On the q-criticality of graphs with respect to secure graph domination

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Abstract

A subset X of the vertex set of a graph G is a *secure dominating set* of G if each vertex of G which is not in X is adjacent to some vertex in X and if, for each vertex u not in X, there is a neighbouring vertex v of u in X such that the swap set $(X - \{v\}) \cup \{u\}$ is again a dominating set of G. The *secure domination number* of G is the cardinality of a smallest secure dominating set of G.

The notion of secure graph domination finds applications in the generic setting where the vertex set of G represents distributed locations in some spatial domain and the edges of G represent links between these locations. Patrolling guards, each stationed at one of these locations, may move along the links in order to protect the graph. A minimum secure dominating set of G then represents a smallest collection of locations at which guards may be stationed so that the entire location complex modelled by G is protected in the sense that if a security concern arises at location u, there will either be a guard stationed at that location who can deal with the problem, or else a guard dealing with the problem from an adjacent location v will still leave the location complex protected after moving from location v to location u in order to deal with the problem.

A graph G is *q-critical* if the smallest arbitrary subset of edges whose removal from G necessarily increases the secure domination number, has cardinality q. The notion of q-criticality is important in applications such as the one mentioned above, because it provides threshold information as to the number of edge failures (perhaps caused by an adversary) that will necessitate the hiring of additional guards to secure the location complex.

Denote by Ω_n the largest value of q for which q-critical graphs of order n exist. It has previously been established that $\Omega_2 = 1$, $\Omega_3 = 2$, $\Omega_4 = 4$, $\Omega_5 = 6$ and $\Omega_6 = 9$. In this paper we present a repository of all q-critical graphs of orders $2, 3, 4, 5$ and 6 for all admissible values of q and we also establish the previously unknown values $\Omega_7 = 12$, $\Omega_8 = 17$ and $\Omega_9 = 23$. These values support an existing conjecture that $\Omega_n = \binom{n}{2} - 2n + 5$ for all $n \geq 7$.

Key words: Secure domination, graph protection, edge criticality.

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1   Introduction

A *dominating set* of a graph G is a subset X of the vertex set of G with the property that each vertex of G not in X is adjacent to at least one vertex in X. A *secure dominating set* of G is a subset X_s of the vertex set of G with the property that X_s forms a dominating set of G and, additionally, for each vertex u not in X_s, there exists a vertex $v \in X_s$ for which the *swap set* $(X_s - \{v\}) \cup \{u\}$ is again a dominating set of G. The *secure domination number* of G, denoted by $\gamma_s(G)$, is the minimum value of $|X_s|$, taken over all secure dominating sets X_s of G (*i.e.* the cardinality of a smallest secure dominating set of G). A number of general bounds have been established for the parameter $\gamma_s(G)$ in [7], and exact values of $\gamma_s(G)$ have also been established for various graph classes, such as paths, cycles, complete multipartite graphs and products of paths and cycles. Various properties of secure dominating sets of graphs have also been studied in [1, 2, 4, 5, 6].

Consider, as an example, the graph G_1 in Figure 1 for which $\gamma_s(G_1) = 2$. A minimum secure dominating set for G_1 is $\{v_1, v_2\}$; vertex v_4 is defended by v_2 while v_3 and v_5 are both defended by v_1.
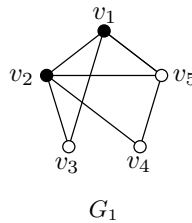
Figure 1:   *A minimum secure dominating set $\{v_1, v_2\}$ for a graph G_1 of order 5.*

The notion of secure graph domination finds applications in the generic setting where the vertex set of G represents a network of distributed locations in some spatial domain and the edges of G represent links between these locations. Patrolling guards, each stationed at one of these locations, may move along the links in order to protect the graph. A minimum secure dominating set of G then represents a smallest collection of locations at which guards may be stationed so that the entire location complex modelled by G is protected in the sense that if a security concern arises at location u, there will either be a guard stationed at that location who can deal with the problem, or else a guard dealing with the problem from an adjacent location v will still leave the location complex protected after moving from location v to location u in order to deal with the problem.

In this setting, the notion of edge removal is important, because one might seek the cost (in terms of the additional number of guards required to protect a location complex modelled by G) if a number of edges in G fail (*i.e.* a number of links are eliminated from the location complex, thereby disqualifying guards from moving along such disabled links).

A graph G is *q-critical* if the smallest arbitrary subset of edges whose removal from G necessarily increases the secure domination number, has cardinality q. Being able to determine the value of q for which a given graph is q-critical is important from an application point of view, because this value may be seen as a robustness threshold in the sense that the failure of some subsets of $q - 1$ edges in G result in graphs that can still be dominated

securely by $\gamma_s(G)$ guards, but this is not true for the failure of q edges in G.

In this paper, we provide empirical evidence in support of a conjecture by Burger *et al.* [3] that the largest value of q for which there exists a graph of order n that is q-critical, is $\binom{n}{2} - 2n + 5$ for all $n \geq 7$. We also provide a repository of all q-critical graphs of order n and size m for all $q \in \{0, 1, \ldots, m\}$ and all $m \in \{1, 2, \ldots, \binom{n}{2}\}$, where $n \in \{2, \ldots, 6\}$.

2 The concept of q-criticality

We denote the set of all non-isomorphic graphs obtained by removing $q \in \{0, 1, \ldots, m\}$ edges from a given graph G of size m by $G - qe$. Furthermore, let $\gamma_s(G - qe)$ denote the set of values of $\gamma_s(H)$ as $H \in G - qe$ varies (for a fixed value of q). We distinguish between the graph obtained by removing a *specific* edge e from a given graph G, by writing $G - e$, and the class of graphs obtained by removing *any* single edge from G, by writing $G - 1e$.

The cost function

$$c_q(G) = \min \gamma_s(G - qe) - \gamma_s(G) \tag{1}$$

is nonnegative and bounded from above by q for all $q \in \{0, 1, \ldots, m\}$ [3]. This cost function measures the *smallest possible* increase in the secure domination number of a member of $G - qe$, relative to the secure domination number of a graph G of size m, when a set of $q \in \{0, 1, \ldots, m\}$ edges are removed from G.

A graph G is *q-critical* if $c_{q-1}(G) = 0$, but $c_q(G) > 0$ (that is, if the smallest arbitrary subset of edges whose removal from G necessarily increases the secure domination number, has cardinality q). The notion of q-criticality partitions the set of all non-isomorphic, nonempty graphs of order n in the sense that any such graph is q-critical for exactly one value of $q \in \{0, 1, 2, \ldots, \binom{n}{2}\}$, as demonstrated in the so-called edge-removal metagraph of the complete graph $\mathcal{K}_n$ of order 4 in Figure 2. The *edge-removal metagraph* of a graph G of size m is a graph whose nodes represent the non-isomorphic members of $G - qe$ for all $q = 0, 1, \ldots, m$. These nodes are arranged in $m+1$ levels, numbered $0, 1, \ldots, m$. The nodes on level q correspond to the members of $G - qe$. A node H on level $q-1$ of this metagraph is joined to a node H' on level q if H' can be obtained by removing one edge from H, for any $q \in \{1, 2, \ldots, m\}$. The only node on level 0 of the edge-removal metagraph of some graph G corresponds to G itself, while the only node on level m corresponds to the empty graph of the same order as G. The edge-removal metagraph of the complete graph $\mathcal{K}_n$ is of particular interest, because it contains nodes corresponding to all the non-isomorphic graphs of order n.

Let $\mathcal{Q}_n^q$ be the class of q-critical graphs of order $n \geq 2$ for some $q \in \{1, \ldots, \binom{n}{2}\}$. Grobler and Mynhardt characterised the graph class $\mathcal{Q}_n^1$ for all $n \in \mathbb{N}$ in 2009 [8, Theorem 2] and used their characterisation to derive a four-step construction process for computing all the members of the class $\mathcal{Q}_n^1$. Because of space constraints we do not give a full description of this (nontrivial) construction process here, but rather refer the reader to [8, Section 3.1] for the details. The following characterisation may be used to compute the class $\mathcal{Q}_n^q$ inductively from the class $\mathcal{Q}_n^{q-1}$ for any integer $n \geq 2$ and all permissible values of $q \geq 2$,

using the above-mentioned 4-step construction process by Mynhardt and Grobler [8] for the class $\mathcal{Q}_n^1$ as base case.

Proposition 1 ([3]) *A graph G of size at least $q > 1$ is q-critical if and only if*
(a) at least one graph $H^* \in G - 1e$ for which $\gamma_s(H^*) = \gamma_s(G)$ is $(q-1)$-critical, and
(b) each graph $H \in G - 1e$ for which $\gamma_s(H) = \gamma_s(G)$ is q^*-critical for some $q^* \leq q - 1$. ∎

The inductive process referred to above is formalised in Algorithm 1. The algorithm commences by considering a graph $H \in \mathcal{Q}_n^{q-1}$ and proceeding to add a single edge $e \notin E(H)$ to H in Step 3, upon which the result of Proposition 1 is used to test whether or not $H + e \in \mathcal{Q}_n^q$. This process is repeated for each edge $e \notin E(H)$ and for each graph $H \in \mathcal{Q}_n^{q-1}$.

In Step 3 of Algorithm 1, another algorithm, Algorithm 2, is called to test whether $G = H + e \in \mathcal{Q}_n^q$. In Algorithm 2, each member of $G - 1e$ is examined. If a member $E \in G - 1e$
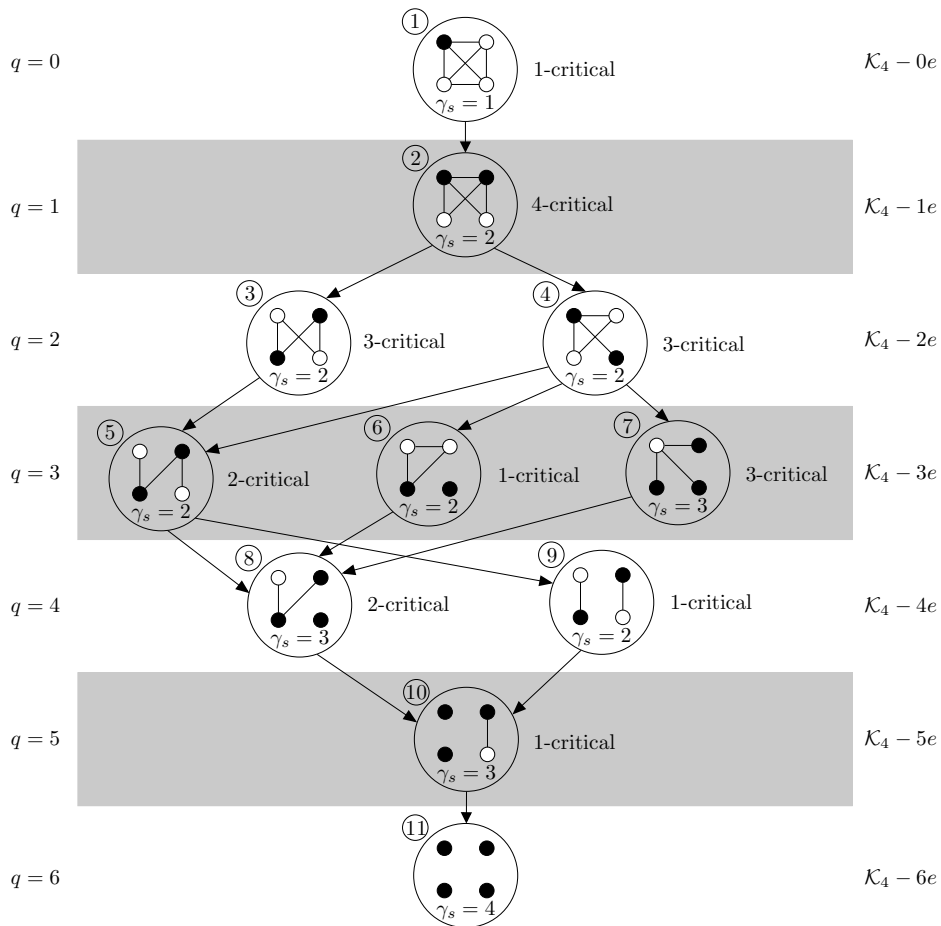
Figure 2: *The edge-removal metagraph of the complete graph $\mathcal{K}_4$ of order 4. The set $\mathcal{K}_4 - qe$ is shown on level q of the graph for all $q = 0, \ldots, 6$. Minimum secure dominating sets of the resulting graphs are denoted by solid vertices in each case. An arrow of the form $G \to H$ from level q to level $q + 1$ means that G is a certificate in $\mathcal{K}_n - qe$ showing that $H \in \mathcal{K}_n - (q+1)e$.*

Algorithm 1: Computing the class of $\mathcal{Q}_n^q$ of q-critical graphs of order n

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Input   : The graph classes $\mathcal{Q}_n^1, \ldots, \mathcal{Q}_n^{q-1}$.
Output : The class $\mathcal{Q}_n^q$ of q critical graphs of order n.

1 **for** *each $H \in \mathcal{Q}_n^{q-1}$* **do**
2   **for** *each $e \notin E(H)$* **do**
3   **if** q-Critical$(H + e, q)$ **then** $\mathcal{Q}_n^q \leftarrow \mathcal{Q}_n^q \cup \{H + e\}$

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is found for which $\gamma_s(E) = \gamma_s(G)$, then $G \notin \mathcal{Q}_n^q$ by Proposition 1. Similarly, if a member $F \in \mathcal{Q}_n^p$ is found for some $p \geq q$, then $G \notin \mathcal{Q}_n^q$ by Proposition 1. If, however, no such graph F is found, then $G \in \mathcal{Q}_n^q$ by Proposition 1, since $H \in \mathcal{Q}_n^{q-1}$.

Algorithm 2: q-Critical(G, q)

Input   : A graph G and the value of q.
Output : A boolean value stating whether G is q-critical.

1 **if** $G \in \mathcal{Q}_n^p$ *for some $p \leq q - 1$* **then**
2   **return** [FALSE]

3 **for** *each $e \in E(G)$* **do**
4   **if** $\gamma_s(G - e) = \gamma_s(G)$ **and** $G - e \notin \mathcal{Q}_n^p$ *for some $p \leq q - 1$* **then**
5   **return** [FALSE]

6 **return** [TRUE]
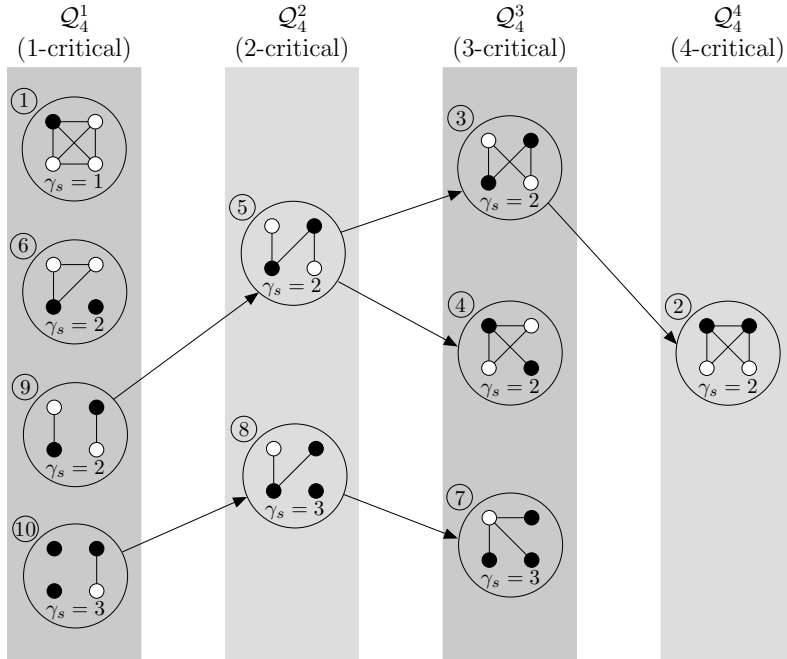
Figure 3:   *The graph classes $\mathcal{Q}_4^1, \mathcal{Q}_4^2, \mathcal{Q}_4^3$ and $\mathcal{Q}_4^4$. Minimum secure dominating sets are denoted by solid vertices in each case. An arrow of the form $H^* \to G$ in the figure denotes the relationship between the graphs G and H^* in Proposition 1.*

The graph classes $\mathcal{Q}_4^1, \ldots, \mathcal{Q}_4^4$ are shown in Figure 3. The classes $\mathcal{Q}_4^5$ and $\mathcal{Q}_4^6$ are both

				Number of q-critical graphs of order n				
$n \rightarrow$	2	3	4	5	6	7	8	9
$\Omega_n \rightarrow$	1	2	4	6	9	12	17	23
$|\mathcal{Q}_n^1|$	1	2	4	7	14	26	52	104
$|\mathcal{Q}_n^2|$		1	2	6	18	50	141	394
$|\mathcal{Q}_n^3|$			3	9	32	111	428	1 514
$|\mathcal{Q}_n^4|$			1	8	34	165	910	4 424
$|\mathcal{Q}_n^5|$				2	28	199	1 484	10 587
$|\mathcal{Q}_n^6|$				1	18	195	1 875	20 144
$|\mathcal{Q}_n^7|$					8	153	2 010	30 849
$|\mathcal{Q}_n^8|$					2	93	1 847	38 831
$|\mathcal{Q}_n^9|$					1	37	1 520	41 620
$|\mathcal{Q}_n^{10}|$						10	1 088	38 341
$|\mathcal{Q}_n^{11}|$						3	627	30 962
$|\mathcal{Q}_n^{12}|$						1	260	22 864
$|\mathcal{Q}_n^{13}|$							76	15 934
$|\mathcal{Q}_n^{14}|$							19	10 053
$|\mathcal{Q}_n^{15}|$							5	5 222
$|\mathcal{Q}_n^{16}|$							2	2 048
$|\mathcal{Q}_n^{17}|$							1	585
$|\mathcal{Q}_n^{18}|$								138
$|\mathcal{Q}_n^{19}|$								34
$|\mathcal{Q}_n^{20}|$								11
$|\mathcal{Q}_n^{21}|$								5
$|\mathcal{Q}_n^{22}|$								2
$|\mathcal{Q}_n^{23}|$								1
Total	1	3	10	33	155	1 043	12 345	274 667
Time	$\ll 1$	$\ll 1$	< 1	2	23	531	27 208	1 069 220

Table 1: *Cardinalities of the graph classes $\mathcal{Q}_n^1, \ldots, \mathcal{Q}_n^{\Omega_n}$ for $n \in \{2, \ldots, 9\}$ as computed on a 3.4 GHz Intel(R) Core(TM) i7-3770 processor with 8 GiB RAM running in* Ubuntu 12.04 *and using a* C++ *implementation of Algorithms 1–2 in conjunction with the* Boost *graph library [11] for graph isomorphism testing. The computation times, shown in the last row, are measured in seconds and represent the time required for determining the graph class $\mathcal{Q}_n^q$ from the graph class $\mathcal{Q}_n^{q-1}$, for all $q \in \{2, \ldots, \Omega_n\}$.*

empty. The 4-step construction of Grobler and Mynhardt [8] was used to compute the class $\mathcal{Q}_n^1$ in the first column of the figure as base case. Thereafter, Algorithm 1 was used to compute the classes $\mathcal{Q}_4^2$, $\mathcal{Q}_4^3$ and $\mathcal{Q}_4^4$ inductively.

Note that it is, in view of Proposition 1 and Algorithms 1–2, not necessary to construct the entire edge-removal metagraph of the complete graph of order n in order to determine the graph class $\mathcal{Q}_n^q$ for a fixed value of q; instead only the classes $\mathcal{Q}_n^1, \ldots, \mathcal{Q}_n^q$ need be constructed inductively which, for values of q that are small compared to $\binom{n}{2}$, can be achieved in a fraction of the time required to construct the entire edge removal metagraph of $\mathcal{K}_n$.

3 Numerical results

Let Ω_n denote the largest value of q for which there exist q-critical graphs of order n. Values of Ω_n have been established for small n. In particular, Burger *et al.* [3] showed that $\Omega_2 = 1$, $\Omega_3 = 2$, $\Omega_4 = 4$, $\Omega_5 = 6$ and $\Omega_6 = 9$. They also conjectured as follows.

Conjecture 1 ([3]) $\Omega_n = \binom{n}{2} - 2n + 5$ *for all* $n \geq 7$.

In this paper we provide further circumstantial evidence in support of Conjecture 1, by proving the conjecture correct for $n \in \{7, 8, 9\}$. In particular, using a C++ implementation of Algorithms 1–2, we confirmed the values of Ω_n for $n \leq 6$ mentioned above, and additionally showed that $\Omega_7 = 12$, $\Omega_8 = 17$ and $\Omega_9 = 23$. The results thus obtained are summarized in Table 1, which contains listings of the cardinalities of the graph classes $\mathcal{Q}_n^q$ for $n \in \{2, \ldots, 9\}$ and $q \in \{1, \ldots, \Omega_n\}$. The classes $\mathcal{Q}_2^1, \ldots, \mathcal{Q}_9^1$ were determined by the 4-step construction process of Mynhardt and Grobler [8], referred to above.

A repository of the members of the graph classes $\mathcal{Q}_n^1, \ldots, \mathcal{Q}_n^{\Omega_n}$ is provided in Table 2 for $n \in \{2, 3, 4, 5, 6\}$. The graphs in this table are presented in the well-known *graph6* format [9], which is ideal for storing class representatives of isomorphism classes of undirected graphs in a compact manner, using only printable ASCII characters. These graphs may be converted to adjacency matrices and other formats using the reader showg, which is available online [9]. The reader showg package is part of nauty, originally designed by McKay and Piperno [10] for graph isomorphism testing.

4 Further work

In addition to attempting a general proof or refutation of Conjecture 1, another interesting problem for future research would be to investigate the *largest* number of arbitrary edges whose removal from a graph *necessarily does not increase* the secure domination number. In this context the cost function

$$C_p(G) = \max \gamma_s(G - pe) - \gamma_s(G)$$

is applicable instead of (1), and a graph G may be defined to be *p-stable* if $C_p(G) = 0$, but $C_{p+1}(G) > 0$. This problem finds application in cases where one seeks threshold information in terms of the largest set of edges whose removal from G does not increase the secure domination number of the resulting graph.

Acknowledgements

Class	Class members										
$\mathcal{Q}_2^1$	A_										
$\mathcal{Q}_3^1$	Bw	B_									
$\mathcal{Q}_3^2$	Bg										
$\mathcal{Q}_4^1$	C~	CJ	CK	C@							
$\mathcal{Q}_4^2$	CL	CB									
$\mathcal{Q}_4^3$	CN	C]	CF								
$\mathcal{Q}_4^4$	C^										
$\mathcal{Q}_5^1$	D~{	DJ[	DBw	DJ_	D@K	D@O	D?C				
$\mathcal{Q}_5^2$	DB{	DFw	DJc	D@S	D@o	D?K					
$\mathcal{Q}_5^3$	DF{	DJk	DK{	DLs	D@[	DBW	D@s	DBg	D?[		
$\mathcal{Q}_5^4$	DJ{	DL{	DNw	DB[	D@{	DBk	DLo	D?{			
$\mathcal{Q}_5^5$	DN{	D]{									
$\mathcal{Q}_5^6$	D^{										
$\mathcal{Q}_6^1$	E~~w	EJ\w	EB^_	E?^o	EJ]?	E@ro	E@Kw	EJaG	E?Lo	E@L?	E?CW	E@Q?
	E?C_	E??G										
$\mathcal{Q}_6^2$	E?^w	E@^o	EJ]G	E?~o	E@rw	E@vo	EBjg	EJaW	E?Lw	E?\o	E?No	E?]o
	E@LG	E@N?	E@QG	E?Cg	E?D_	E??W						
$\mathcal{Q}_6^3$	E@^w	EB^g	EJ]W	E?~w	E@vw	E@~o	EBjw	EBng	EBzg	EBzo	EJfo	EJnO
	EJeW	EJeg	E?\w	E?Nw	E?]w	E@LW	E@Pw	E@Tg	E@NG	E@QW	E@Qo	E@U_
	EBY?	E?Cw	E?Ko	E?Dg	E?LO	E?F_	E?N?	E??w				
$\mathcal{Q}_6^4$	EB^w	EJ]w	E@~w	EBnw	EBzw	EB~o	EJno	EFzg	ELrw	EJew	EJbw	EJfg
	ELv_	E@Lw	E@Tw	E@\o	E@NW	E@UW	E@Qw	E@Ug	E@Uo	E@V_	E@^?	EBYG
	EB]?	EBj?	E?Kw	E?Dw	E?LW	E@T_	E?Fg	E?NG	E?NO	E?@w		
$\mathcal{Q}_6^5$	EJ^w	EB~w	EJ~o	EFzw	EJmw	EJfw	EJnW	EK~o	ELvg	E@\w	EBXw	E@Nw
	E@Uw	E@]o	E@Rw	E@Vg	E@^G	E@^O	EBYW	EB]G	EBYg	EB]_	E@v_	EBjG
	EBj_	E?Fw	E?NW	E?Bw								
$\mathcal{Q}_6^6$	EF~w	EJnw	EK~w	ELvw	El~o	ENzg	EB\w	E@]w	E@Vw	E@^W	EB]W	EBYw
	EB]g	E@vg	EBjW	EBn_	EBz_	EJf_						
$\mathcal{Q}_6^7$	EJ~w	EL~w	ENzw	E]~o	EB]w	EBZw	EBnW	EFz_				
$\mathcal{Q}_6^8$	EN~w	E]~w										
$\mathcal{Q}_6^9$	E^~w										

Table 2: *The graph classes $\mathcal{Q}_n^1, \ldots, \mathcal{Q}_n^{\Omega_n}$ for $n \in \{2, \ldots, 6\}$. Class members are presented in the well-known graph6 format, which is ideal for storing undirected graphs in a compact manner, using only printable ASCII characters. These graph representations may be converted to adjacency matrices (or other formats) using the reader* showg *which is available online [9].*

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