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Quadratic Irrationals under the Action of Subgroups of Hecke Group $H(\sqrt{3})$

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ABSTRACT

This paper is concerned with the natural action (as Möbius transformations) of some subgroups of $M = H(\sqrt{3}) = \langle x, y : x^2 = y^6 = 1\rangle$ on the elements of quadratic number field over the rational numbers. We start with two subgroups, $M_2 = \langle t, y : t^2 = y^6 = 1\rangle$, where $t = xy^3x$ and $M_3 = \langle t, y : t^6 = y^6 = 1\rangle$, where $t = xy^5x$ and discuss some number theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of these subgroups acting on different subsets of $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$. We also give a comparison of our results obtained from M_3 with the subgroup $M_1 = \langle t, y : t^6 = y^6 = 1\rangle$, $t = xyx$, and find a new possibility for finding the M-orbits of $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$.

KEYWORDS: Ambiguous number, Totally positive (negative) number, Möbius transformations, Hecke groups, Coset diagram.

AMS Mathematics subject classification (2000): 05C25, 11E04, 20G15

1 INTRODUCTION

In 1936 Erich Hecke (see [1]) introduced the groups $H(\lambda)$ generated by two linear-fractional transformations $x(z) = \frac{-1}{z}$ and $y(z) = \frac{-1}{z+\lambda}$. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \in \mathbb{N}$, $q \geq 3$ or $\lambda \geq 2$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic group of order 2 and q, and it has a presentation

$$H(\lambda_q) = \langle x, y : x^2 = y^q = 1\rangle \cong C_2 * C_q$$

The first few of these groups are $H(\lambda_3) = PSL(2, \mathbb{Z})$, the modular group, $H(\lambda_4) = H(\sqrt{2}) = \langle x, y : x^2 = y^4 = 1\rangle$, where $x(z) = \frac{-1}{2z}$ and $y(z) = \frac{-1}{2(z+1)}$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$ and $H(\lambda_6) = H(\sqrt{3}) = \langle x, y : x^2 = y^6 = 1\rangle$. One of the main reasons for $H(\sqrt{2})$ and $H(\sqrt{3}) = M$ (say) to be two of the most important Hecke groups is that part of the modular group, they are the only Hecke groups $H(\lambda_q)$ whose elements can be completely described. A non-empty set Ω with an action of the group G on it, is said to be a G-set. We say that Ω is a transitive G-set if, for any p, q in Ω there exists a g in G such that $p^g = q$. In our case the set $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ is a G-set and throughout this paper we take m as a square free positive integer. Since every element of $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q} = \{t + w\sqrt{m} : t, 0 \neq w \in \mathbb{Q}\}$ can be expressed uniquely as $\frac{a+\sqrt{n}}{c}$, where $n = k^2m$, k is any positive integer and $a, \frac{a^2-n}{c}$ and c are relatively prime integers and we denote it by $\alpha(a,b,c)$. Then

$$\mathbb{Q}^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \neq 0, b = \frac{a^2-n}{c} \in \mathbb{Z} \text{ and } (a,b,c) = 1\}$$

is the set of all roots of primitive second degree equations $cx^2 + 2ax + b = 0$ with reduced discriminant

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$\Delta = a^2 - bc$ equal to n and $Q(\sqrt{m})\setminus Q$ is a disjoint union of $Q^*(\sqrt{k^2m})$ for all $k \in N$. Thus it reduces the study of action on $Q(\sqrt{m})$ to the study of action on $Q^*(\sqrt{n})$. If $\alpha(a,b,c) \in Q^*(\sqrt{n})$ and its conjugate $\overline{\alpha}$ have opposite signs then α is called an ambiguous number [2]. If they are both negative, then we call α is totally negative number and if α, $\overline{\alpha}$ both are positive, then α is called totally positive number. The actual number of ambiguous numbers in $Q^*(\sqrt{n})$ has been discussed in [3] as a function of n. The classification of the elements of $Q^*(\sqrt{n})$ in the form $[a,b,c]$ modulo p has been given in [4].

This paper is concerned with the natural action (as Möbius transformations) of some subgroups of $PGL_2(Z)$ on the projective line over rationals with emphases on irrationals of the form $\frac{a+\sqrt{n}}{c}$ with $(a, \frac{a^2-n}{c}, c) = 1$. We start with two subgroups- $M_2 = \langle t, y : t^2 = y^6 = 1 \rangle$, where $t = xy^3x$ and $M_3 = \langle t, y : t^6 = y^6 = 1 \rangle$, where $t = xy^5x$ of the group of Möbius transformations $M = \langle x, y : x^2 = y^6 = 1 \rangle$, $x(z) = \frac{-1}{3z}$ and $y(z) = \frac{-1}{3(z+1)}$ (see [1]). We discuss some number theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of these subgroups acting on different subsets of $Q(\sqrt{m})\setminus Q$. Also we give comparison of our results obtained from M_3 with subgroup $M_1 = \langle t, y : t^6 = y^6 = 1 \rangle$, $t = xyx$, which opens a possibility to look at structure of M -orbits of $Q(\sqrt{m})\setminus Q$.

2 Preliminaries

We quote from, [2], [5] and [6] the following results for later reference. Also we tabulate the actions on $\alpha(a,b,c) \in Q'''(\sqrt{n})$ of x, y in Table 1.

Table 1: The action of elements of M on $\alpha(a,b,c) \in Q'''(\sqrt{n})$

$\alpha = \frac{a+\sqrt{n}}{c}$	a	b	c
$x(\alpha) = \frac{-1}{3\alpha}$	$-a$	$\frac{c}{3}$	$3b$
$y(\alpha) = \frac{-1}{3\alpha+3}$	$-a-c$	$\frac{c}{3}$	$3(2a+b+c)$
$y^2(\alpha) = \frac{-(\alpha+1)}{3\alpha+2}$	$-5a-3b-2c$	$2a+b+c$	$12a+9b+4c$
$y^3(\alpha) = \frac{-(3\alpha+2)}{6\alpha+3)}$	$-7a-6b-2c$	$\frac{12a+9b+4c}{3}$	$3(4a+4b+c)$
$y^4(\alpha) = \frac{-(2\alpha+1)}{(3\alpha+1)}$	$-5a-6b-c$	$4a+4b+c$	$6a+9b+c$
$y^5(\alpha) = \frac{-(3\alpha+1)}{3\alpha}$	$-a-3b$	$\frac{6a+9b+c}{3}$	$3b$
$yx(\alpha) = \frac{\alpha}{-3\alpha+1}$	$a-3b$	b	$-6a+9b+c$
$y^2x(\alpha) = \frac{-3\alpha+1}{6\alpha-3}$	$5a-6b-c$	$\frac{-6a+9b+c}{3}$	$3(-4a+4b+c)$
$y^3x(\alpha) = \frac{-2\alpha+1}{3\alpha-2}$	$7a-6b-2c$	$-4a+4b+c$	$-12a+9b+4c$
$y^4x(\alpha) = \frac{-3\alpha+2}{6\alpha+3}$	$5a-3b-2c$	$\frac{-12a+9b+4c}{3}$	$3(-2a+b+c)$
$y^5x(\alpha) = \alpha-1$	$5a-3b-2c$	$-2a+b+c$	c

Theorem 2.1 ([5]) An $\alpha = \dfrac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally positive if and only if either $a,b,c > 0$ or $a,b,c < 0$

Theorem 2.2 ([5]) An $\alpha = \dfrac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative number if and only if either $b,c > 0$ and $a < 0$ or $b,c < 0$ and $a > 0$.

Theorem 2.3 ([5]) An $\alpha = \dfrac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number number if and only if either $c > 0$ and $b < 0$ or $c < 0$ and $b > 0$.

Theorem 2.4 ([2]) If $\alpha = \dfrac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally positive number, then $(\alpha)y^i$, $1 \le i \le 5$, are totally negative numbers.

Theorem 2.5 ([6]) $Q^{'''}(\sqrt{n}) = \{\dfrac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1,3\}$ is invariant under the action of M.

3 Action $M_2 = \langle t, y : t^2 = y^6 = 1 \rangle$ on $Q^{'''}(\sqrt{n})$

We start this section by defining the coset diagrams (see [2]) for the group M as follows: the 6-cycles of the transformations y are denoted by six vertices of a hexagon permuted anti-clockwise by y, and the two vertices which are interchanged by x are joined by an edge. Fixed points of x and y, if they exist, are denoted by heavy dots. Thus in the case of M the diagram consists of a set of small hexagons representing the action of $C_6 = \langle y : y^6 = 1 \rangle$ and a set of edges representing the action of $C_2 = \langle x : x^2 = 1 \rangle$ (see Fig.1b).

Let $\alpha(a,b,c) \in Q^{'''}(\sqrt{n})$. The transformation $t = xy^2x$ defined as $t(\alpha) = \dfrac{-3\alpha+2}{3(-2\alpha+1)}$ has order two and $y(\alpha) = \dfrac{-1}{3\alpha+3}$ has order six, then for each α, $(\alpha)y$, $(\alpha)y^2$, $(\alpha)y^3$, $(\alpha)y^4$, $(\alpha)y^5$ and $(\alpha)y^6 = \alpha$, form vertices of a small hexagon. We prove that if α is an ambiguous number, then $t(\alpha)$ is not an ambiguous number. Therefore each hexagon having two vertices ambiguous, will not form the closed path in the coset diagram. The general fragment of coset diagram for the action of subgroup M_2 will be as shown in Fig. 1b.

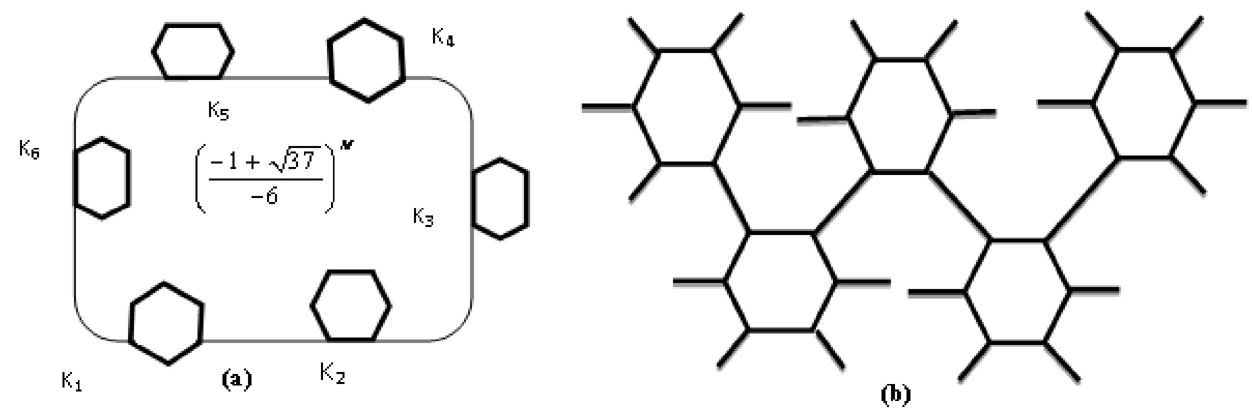

Figure 1: The coset diagram for the action of M on $k_1 = \dfrac{-1+\sqrt{37}}{-6}$ and M_2

Also we observe that if α is totally negative number, then $t(\alpha)$ is totally positive number and when α is totally positive number, then $t(\alpha)$ may or may not be totally negative in general.

Theorem 3.1 If $\alpha = \dfrac{a+\sqrt{n}}{c} \in Q^{'''}(\sqrt{n})$ is totally negative real number, then $t(\alpha)$ is totally positive number.

Proof. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{'''}(\sqrt{n})$ with $b = \frac{a^2-n}{c}$ is totally negative real number. We have to show that $t(\alpha)$ is totally positive number. For this, we consider $t(\alpha) = \frac{-3\alpha+2}{-6\alpha+3} = \frac{a_1+\sqrt{n}}{c_1}$ where $a_1 = -7a+6b+c$, $b_1 = \frac{-12a+9b+4c}{3}$ and $c_1 = 3(-4a+4b+c)$. If $\alpha(a,b,c)$ is totally negative number, then either $b,c>0$ and $a<0$ or $b,c<0$ and $a>0$. This implies that either $a_1,b_1,c_1>0$ (in case of $b,c>0$ and $a<0$) or $a_1,b_1,c_1<0$ (in case of $b,c<0$ and $a>0$). But by Lemma 2.1, $t(\alpha)$ is totally positive number. This completes the proof.

Theorem 3.2 Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{'''}(\sqrt{n})$ be an ambiguous number. Then $t(\alpha)$ is not an ambiguous number.

Proof. Note that if α is an ambiguous number then by definition either $c>0$, $b<0$ or $c<0$, $b>0$, that is $bc<0$. Thus $b = \frac{a^2-n}{c}$ implies that $a^2-n<0$. So that $a>0$ or $a<0$. Since $t(\alpha) = \frac{-3\alpha+2}{-6\alpha+3} = \frac{a_1+\sqrt{n}}{c_1}$ where $a_1 = -7a+6b+2c$, $b_1 = \frac{-12a+9b+4c}{3}$ and $c_1 = 3(-4a+4b+c)$.

Case 1(a) ($a<0$, $b<0$, $c>0$) then $-7a+2c>0$ and $6b<0$ given that $a_1 = -7a+6b+2c$ either positive or negative, if $a_1>0$ we have $-7a+2c>6b$ implies that $-12a+3c>12b$ or $-12a+12b+3c>0$ implies $c_1>0$ as $c_1 = -12a+12b+3c$. But we are not sure about b_1, whether it is positive or negative., we conclude that $t(\alpha)$ is not an ambiguous number.

For $a_1<0$, $-7a+2c<6b$ since $-12+3c>0$ and $12b<0$ therefore either $-12a+12b+3c>0$ or $-12+12b+3c<0$ but we are not sure about b_1 whether it is positive or negative. So we conclude that $t(\alpha)$ is not an ambiguous number.

Case 1(b) ($a>0$, $b<0$, $c>0$) We note that $-7a+6b<0$ and $2c>0$ This implies that $-7a+6b+2c>0$ or $-7a+6b+2c<0$, that is $a_1>0$, or $a_1<0$. For $a_1>0$, we have $c_1>0$. But we are not sure about b_1, whether it is positive or negative. so we conclude that $t(\alpha)$ is not an ambiguous number.

Similarly for $a_1<0$, we have $-7a+6b>2c$ which implies $a_1<0$, $-7a+6b>2c$ implies $-12a+12b>3c$ that is $c_1<0$. But this does not implies $b_1>0$ or $b_1<0$, therefore we conclude that $t(\alpha)$ is not an ambiguous number.

Case 2(a) ($a<0$, $c<0$, $b>0$) $-7a+6b>0$ and $c<0$ which implies that $-7a+6b+c>0$ or $-7a+6b+c<0$. that is either a_1 is positive or a_1 is negative. For $a<0$ $-12a+12b>3c$, that is $c_1>0$. for $a_1<0$ we have $-12a+12b+3c>0$ that is $c_1<0$ or $c_1>0$. but this does not imply that $b_1>0$ or $b_1<0$. Therefore we conclude that $t(\alpha)$ is not an ambiguous number.

Case 2(b) ($a>0$, $b>0$, $c<0$) $-7a+2c<0$ and $6b>0$ which implies that $-7a+6b+2c>0$ or $-7a+6b+2c<0$, that is $a_{>}0$ or $a_1<0$ we have $12b>-12a+3c$ which implies that $12b-12a+3c>0$, that is $c_1>0$. for $a_1<0$, we have $a2b-12a+3c<0$ or $12b-12a+3c>0$ that is $c_1<0$ or $c_1>0$. Therefore we conclude that $t(\alpha)$ is not an ambiguous number. Thus in all the all cases $t(\alpha)$ is not an ambiguous number. This completes the proof.

Theorem 3.3 Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{'''}(\sqrt{n})$ be totally positive number.

1. If $t(\alpha)$ is totally positive number, then either $|6b+2c| > 7a$, $|12b+3c| > 12a$ or $|6b+2c| < 7a$, $|12b+3c| < 12a$.

2. If $t(\alpha)$ is totally negative number, then either $|6b+2c| > 7a$, $|12b+3c| < 12a$ or $|6b+2c| < 7a$, $|12b+3c| > 12a$.

Proof. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{'''}(\sqrt{n})$ with $b = \frac{a^2-n}{c}$ is totally positive number, then by Lemma 2.1, either $a,b,c > 0$ or $a,b,c < 0$. Since we know that $t(\alpha) = \frac{-3\alpha+2}{-6\alpha+3} = \frac{a_1+\sqrt{n}}{c_1}$, where $a_1 = -7a+6b+c$, $b_1 = \frac{-12a+9b+4c}{3}$ and $c_1 = 3(-4a+4b+c)$. Suppose $t(\alpha)$ is totally positive number then, by Lemma 2.1, either $a_1 > 0$, $b_1 > 0$, $c_1 > 0$ or $a_1 < 0$, $b_1 < 0$, $c_1 < 0$. Now suppose that $t(\alpha)$ is totally negative then by Lemma 2.2, either $a_1 > 0$, $b_1 < 0$, $c_1 < 0$ or $a_1 < 0$, $b_1 > 0$, $c_1 > 0$. Now for the case $a > 0$, $b > 0$, $c > 0$ we observe that

$$a_1 = -7a+6b+2c > 0 \text{ implies that } 6b+2c > 7a \quad (1)$$
$$a_1 = -7a+6b+2c < 0 \text{ implies that } 6b+2c < 7a \quad (2)$$
$$c_1 = -12a+12b+3c > 0 \text{ implies that } 12b+3c > 12a \quad (3)$$
$$c_1 = -12a+12b+3c < 0 \text{ implies that } 12b+3c < 12a \quad (4)$$

and for $a < 0, b < 0, c < 0$, we have

$$a_1 = -7a+6b+2c > 0 \text{ implies that } 6b+2c < -7a \quad (5)$$
$$a_1 = -7a+6b+2c < 0 \text{ implies that } 6b+2c > -7a \quad (6)$$
$$c_1 = -12a+12b+3c > 0 \text{ implies that } 12b+3c < -12a \quad (7)$$
$$c_1 = -12a+12b+3c < 0 \text{ implies that } 12b+3c > -12a \quad (8)$$

From (1) and (5) we see that $a_1 > 0$ only when $7a < 6b+2c < -7a$ that is $|6b+2c| > 7a$ and (2) and (6) implies $a_1 < 0$ only when $-7a < 6b+2c < 7a$ that is $|6b+2c| < 7a$. From (3) and (7) we see that $c_1 > 0$ only when $12a < 12b+3c < -12a$ that is $|12b+3c| > 12a$ and (4) and (8) gives $c_1 < 0$ only when $-12a < 12b+2c < 12a$ that is $|12b+3c| < 12a$. whenever $t(\alpha)$ is totally positive, we have either $|6b+2c| > 7a$, $|12b+3c| > 12a$ (in case of $a_1 > 0$, $b_1 > 0$, $c_1 > 0$) or $|6b+2c| < 7a$, $|12b+3c| < 12a$ (in case of $a_1 < 0$, $b_1 < 0$, $c_1 < 0$).

whenever $t(\alpha)$ is totally negative, we have either $|6b+2c| > 7a$, $|12b+3c| < 12a$ (in case of $a_1 > 0$, $b_1 < 0$, $c_1 < 0$) or $|6b+2c| < 7a$, $|12b+3c| > 12a$ (in case of $a_1 < 0$, $b_1 > 0$, $c_1 > 0$).

4 Action of $M_3 = \langle t, y : t^6 = y^6 = 1 \rangle$ on $Q^{'''}(\sqrt{n})$

In this section we discuss some number theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of subgroup $M_3 = \langle t, y : t^6 = y^6 = 1 \rangle$ where $t = xy^5x$, acting on $Q^{'''}(\sqrt{n})$. Also we give the comparison of our results obtained from M_3 with subgroup

$M_1 = \langle t, y : t^6 = y^6 = 1 \rangle$, $t = xyx$ which was discussed by Aslam and Mushtaq in 2006.

Table 2: The action of $t = xy^5x$ on $\alpha(a,b,c) \in Q'''(\sqrt{n})$

$\alpha = \frac{a+\sqrt{n}}{c}$	a	b	c
$t(\alpha) = \frac{-1}{3(\alpha-1)}$	$-a+c$	$\frac{c}{3}$	$-6a+3b+3c$
$t^2(\alpha) = \frac{-\alpha+1}{3\alpha+2}$	$-5a+3b+2c$	$-2a+b+c$	$-12a+9b+4c$
$t^3(\alpha) = \frac{3\alpha-2}{3(2\alpha-1)}$	$-7a+6b+2c$	$\frac{-12a+9b+4c}{3}$	$-12a+12b+3c$
$t^4(\alpha) = \frac{-2\alpha+1}{3\alpha+1}$	$-5a+6b+c$	$-4a+4b+c$	$-6a+9b+c$
$t^5(\alpha) = \frac{3\alpha-1}{3\alpha}$	$-a+3b$	$\frac{-6a+9b+c}{3}$	$3b$

Theorem 4.1 Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q'''(\sqrt{n})$ be totally negative number, then $t^i(\alpha)$ is totally positive number for each $i \in \{1,2,3,4,5\}$.

Proof. If $\alpha = \frac{a+\sqrt{n}}{c} \in Q'''(\sqrt{n})$, then $t^i(\alpha)$ for each $i \in \{1,2,3,4,5\}$ is given in Table 2. If $\alpha = \frac{a+\sqrt{n}}{c} \in Q'''(\sqrt{n})$ is totally negative number then by Lemma 2.2, either $b > 0$, $c > 0$ and $a < 0$ or $b < 0$, $c < 0$ and $a > 0$. When $b > 0$, $c > 0$ and $a < 0$, we see that, by Table 2., for each i, the new values of a, b and c are positive. Hence by lemma 2.1 $t^i(\alpha)$ are totally positive. Similarly, for $b < 0$, $c < 0$ and $a > 0$, the new values of a, b and c are negative for each $i = 1,2,3,4$ or 5. Thus by lemma 2.1 each $t^i(\alpha)$ is totally positive. This completes the proof.

Theorem 4.2 If $\alpha = \frac{a+\sqrt{n}}{c} \in Q'''(\sqrt{n})$ is an ambiguous number, then for $i \in \{1,2,3,4,5\}$, one of $t^i(\alpha)$ is an ambiguous number and other four are totally positive.

Proof. Case 1. Let α is negative number. Then the possibilities for signs of $\overline{\alpha} = \frac{a-\sqrt{n}}{c}$, $\overline{t(\alpha)}$, $\overline{t^2(\alpha)}$, $\overline{t^3(\alpha)}$, $\overline{t^4(\alpha)}$ and $\overline{t^5(\alpha)}$ are given in Table 3.

Case 2. Let α is positive number. Then the possibilities for signs of $\overline{\alpha} = \frac{a-\sqrt{n}}{c}$, $\overline{t(\alpha)}$, $\overline{t^2(\alpha)}$, $\overline{t^3(\alpha)}$, $\overline{t^4(\alpha)}$ and $\overline{t^5(\alpha)}$ are given in Table 4. Thus, for $i = 1,2,3,4,5$, we observe that one of $t^i(\alpha)$ is an ambiguous number and other four are totally positive.

Table 3: Possibilities for sign of $\overline{t^i(\alpha)}$, α is negative, for $i = 1,2,3,4,5$ and 6.

α	$t(\alpha)$	$t^2(\alpha)$	$t^3(\alpha)$	$t^4(\alpha)$	$t^5(\alpha)$	$\overline{\alpha}$	$\overline{t(\alpha)}$	$\overline{t^2(\alpha)}$	$\overline{t^3(\alpha)}$	$\overline{t^4(\alpha)}$	$\overline{t^5(\alpha)}$
−	+	+	+	+	+	+	−	+	+	+	+
						+	+	−	+	+	+
						+	+	+	−	+	+
						+	+	+	+	−	+
						+	+	+	+	+	−

Table 4: Possibilities for sign of $\overline{t^i(\alpha)}$, α is positive, for $i = 1,2,3,4,5$ and 6

α	$t(\alpha)$	$t^2(\alpha)$	$t^3(\alpha)$	$t^4(\alpha)$	$t^5(\alpha)$	$\overline{\alpha}$	$\overline{t(\alpha)}$	$\overline{t^2(\alpha)}$	$\overline{t^3(\alpha)}$	$\overline{t^4(\alpha)}$	$\overline{t^5(\alpha)}$
+	−	+	+	+	+	−	+	+	+	+	+
+	+	−	+	+	+						
+	+	+	−	+	+						
+	+	+	+	−	+						
+	+	+	+	+	−						

Since $y:\alpha \to \frac{-1}{3(\alpha+1)}$, (Mushtaq and Aslam, 1998) studied that if $k \notin \{0, \frac{-1}{3}, \frac{-1}{2}, \frac{-2}{3}, -1, \infty\}$ and is a vertex of a hexagon in the coset diagram, then

(i) $\alpha < -1$ then $(\alpha)\alpha > 0$.

(ii) $\alpha > 0$ then $\frac{-1}{3} < (\alpha)\alpha < 0$.

(iii) $\frac{-1}{3} < \alpha < 0$ then $\frac{-1}{2} < (\alpha)y < \frac{-1}{3}$.

(iv) $\frac{-1}{2} < \alpha < \frac{-1}{3}$ then $\frac{-2}{3} < (\alpha)y < \frac{-1}{2}$.

(v) $\frac{-2}{3} < \alpha < \frac{-1}{2}$ then $-1 < (\alpha)y < \frac{-2}{3}$.

(vi) $-1 < \alpha < \frac{-2}{3}$ then $(\alpha)y < -1$.

That is, of the vertices k, ky, ky^2, ky^3, ky^4 and ky^5 one is positive and five are negative. It have been studied by Aslam (2006) that if $k \notin \{0, \frac{2}{3}, \frac{1}{3}, \frac{1}{2}, 1, \infty\}$ and is a vertex of a hexagon in the coset diagram, then

(i) $\alpha < 0$ then $t(\alpha) > 1$.

(ii) $\alpha > 1$ then $\frac{2}{3} < t(\alpha) < 1$.

(iii) $\frac{2}{3} < \alpha < 1$ then $\frac{1}{2} < t(\alpha) < \frac{2}{3}$.

(iv) $\frac{1}{2} < \alpha < \frac{2}{3}$ then $\frac{1}{3} < t(\alpha) < \frac{1}{2}$.

(v) $\frac{1}{3} < \alpha < \frac{1}{2}$ then $0 < t(\alpha) < \frac{1}{3}$.

(vi) $0 < \alpha < \frac{1}{3}$ then $t(\alpha) < 0$.

That is, of the vertices k, tk, t^2k, t^3k, t^4k and t^5k one is negative and five are positive.

In our case, we study that if $k \notin \{0, \frac{2}{3}, \frac{1}{3}, \frac{1}{2}, 1, \infty\}$ and is a vertex of a hexagon in the coset diagram, then

(i) $\alpha < 0$ then $0 < t(\alpha) < \frac{1}{3}$.

(ii) $0 < \alpha < \frac{1}{3}$ then $\frac{1}{3} < t(\alpha) < \frac{1}{2}$.

(iii) $\frac{1}{3} < \alpha < \frac{1}{2}$ then $\frac{1}{2} < t(\alpha) < \frac{2}{3}$.

(iv) $\frac{1}{2} < \alpha < \frac{2}{3}$ then $\frac{2}{3} < t(\alpha) < 1$.

(v) $\frac{2}{3} < (\alpha) < 1$ then $1 < t(\alpha) < \frac{1}{3}$.

(vi) $1 < \alpha < \frac{1}{3}$ then $t(\alpha) < 0$.

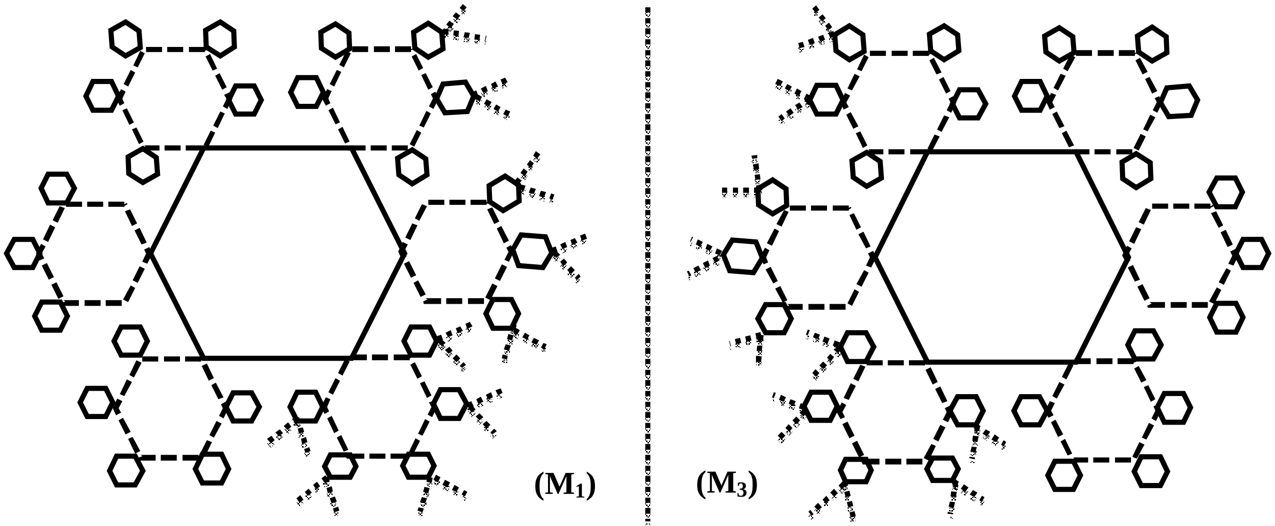

Figure 2: Fragments of coset diagram for the action of M_1 and M_3

That is, if vertices α, $t(\alpha)$, $t^2(\alpha)$, $t^3(\alpha)$, $t^4(\alpha)$ and $t^5(\alpha)$ of a hexagon are not $0, \frac{2}{3}, \frac{1}{3}, \frac{1}{2}, 1, \infty$ then one of six vertices is negative and five are positive. Therefore the number theoretic properties belonging to the orbit of subgroup $M_3 = \langle t, y : y^6 = t^6 = 1 \rangle$, where $t = xy^5x$, are same as of the subgroup M_1. The coset diagram (Fig. 2) concludes the whole discussion. That is, 6-cycles of y are defined six solid edges and 6-cycles of the transformation $t = xy^5x$ are defined by six broken edges of a small hexagon permuted anti-clockwise by both y and t. The vertices of 6-cycles of transformation $t = xy^5x$ are permuted in the opposite direction to the vertices of $t = xyx$.

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