_l(u,v)$ cannot hold for $l\notin L$ (otherwise, any such $z_l=0$ could be changed to $z_l=1$ without violating constraints while increasing the objective function). Thus $R_k^{\geq}(u,v)$ does not contain elements that are outside of L. It follows that $L=R_k^{\geq}(u,v)$ and $r_k^{\max}\geq 1+|R_k^{\geq}(u,v)|=1+\sum_{l\neq k} z_l$.

Proof of Lemma 1. To prove that $u'\in S_u$, note that $u\in S_u$ implies $A_u u\leq 0$ and hence $A_u u'=A_u c_u u=c_u(A_u u)\leq 0$; similarly, $v'\in S_v$. The last claim follows from

$$\begin{aligned}D_{k,l}(u',v') &= \frac{E_k(u',v')}{E_l(u',v')}=\frac{\sum_n u'_n y_{nk}}{\sum_m v'_m x_{mk}}\frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}}\\ &= \frac{c_u\sum_n u_n y_{nk}}{c_v\sum_m v_m x_{mk}}\frac{c_v\sum_m v_m x_{ml}}{c_u\sum_n u_n y_{nl}}=\frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}}\frac{\sum_m v_m x_{ml}}{\sum_n u_n y_{nl}}=D_{k,l}(u,v).\end{aligned}$$

Proof of Theorem 3. Choose $(u^*,v^*)\in(S_u,S_v)$ such that $D_{k,l}(u^*,v^*)\geq D_{k,l}(u,v)\ \forall(u,v)\in(S_u,S_v)$. Define v' so that $v'_m=v^*_m/[\sum_i v^*_i x_{ik}]$. By construction, $v'\in S_v$ and $\sum_m v'_m x_{mk}=1$. Define $u'=c_u u^*\in S_u$ such that $\sum_n u'_n y_{nl}=\sum_m v'_m x_{ml}$ (this is possible, because $\sum_n u^*_n y_{nl}$ and $\sum_m v'_m x_{ml}$ are positive). The weights (u',v') satisfy constraints (14)–(16), while the repeated application of Lemma 1 gives $D_{k,l}(u^*,v^*)=D_{k,l}(u^*,v')=D_{k,l}(u',v')=\sum_n u'_n y_{nk}$, proving that the maximum of (13) over (14)–(16) is at least as high as $D_{k,l}(u^*,v^*)$.

Assume that the maximum of (13) is attained at (u^o,v^o). For these weights $(u^o,v^o)\in(S_u,S_v)$, we have

$$D_{k,l}(u^o,v^o)=\frac{E_k(u^o,v^o)}{E_l(u^o,v^o)}=\frac{\sum_n u^o_n y_{nk}}{\sum_m v^o_m x_{mk}}\frac{\sum_m v^o_m x_{ml}}{\sum_n u^o_n y_{nl}}=\sum_n u^o_n y_{nk},$$

because the weights (u^o,v^o) satisfy (14)–(15). Thus, the maximum of $D_{k,l}(u,v)$ over (S_u,S_v) cannot be smaller than the solution to the maximization problem in Theorem 3. The minimization case can be shown analogously.

Proof of Theorem 4. Let the maximum of (17) be ζ^* so that this optimum is attained at (u^*, v^*). There then exists some $l^* \in L$ such that $E_{l^*}(u^*, v^*) \geq E_l(u^*, v^*)\ \forall\ l \in L$. Choose $v' = v^*/[\sum_m v^* x_{mk}]$ so that $\sum_m v'_m x_{mk} = 1$. Also, choose a constant $c_u > 0$ so that $\sum_n u'_n y_{nl^*} = \sum_m v'_m x_{ml^*}$ for $u' = c_u u^*$. For any $l \in L$, we have

$$1 \geq D_{l,l^*}(u^*, v^*) = D_{l,l^**}(u', v') = \frac{E_l(u', v')}{E_{l^*}(u', v')} = \frac{\sum_n u'_n y_{nl}}{\sum_m v'_m x_{ml}}$$

so that the constraint (20) is satisfied by (u', v'). By construction, $\zeta^* = \max_{u,v} D_{k,\overline{L}}(u, v) = D_{k,l^*}(u', v') = \sum_n u'_n y_{nk}$, which shows that the maximum of (19) is at least as high as that of (17).

Conversely, assume that the maximum of (19) ζ' is attained at (u', v') and choose $l' \in L$ so that the constraint in (20) is binding (such l' exists, for otherwise u' could be increased to improve the value of the objective function, which would be in violation of the optimality assumption). Now,

$$\max_{u,v} D_{k,\overline{L}}(u, v) \geq \frac{E_k(u', v')}{E_{l'}(u', v')} = \zeta'$$

so that the maximum (17) must be at least as high as that of (19).

Proof of Theorem 5. Let the minimum of (18) ζ^* be attained at (u^*, v^*). There then exists some l^* such that $E_{l^*}(u^*, v^*) \leq E_l(u^*, v^*), l \in L$ and $\zeta^* = \min_{u,v} D_{k,\underline{L}}(u, v) = E_k(u^*, v^*)/E_{l^*}(u^*, v^*)$. As in the proof of Theorem 4, use (u^*, v^*) in defining normalized valuation vectors (u', v') such that $\sum_m v'_m x_{mk} = 1$ and $E_{l^*}(u', v') = 1$. The choice of l^* guarantees that $1 \leq E_l(u', v')$ so that first constraint in constraint (22) holds for all $l \in L$. Because

$$\zeta^* = \frac{E_k(u^*, v^*)}{E_{l^*}(u^*, v^*)} = \frac{E_k(u', v')}{E_{l^*}(u', v')} = \sum_n u'_n y_{nk},$$

the minimum to (21) is at least as small as the minimum to (18).

Assume that ζ', the minimum of (21), is obtained at (u', v'). Choose l' such that the constraint in (22) is binding (such l' must exist, for otherwise the assumption of optimality would be violated). Then $E'_l(u', v') = 1$ while constraint (22) implies that $E_l(u', v') \geq 1$ for any other $l \in L$; hence $E'_l(u', v') \leq E_l(u', v')$. It follows that

$$\min_{u,v} D_{k,\underline{L}}(u, v) \leq D_{k,\underline{L}}(u', v') = \frac{E_k(u', v')}{\min_{l \in L} E_l(u', v')} = \frac{E_k(u', v')}{E_{l'}(u', v')} = \zeta',$$

proving that the minimum of (18) is at least as small as the optimum to (21).

Proof of Theorem 6. To prove that the optimum exists, first choose any $(u^\circ, v^\circ) \in (S_u, S_v)$, let $\zeta^\circ = \max_{l \neq k}[\sum_n u^\circ y_{nl}]/[\sum_m v^\circ_m x_{ml}]$ and define the revised outputs of DMU_k by $y'_{nk} = \zeta^\circ y_{nk}$. Then, when y_{nk} is replaced by y'_{nk}, all the constraints in Theorem 6 are satisfied with $z_l = 0, l \neq k$. Also, any feasible solution ζ must be greater than one, because inserting $\zeta = 1$ to (26) would associate an efficiency ratio of one with DMU_k while constraints (24)–(25) would imply that there would be no more than $R^*_k - 1$ other DMUs with a strictly better efficiency ratio, thus contradicting the initial assumption $r^{\min}_k \leq R^*_k$). We may therefore introduce the constraints $1 \leq \zeta \leq \zeta^\circ$ which, when stated as $1 \leq \sum_n u_n y_{nk} \leq \zeta^\circ$, show that the optimum solution is contained in a compact set of weights so that the optimum ζ^* exists.

For the revised $y'_{nk} = \zeta^* y_{nk}$, constraints (24)–(25) imply that there will be no more than $R^*_k - 1$ other DMUs with a strictly better efficiency ratio, and thus the best ranking of the revised DMU_k will be R^*_k or better. Also, for $\zeta'' < \zeta^*$, the optimality of ζ^* would require that that constraint (24) would be violated for any $z_l, l \neq k$ such that constraint (25) is satisfied, which in turn would imply that the ranking R^*_k would not be attained by the revised DMU_k with outputs $y'_{nk} = \zeta'' y_{nk}$.

Proof of Theorem 7. Analogous to the proof of Theorem 6.