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The Largest Cartesian Closed Category of Domains, Considered Constructively

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A conjecture of Smyth is discussed which says that if D and $[D \to D]$ are effectively algebraic directed-complete partial orders with least element (cpo's), then D is an effectively strongly algebraic cpo, where it was not made precise what is meant by an effectively algebraic and an effectively strongly algebraic cpo.
Notions of an effectively strongly algebraic cpo and an effective SFP domain are introduced and shown to be (effectively) equivalent. Moreover, the conjecture is shown to hold if instead of being effectively algebraic, $[D \to D]$ is only required to be ω-algebraic and D is forced to have a completeness test, that is a procedure which decides for any two finite sets X and Y of compact cpo elements whether X is a complete set of upper bounds of Y. As a consequence, the category of effective SFP objects and continuous maps turns out to be the largest Cartesian closed full subcategory of the category of ω-algebraic cpo's that have a completeness test.
It is then studied whether such a result also holds in a constructive framework, where one considers categories with constructive domains as objects, that is, domains consisting only of the constructive (computable) elements of an indexed ω-algebraic cpo, and computable maps as morphisms. This is indeed the case: the category of constructive SFP domains is the largest constructively Cartesian closed weakly indexed effectively full subcategory of the category of constructive domains that have a completeness test and satisfy a further effectivity requirement.

1. Introduction

In his seminal paper (Smyth 1983) Smyth showed that the category **SFP** introduced by Plotkin (Plotkin 1976) is the largest Cartesian closed category of domains, thus confirming a conjecture of Plotkin. In this paper we treat Plotkin's conjecture for the case of effectively given domains.

For various reasons one mostly uses the term *domain* to mean ω-algebraic directed-complete partial order with least element (cpo) in studies of programming language

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semantics. Unfortunately, the class of domains is not closed under an important construction needed e.g. for the interpretation of higher-type procedures: the space $[D \to E]$ of continuous maps between two domains D and E must not be a domain again.

To circumvent this problem, people often restrict themselves to bounded-complete domains, the class of which is closed under the function space construction. However, also this class is not closed under all constructions needed in semantics: the Plotkin or convex powerdomain of a bounded-complete domain is not bounded-complete in general. Powerdomains are used for the interpretation of nondeterministic programs. Plotkin therefore introduced the larger class of SFP domains and showed that it is closed under the construction of his powerdomain as well as the function space. Moreover, he conjectured that if D and $[D \to D]$ are domains, then D is SFP. The conjecture, proved by Smyth, indicates that the category **SFP** of SFP domains and continuous maps is the largest category of domains closed under the constructions of interest.

The question which category has to be considered instead of **SFP**, if the term *domain* is allowed to mean some more general kind of directed-complete partial order, has extensively been studied by Jung (Jung 1988; Jung 1989; Jung 1990; Abramsky and Jung 1994). If instead of the space of continuous maps one confines to the space of stable maps, the corresponding problem has been dealt with by Amadio (Amadio 1991), Zhang (Zhang 1996) as well as Zhang et al. (Zhang *et al.* 2002).

In his paper Smyth conjectured that with respect to natural notions of effectively algebraic and effectively strongly algebraic the following statement be true: If D and $[D \to D]$ are effectively algebraic cpo's, then D is an effectively strongly algebraic cpo. The study of effectiveness is important in a theory of the foundations of programming. "One reason", said Smyth (Smyth 1980), "has to do with the systematic study of the power of specification techniques. We cannot require of a general purpose programming language that it be able to specify (define) all number-theoretic functions, but only (at most) those which are partial recursive. A corresponding distinction must be made for all the 'data types' which one may wish to handle. And the problem is not simply that of picking out the computable functions over a given data type; we have the problem of specifying the data types themselves, and thus of determining the 'computable', or effectively given, data types (i.e. the types which should in principle be specifiable)."

We first introduce the notions of an effectively strongly algebraic domain and of an effective SFP domain and derive their (effective) equivalence, which shows that we have obtained a stable effectivity notion for SFP domains. Vickers has recently studied this equivalence in a topos-theoretic setting (Vickers 2001).

Plotkin introduced SFP domains as colimits of ω-chains of finite domains with embeddings as connecting morphisms. Then he proved that they are exactly the strongly algebraic domains, that is, those domains for which for any finite set X of compact elements, the least set containing X and closed under the operation of taking all minimal upper bounds of subsets of X is finite. Here, we encode the finite domains and the embeddings between them in a canonical way and consider effective ω-chains. These are such that for a given natural number n one can compute both the index of the nth domains and the index of the nth embedding in the chain. Effective SFP domains are then defined to be colimits of such effective chains. An effectively strongly algebraic domain is a strongly algebraic domain which has an indexing of its compact elements such that for any finite set X of compact elements a canonical index of the set of its minimal upper bounds can be computed from a canonical index of X.

Effective SFP domains have also been studied by Kanda in his dissertation (Kanda 1979b). But whereas in the present paper the effective SFP domains are the constructive objects of the category of indexed domains, this is not the case in Kanda's treatment. Note that a domain is indexed if it comes with a fixed numbering of its compact elements. An object in a concrete category is constructive if it can be obtained (constructed) in an effective way from its finite parts. Kanda does not code the finite domains by canonical or explicit indices, from which the domains can easily be recovered. Instead he codes finite domains in the same way as effectively given domains in general. This coding contains only partial information about the domain. (See also the remark of Smyth in (Smyth and Plotkin 1982, Section 5).)

In effectivity considerations of domains it is usual to require that the domain order be decidable on the compact elements. Here, we use a stronger requirement. A domain is said to have a completeness test if there is a procedure which decides for any two finite sets X and Y of compact elements whether X is a complete set of upper bounds of Y. We show that if D and $[D \to D]$ are domains such that D has a completeness test, then D is an effective SFP domain. It is not known, whether the condition of having a completeness test can be weakened in this result. As in Smyth (Smyth 1983) it follows that the category of effective SFP domains and continuous maps is the largest Cartesian closed full subcategory of the category of domains having a completeness test.

Next, it is studied whether a result of this kind also holds in a constructive framework, or, to be more precise, in the framework of recursive mathematics. Here, one considers categories with constructive domains as objects, that is, domains consisting only of the constructive (computable) elements of an indexed ω-algebraic cpo, and computable maps as morphisms. It is shown that the category of constructive SFP domains is the largest constructively Cartesian closed weakly indexed effectively full subcategory of the category of constructive domains having a completeness test and satisfying a further effectivity requirement.

The effectivity requirements that have to be satisfied by the category are rather weak compared with the conditions considered by Kanda (Kanda 1979a) and Smyth (Smyth 1980) in their approaches to effectiveness in categories. We only require that for any two objects the corresponding morphism set is indexed in such a way that the universality statement in the definition of a categorical product holds effectively.

The additional effectivity condition that has to be fulfilled by the domains in the category we will consider holds trivially in the case of constructive SFP domains. It also deals with completeness and demands for every pair x_1, x_2 of compact elements that if its set $\mathcal{U}(\{x_1, x_2\})$ of minimal upper bounds is not complete then we must be able to effectively find a witness for this, i.e., an upper bound of x_1 and x_2 below which there is no minimal upper bound of x_1 and x_2.

The proof of the maximality result of the category of constructive SFP domains consists of four main steps. The first deals with the question whether in the categories under consideration the space of all computable maps on a constructive domain D is the exponent of D with itself. In the other steps it is shown that the set $\mathcal{U}(X)$ of minimal upper bounds of a finite set X of compact elements is complete for X and finite. Due to a result of Plotkin (cf. (Smyth 1983)) it is sufficient here to consider only sets X of cardinality two. In addition, it is demonstrated that the process of iteratively taking minimal upper bounds of subsets of X terminates after finitely many steps.

In the case of the last step the proofs given in (Smyth 1983; Jung 1989) use König's Lemma and/or the Heine-Borel Theorem. Both statements do not hold in recursive mathematics (Beeson 1985). Here, a proof is given which does not use such results. Similarly, in the proofs of the finiteness of $\mathcal{U}(X)$ given in the literature, under the assumption that $\mathcal{U}(X)$ is infinite a contradiction is derived by showing that $[D \to D]$ has uncountably many compact elements. This construction is of no use in the constructive setting of the present paper. Therefore another contradiction is derived. The general idea is to show that if $\mathcal{U}(X)$ is infinite then an effective enumeration of all computable maps on $\mathcal{U}(X)$ can be constructed and to use a simple diagonalization argument to show that this is impossible.

Jung (Jung 1990) has given another proof of Smyth's result that also makes no use of the Axiom of Choice. But here, as in the other approaches not dealing with a constructive version of this result, the full continuous function space is considered. In order to apply the method to the constructive framework of the present paper one would have to derive effective versions of his results. A closer analysis shows that in this case too one has to restrict oneself to domains that satisfy the above mentioned witness condition for sets of minimal upper bounds that are not complete.

Preliminary versions of the results at hand have been presented at the Dagstuhl Seminar "Domain Theory and Its Applications" 1998 and the Seventh Workshop on Logic, Language, Information and Computation 2000.

The rest of the paper is organized as follows. In Section 2 basic definitions and results from domain theory are given. Section 3 is its effective counterpart. Here, the definition of an effectively given SFP domain is given and some properties are derived. Smyth's conjecture is treated in Section 4 and in Section 5 the corresponding question for constructive domains is considered.

2. Domains

Let $(D, \sqsubseteq)$ be a partial order with smallest element $\perp$. For a subset S of D, $\downarrow S = \{\, x \in D \mid (\exists y \in S) x \sqsubseteq y \,\}$ is the *lower set* generated by S. The subset S is called *compatible* if it has an upper bound. S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete* partial order (cpo) if every directed subset S of D has a least upper bound $\bigsqcup S$ in D.

An element x of a cpo D is *compact* if for any directed subset S of D the relation $x \sqsubseteq \bigsqcup S$ always implies the existence of an element $u \in S$ with $x \sqsubseteq u$. We write D^0 for the set of compact elements of D. If D^0 is countable and for every $y \in D$ the set $\downarrow\{y\} \cap D^0$ is directed with $y = \bigsqcup(\downarrow\{y\} \cap D^0)$, the cpo D is *ω-algebraic* or, as we prefer to say, a *domain.* Standard references for domain theory are (Gunter and Scott 1990;

Gunter 1992; Abramsky and Jung 1994; Stoltenberg-Hansen *et al.* 1994; Amadio and Curien 1998).

The *product* $D \times E$ of two cpo's D and E is the Cartesian product of the underlying sets ordered coordinatewise. Obviously, $D \times E$ is a domain again with $(D \times E)^0 = D^0 \times E^0$, if D and E are domains.

Definition 2.1. A map $F\colon D \to E$ between cpo's D and E is *continuous* if it is monotone and for any directed subset S of D,

$$F(\bigsqcup S) = \bigsqcup F(S).$$

Let $[D \to E]$ denote the set of all continuous maps from D to E. Endowed with the *pointwise order*, that is $F \sqsubseteq G$ if $F(x) \sqsubseteq G(x)$, for all $x \in D$, it is a cpo again, but in general it need not be a domain. This means that the category **DOM** of domains and continuous maps is not Cartesian closed. Therefore one considers subclasses of domains which have this property, when using domains in programming language semantics, *e.g.* SFP domains. To introduce this kind of domains we need the following definitions.

Definition 2.2. An *embedding/projection* (F, G) from a cpo D to a cpo E is a pair of maps $F \in [D \to E]$ and $G \in [E \to D]$ such that $G \circ F = \mathrm{Id}_D$, the identity map on D, and $F \circ G \sqsubseteq \mathrm{Id}_E$. The map F is called *embedding* and G *projection*.

Note that the map G is uniquely determined by F, and vice versa (Smyth and Plotkin 1982). Therefore, we also write F^R instead of G. Embeddings are one-to-one and preserve compactness (Plotkin 1976).

Lemma 2.3. Let D and E be domains and $F\colon D \to E$. Then F is an embedding if and only if there is a monotone and one-to-one map $F_0\colon D^0 \to E^0$ such that for all $y \in E$ and all $u, u' \in D^0$, if $F_0(u), F_0(u') \sqsubseteq y$ then there exists some $\bar{u} \in D^0$ so that $u, u' \sqsubseteq \bar{u}$ and $F_0(\bar{u}) \sqsubseteq y$.

Suppose (F, F^R) is an embedding/projection from C to D and (G, G^R) is an embedding/projection from D to E. Then the composition of (F, F^R) and (G, G^R) is defined by

$$(G, G^R) \circ (F, F^R) = (G \circ F, F^R \circ G^R).$$

Let $\mathbf{DOM^e}$ denote the category of domains and embeddings.

By an ω-chain in $\mathbf{DOM^e}$ we understand a diagram of the form $\mathcal{D} = D_0 \overset{F_0}{\to} D_1 \overset{F_1}{\to} \dots$ (that is, a functor from ω to $\mathbf{DOM^e}$). As is well known, the category $\mathbf{DOM^e}$ is ω-cocomplete: every ω-chain in $\mathbf{DOM^e}$ has a colimit. Up to isomorphism this is given by the set

$$D_\infty = \{\, x \in \Pi_{m \in \omega} D_m \mid (\forall m \in \omega) x_m = F_m^R(x_{m+1}) \,\}$$

endowed with the componentwise partial order, that is

$$x \sqsubseteq y \Leftrightarrow (\forall m \in \omega) x_m \sqsubseteq_{D_m} y_m.$$

Note that

$$D_\infty^0 = \{\, u \in D_\infty \mid (\exists m \in \omega) u_m \in D_m^0 \wedge (\forall n \geq m) u_{n+1} = F_n(u_n) \,\}.$$

Definition 2.4. An *SFP domain* is a colimit of an ω-chain in $\mathbf{DOM^e}$, where all domains in the chain are finite.

In (Plotkin 1976) Plotkin gave an alternative, purely order-theoretic characterization of SFP domains, which is quite useful in many cases.

Definition 2.5. Let D be a partial order, X be a subset of D and $\mathrm{UB}(X)$ be the set of all upper bounds of X.

1 An element x of D is a *minimal upper bound* of X if it is an upper bound of X and it is not strictly greater than any other upper bound of X.
2 A subset Y of $\mathrm{UB}(X)$ is *complete* for X if whenever $x \in \mathrm{UB}(X)$, then $x \sqsupseteq y$ for some $y \in Y$.

Let $\mathcal{U}(X)$ be the set of minimal upper bounds of X. Then $\mathcal{U}(X)$ is included in every subset Y of $\mathrm{UB}(X)$ that is complete for X. Moreover, if D is a domain and X contains only compact elements, the same is true for $\mathcal{U}(X)$ (Jung 1989, Proposition 1.9). Define $\mathcal{U}^*(X)$ to be the union of all sets $\mathcal{U}^n(X)$, where

$$\mathcal{U}^0(X) = X \quad \text{and}$$
$$\mathcal{U}^{n+1}(X) = \bigcup \{\, \mathcal{U}(Z) \mid Z \text{ a finite subset of } \mathcal{U}^n(X) \,\}.$$

A domain D is called *strongly algebraic* if for each finite subset X of D^0, $\mathcal{U}(X)$ is complete for X and $\mathcal{U}^*(X)$ is finite.

Theorem 2.6. A domain is an SFP domain if and only if it is strongly algebraic.

3. Effectively given domains

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a recursive pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$, and let Δ be a standard coding of all finite subsets of natural numbers. We extend the pairing function in the usual way to an n-tuple encoding. Moreover, let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) recursive functions, and let W_i be the domain of the ith partial recursive function φ_i with respect to some Gödel numbering φ. In this case i is called an *r.e. index* of the recursively enumerable (r.e.) set W_i. We let $\varphi_i(a)\!\downarrow$ mean that the computation of $\varphi_i(a)$ stops and $\varphi_i(a)\!\downarrow \in C$ that it stops with value in C.

Let S be a nonempty set. A *(partial) numbering* ν of S is a partial map $\nu\colon \omega \rightharpoonup S$ (onto) with domain $\mathrm{dom}(\nu)$. The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$. The pair (S, ν) is called *numbered set.* Note that instead of numbering and numbered set, respectively, we also say *indexing* and *indexed set.*

Definition 3.1. Let ν and κ be numberings of the set S.

1 $\nu \le \kappa$, read ν is *reducible* to κ, if there is some function $g \in P^{(1)}$ with $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_m = \kappa_{g(m)}$, for all $m \in \mathrm{dom}(\nu)$.
2 $\nu \equiv \kappa$, read ν is *equivalent* to κ, if $\nu \le \kappa$ and $\kappa \le \nu$.

A map $F\colon S \to S'$ from a numbered set (S,ν) to a numbered set (S',ν') is *effective* if there is some function $f \in P^{(1)}$ such that $f(i){\downarrow} \in \mathrm{dom}(\nu')$ and $F(\nu_i) = \nu'_{f(i)}$, for all $i \in \mathrm{dom}(\nu)$. The function f is said to *realize* f.

The following definition is essentially due to Smyth (Smyth 1980).

Definition 3.2. An *effectively given* category is a concrete category **K** together with a total indexing ϱ of its finite objects (i.e., those with finite underlying set) and a total indexing ϑ of the morphisms between finite objects such that the following conditions hold:

1 The set $\{\, \langle m,n\rangle \mid \varrho_m = \varrho_n \,\}$ is recursive.
2 The set $\{\, m \in \omega \mid \vartheta_m$ is an identity morphism$\,\}$ is recursive.
3 There are functions $d, c \in R^{(1)}$ such that $\varrho_{d(m)}$ and $\varrho_{c(m)}$, respectively, are the domain and codomain of ϑ_m.
4 There is a function comp $\in P^{(2)}$ such that for all $m,n \in \omega$ for which the codomain of ϑ_m is the domain of ϑ_n, $\mathrm{comp}(m,n){\downarrow} \in \mathrm{dom}(\vartheta)$ and $\vartheta_n \circ \vartheta_m = \vartheta_{\mathrm{comp}(m,n)}$.

An ω-chain $(A_m, F_m)_{m\in\omega}$ of finite objects in **K** is *effective* if there is a function $t \in R^{(1)}$ such that $A_m = \varrho_{\pi_1(t(m))}$ and $F_m = \vartheta_{\pi_2(t(m))}$, for all $m \in \omega$. A *constructive* object A of **K** is then a colimit of an effective ω-chain of finite objects in **K**.

Note that Smyth (Smyth 1980) does not restrict his considerations to concrete categories. By mimicking the compactness definition for domain elements the notion of a finite object in a category is introduced. Unfortunately, the thus defined finite objects of the category $\mathbf{DOM^e}$ are not the finite domains. The same holds if we confine ourselves to the subcategory $\mathbf{IDOM^{ce}}$ of indexed domains and computable embeddings. Here, an *indexed domain* (D,δ) is a domain D with a fixed total numbering δ of its compact elements. Moreover, a *computable embedding* is the left part of an embedding/projection pair (F,G) such that both F and G are computable.

Definition 3.3. Let (D,δ) and (E,ε) be indexed domains. A map $F \in [D \to E]$ is *computable* if the set graph_F with $\mathrm{graph}_F = \{\, \langle i,j\rangle \mid \varepsilon_j \sqsubseteq F(\delta_i) \,\}$ is r.e.

The numbering of the compact elements is used to impose certain effectivity requirements on these elements. A condition that we shall always use is the decidability of the domain order.

Definition 3.4. A domain D with a total numbering δ of its compact elements is *effectively given* if the set $\{\, \langle i,j\rangle \mid \delta_i \sqsubseteq \delta_j \,\}$ is recursive.

Note that an embedding F from an effectively given domain (D,δ) into another effectively given domain (E,ε) is computable exactly if its restriction F_0 to the compact elements is effective. Moreover, a Gödel number of the function witnessing effectivity can be computed from r.e. indices of graph_F and graph_{F^R}, and vice versa.

In order to see this, observe that

$$\varepsilon_i = F_0(\delta_j) \Leftrightarrow \varepsilon_i \sqsubseteq F(\delta_j) \wedge \delta_j \sqsubseteq F^R(\varepsilon_i).$$

It follows that the set $\{\, \langle i,j\rangle \mid \varepsilon_i = F_0(\delta_j) \,\}$ is r.e. For $n \in \omega$, let $\langle m,n\rangle$ be the first element $\langle i, j\rangle$ enumerated with respect to some fixed enumeration such that $j = n$. Set $f(n) = m$. Then $f \in R^{(1)}$ and $F_0(\delta_n) = \varepsilon_{f(n)}$.

Conversely, if there is a computable function $f \in R^{(1)}$ such that $F_0(\delta_n) = \varepsilon_{f(n)}$, then graph_F and graph_{F^R} are both r.e., as E is effectively given.

If (D, δ) and (E, ε) are indexed domains, define the numbering $\delta \times \varepsilon$ by $(\delta \times \varepsilon)_{\langle i,j\rangle} = (\delta_i, \varepsilon_j)$. Then $(D \times E, \delta \times \varepsilon)$ is an indexed domain again. If (D, δ) and (E, ε) are effectively given, the same is true for $(D \times E, \delta \times \varepsilon)$.

Domain elements of particular interest are those which can be approximated effectively.

Definition 3.5. Let (D, δ) be an indexed domain. An element x of D is called *constructive* if the set $\{\, i \in \omega \mid \delta_i \sqsubseteq x \,\}$ is r.e.

Observe that computable maps map constructive elements to constructive elements. We denote the set of constructive elements of D by D_c. With respect to the restriction of the domain order it is a partial order, which we call *constructive domain*. Note that such domains no longer have least upper bounds for all directed sets S of compact elements of D, but only for those which are *completely enumerable* (c.e.), i.e., for which $\{\, i \mid \delta_i \in S \,\}$ is r.e. Similarly, the restriction of a computable map to the constructive domain elements preserves only least upper bounds of such directed c.e. sets. We call such restrictions computable as well.

For two indexed domains (D, δ) and (E, ε) denote the space of all computable maps from D into E by $[D \to_c E]$ and, similarly, the space of all computable maps from D_c into E_c by $[D_c \to_c E_c]$. Since D_c contains all compact elements of D, every computable map on D_c has a unique computable extension to D. Hence, there is an order-isomorphism between both spaces.

Let us now introduce canonical indexings of the finite domains and the embeddings between finite domains. In order not to have to deal with isomorphic copies we consider only finite domains that have natural numbers as elements. For $m \in \omega$ set

$$E_m = \{\, \langle m, a\rangle \mid a \in \pi_1(\Delta_m) \cup \pi_2(\Delta_m) \,\}$$

and order it by

$$\langle m, a\rangle \sqsubseteq_m \langle m, b\rangle \Leftrightarrow \langle a, b\rangle \in \Delta_m.$$

In case that E_m is a partial order with smallest element, all elements are compact. We enumerate them in the following way:

$$\eta^m_a = \begin{cases} \langle m, a\rangle & \text{if } \langle m, a\rangle \in E_m, \\ n_\perp & \text{otherwise,} \end{cases} \qquad (a \in \omega).$$

Here $n_\perp$ is the smallest element of E_m. Then (E_m, η^m) is an effectively given domain. Now, let $\zeta_m = (E_m, \eta^m)$, if E_m is a partial order with smallest element, and let $\zeta_m = (\{0\}, \{\langle 0, 0\rangle\}, \lambda a.0)$, otherwise. Moreover, for natural numbers $\langle m, i, n\rangle$ such that there is some embedding $F \in [E_m \to E_n]$ with $\Delta_i = \{\, \langle a, b\rangle \mid \eta^n_b = F(\eta^m_a) \,\}$, define $\theta_{\langle m,i,n\rangle} = F$. In any other case set $\theta_{\langle m,i,n\rangle} = \mathrm{Id}_{\{0\}}$. Then ζ and θ, respectively, are numberings of the finite domains and the embeddings between these such that the category $\mathbf{IDOM^{ce}}$ is effectively given.

Definition 3.6. An *effective* SFP domain is a colimit of an effective ω-chain in $\mathbf{IDOM^{ce}}$.

Let $\mathcal{D} = \big((D_m, \delta^m), F_m\big)_{m\in\omega}$ be an effective ω-chain in $\mathbf{IDOM^{ce}}$. Set $F_{mn} = F_{n-1} \circ \cdots \circ F_m$, for $m < n$, and $F_{mm} = \mathrm{Id}_{D_m}$. Moreover, let $\mathrm{in}_m \colon D_m \to D_\infty$, defined by

$$\mathrm{in}_m(x)(n) = \begin{cases} F_{mn}(x) & \text{if } m \leq n, \\ F^R_{nm}(x) & \text{otherwise,} \end{cases}$$

for $x \in D_m$, be the canonical embedding of D_m into D_∞. For $\langle m, a\rangle \in \omega$ set $\delta^\infty_{\langle m,a\rangle} = \mathrm{in}_m(\delta^m_a)$. Then δ^∞ is an indexing of D^0_∞ such that D_∞ is effectively given.

If D is a colimit of $\mathcal{D}$ there is a computable isomorphism $H \in [D \to D_\infty]$. Isomorphisms are embeddings and as we have seen in Lemma 2.3, these are determined by their values on the compact elements. Moreover, they are computable just if their restriction to the compact elements is effective. Let φ_i and φ_j witness that the restrictions of H and H^{-1}, respectively, to D^0 and D^0_∞ are effective. Moreover, let φ_c witness that the ω-chain $\mathcal{D}$ is effective. Then $\langle i, j, c\rangle$ is an *index* of D. This defines a partial indexing σ of the effective SFP domains.

As we shall see next, Plotkin's order-theoretic characterization of the SFP domains also holds in the effective setting.

Definition 3.7. A domain D with total numbering δ of its compact elements is *effectively strongly algebraic* if it is strongly algebraic and the operation $\mathcal{U}$ is effective, that is, there is some function $g \in R^{(1)}$ such that $\mathcal{U}(\delta(\Delta_i)) = \delta(\Delta_{g(i)})$, for all $i \in \omega$.

As we know that for any finite X, $\mathcal{U}^*(X)$ is finite, we can iterate $\mathcal{U}$ a finite number of times until this process gets stable. This shows that in an effectively strongly algebraic domain also the operation $\mathcal{U}^*$ is effective. Moreover, since $\delta_m \sqsubseteq \delta_n$ if and only if $\delta_n \in \mathcal{U}(\{\delta_m, \delta_n\})$, we have that every such domain (D, δ) is effectively given. If i is a Gödel number of the function g witnessing the effectivity of $\mathcal{U}$, then i is called an *index* of D. Let τ denote the indexing of the effectively strongly algebraic domains thus obtained.

Theorem 3.8. Every effective SFP domain is an effectively strongly algebraic domain, and vice versa. Moreover, this equivalence holds effectively, that is, $\sigma \equiv \tau$.

Proof. Let $(D_m, \delta^m), F^m)_{m\in\omega}$ be an effective ω-chain in $\mathbf{IDOM^{ce}}$ and let this be witnessed by $\varphi_c \in R^{(1)}$. Moreover, let $\mathcal{U}_m$ and $\mathcal{U}_\infty$, respectively, denote the operation $\mathcal{U}$ in D_m $(m \geq 0)$ and D_∞.

Now, assume that X is a finite subset of D^0_∞ and let $a \in \omega$ such that $X = \delta^\infty(\Delta_a)$. Then $X = \mathrm{in}_r(X_r)$, where $r = \max \pi_1(\Delta_a)$ and $X_r = \{\, F_{mr}(\delta^m_i) \mid \langle m, i\rangle \in \Delta_a \,\}$. As is pointed out in (Plotkin 1976), $\mathcal{U}_\infty(X) = \mathrm{in}_r(\mathcal{U}_r(X_r))$. Obviously, r can be computed from a. Moreover, an index b can be computed from a and c such that $X_r = \delta^r(\Delta_b)$. Similarly, since X_r is contained in the finite domain D_r an index e can be computed from b and c with $\mathcal{U}_r(\delta^r(\Delta_b)) = \delta^r(\Delta_e)$. Hence,

$$\mathcal{U}_\infty(X) = \mathrm{in}_r(\mathcal{U}_r(X_r)) = \mathrm{in}_r(\mathcal{U}_r(\delta^r(\Delta_b)) = \mathrm{in}_r(\delta_r(\Delta_e)) = \delta^\infty(\{\, \langle r, i\rangle \mid i \in \Delta_e \,\}).$$

Therefore, there is some function $h \in R^{(2)}$ so that $\mathcal{U}_\infty(\delta^\infty(\Delta_a)) = \delta^\infty(\Delta_{h(a,c)})$.

Let (D, δ) be a colimit of $((D_m, \delta^m), F^m)_{m\in\omega}$ with index $\langle i, j, c\rangle$, then there exists a computable isomorphism $H \in [D \to D_\infty]$ so that φ_i and φ_j, respectively, realize the restriction of H and H^{-1} to D^0 and D^0_∞. Thus, there are functions $f, g \in R^{(2)}$ such that for any $a \in \omega$, $H(\delta(\Delta_a)) = \delta^\infty(\Delta_{f(i,a)})$ and $H^{-1}(\delta^\infty(\Delta_a)) = \delta(\Delta_{g(j,a)})$. Let $k \in R^{(1)}$ with $\varphi_{k(\langle i,j,c\rangle)}(a) = g(j, h(f(i,a),c))$. Then $\mathcal{U}(\delta(\Delta_a)) = \delta(\Delta_{\varphi_{k(\langle i,j,c\rangle)}(a)})$, which shows that $\sigma \leq \tau$.

The converse implication follows as in Plotkin's proof (Plotkin 1976, Theorem 5(i)). Canonical indices of the finite domains as well as of the embeddings can be computed since the operation $\mathcal{U}^*$ is effective and the domain order is decidable. Realizers of both, the operation and the decision procedure, are recursive in any realizer for $\mathcal{U}$. This shows that also $\tau \leq \sigma$. □

In the introduction it has already been mentioned that in his dissertation (Kanda 1979b) Kanda studied SFP domains in an effective setting. But he does not work with canonical indexings of the finite domains and embeddings. As a consequence of this, the numbering of his effective SFP domains is weaker than the one used here. He has no equivalence between the numberings of the effective SFP domains and the effectively strongly algebraic domains, respectively, as above.

As is well known, the category **SFP** of SFP domains and continuous maps is Cartesian closed: the one-point domain $\{\bot\}$ is the terminal object, the domain product is the categorical product and the space of continuous maps between two SFP domains is the categorical exponent. Note that for two SFP domain D and E, $[D \to E]$ is an SFP domain again.

Definition 3.9. Let D and E be SFP domains. A finite subset T of $D^0 \times E^0$ is called *joinable* if

$$(\forall T' \subseteq T)[(\forall u \in \mathcal{U}^D(\mathrm{pr}_1(T')))(\exists v \in \mathcal{U}^E(\mathrm{pr}_2(T')))(u,v) \in T].$$

Here pr_i is the projection onto the ith component.

For elements $u \in D^0$ and $v \in E^0$ define the *step function* $(u \searrow v)\colon D \to E$ by

$$(u \searrow v)(x) = \begin{cases} v & \text{if } u \sqsubseteq x, \\ \bot & \text{otherwise,} \end{cases} \qquad (x \in D).$$

Then the compact elements of $[D \to E]$ are exactly the maps of the form $\bigsqcup\{\, (u_i \searrow v_i) \mid i \in I \,\}$, where $u_i \in D^0$ and $v_i \in E^0$, for $i \in I$, so that $\{\, (u_i, v_i) \mid i \in I \,\}$ is joinable.

If (D, δ) and (E, ε) are effective SFP domains, then it follows with Theorem 3.8 that the set $\{\, i \in \omega \mid \{\, (\delta_m, \varepsilon_n) \mid \langle m, n\rangle \in \Delta_i \,\} \text{ is joinable} \,\}$ is recursive. Thus we can define a numbering γ of $[D \to E]^0$ by setting

$$\gamma_i = \begin{cases} \bigsqcup\{\, (\delta_m \searrow \varepsilon_n) \mid \langle m,n\rangle \in \Delta_i \,\} & \text{if } \{\, (\delta_m, \varepsilon_n) \mid \langle m,n\rangle \in \Delta_i \,\} \text{ is joinable,} \\ (\bot_D \searrow \bot_E) & \text{otherwise,} \end{cases}$$

for $i \in \omega$. Then it is easily verified that $([D \to E], \gamma)$ is effectively given. It is even an effective SFP domain. In addition, we have the important property that an element F of $[D \to E]$ is constructive exactly if it is a computable map. Note that F is uniquely determined by its values on the computable elements.

Define a *constructive SFP domain* to be the constructive domain obtained from an effective SFP domain, then we achieve the following result.

Theorem 3.10. The categories **ESFP** of effective SFP domains and continuous maps and **CSFP** of constructive SFP domains and computable maps are both Cartesian closed.

4. The conjecture

In his paper (Smyth 1983) Smyth conjectured that the proof of his Theorem 1 may be used to show that with respect to appropriate effectivity notions the following statement be true:

If D and $[D \to D]$ are effectively algebraic domains, then D is an effectively strongly algebraic domain.

In effectivity considerations of domains these usually have to be effectively given. The effectivity requirement in the definition of effectively given domains is quite weak. It is not clear to us how the set of minimal upper bounds of a finite set of compact elements can be computed in the case of such domains. We therefore strengthen this condition.

Definition 4.1. A domain D with a total numbering δ of its compact elements has a *completeness test* if the set

$$\{ \langle i,j \rangle \mid \delta(\Delta_j) \subseteq \mathrm{UB}(\delta(\Delta_i)) \wedge \delta(\Delta_j) \text{ is complete for } \delta(\Delta_i) \}$$

is recursive.

Lemma 4.2. Let (D, δ) be an indexed domain that has a completeness test. Then the following three statements hold:

1 (D, δ) is effectively given.
2 The set $\{ i \mid \delta(\Delta_i))$ is compatible$\}$ is recursive.
3 If for any finite set X of compact elements of D, $\mathcal{U}(X)$ is finite and complete for X, then $\mathcal{U}$ is effective.

Proof. (1). Observe that $\delta_i \sqsubseteq \delta_j$ if and only if $\{\delta_j\}$ is complete for $\{\delta_i, \delta_j\}$.

(2). Note that $\delta(\Delta_i)$ is compatible exactly if the empty set is not complete for $\delta(\Delta_i)$. This is obvious if $\delta(\Delta_i)$ is not empty. In the other case every domain element is an upper bound and hence a set that is complete for $\delta(\Delta_i)$ must contain the least domain element.

(3). Let g be the function computed by the following algorithm:

input: `i`;
`n := 0;`
`Z :=` $\delta(\Delta_\mathtt{i})$;
<u>while</u> `not[Z` $\subseteq$ `UB(`$\delta(\Delta_\mathtt{i})$`) and Z complete for` $\delta(\Delta_\mathtt{i})$`]` <u>do</u>
  `Z :=` <u>if</u> $\delta_n \in$ `UB(`$\delta(\Delta_i)$`)` <u>then</u> `Z` $\cup$ $\{\delta_\mathtt{n}\}$ <u>else</u> `Z;`
  `n := n+1`
<u>od</u>;
`find the set` $\delta(\Delta_\mathtt{j})$ `of all minimal elements of` Z;
output: `j`

Since $\mathcal{U}(\delta(\Delta_i))$ is finite as well as complete for $\delta(\Delta_i)$ and any set of upper bounds of $\delta(\Delta_i)$ that includes $\mathcal{U}(\delta(\Delta_i))$ is also complete for $\delta(\Delta_i)$, the `while`-loop always terminates. Hence g is total and realizes $\mathcal{U}$. □

Let D and E be two indexed domains with completeness test. We have already seen that $D \times E$ is an indexed domain as well. Moreover, it is the categorical product of D and E in the category of indexed domains with continuous maps. As follows from the next lemma, it also has a completeness test.

Lemma 4.3. Let X be a finite compatible set of compact elements of $D \times E$. Then a finite subset Z of upper bounds of X is complete for X if and only if the following three conditions hold:

1 $\mathrm{pr}_D(Z)$ is complete for $\mathrm{pr}_D(X)$.
2 $\mathrm{pr}_E(Z)$ is complete for $\mathrm{pr}_E(X)$.
3 For all $(z_1, z_2) \in \mathrm{pr}_D(Z) \times \mathrm{pr}_E(Z)$ there is some $z \in Z$ with $z \sqsubseteq (z_1, z_2)$.

Proof. Obviously, a subset S of $D \times E$ is compatible just if both $\mathrm{pr}_D(S)$ and $\mathrm{pr}_E(S)$ are compatible. Now, assume that Z is complete for X and let $x_1 \in \mathrm{UB}(\mathrm{pr}_D(X))$. Since $\mathrm{pr}_E(X)$ compatible there is some $x_2 \in \mathrm{UB}(\mathrm{pr}_E(X))$. It follows for all $(y_1, y_2) \in \mathrm{pr}_D(X) \times \mathrm{pr}_E(X)$ that $(y_1, y_2) \sqsubseteq (x_1, x_2)$, which implies $(x_1, x_2) \in \mathrm{UB}(X)$. Thus, there is some $(z_1, z_2) \in Z$ with $(z_1, z_2) \sqsubseteq (x_1, x_2)$. This shows that for all $x_1 \in \mathrm{UB}(\mathrm{pr}_D(X))$ there is some $z_1 \in \mathrm{pr}_D(Z)$ so that $z_1 \sqsubseteq x_1$, i.e., $\mathrm{pr}_D(Z)$ is complete for $\mathrm{pr}_D(X)$. In the same way the second condition follows. For the third requirement note that $\mathrm{pr}_D(Z) \times \mathrm{pr}_E(Z)$ is a set of upper bounds of X.

The converse implication is easily shown. If $(x_1, x_2) \in \mathrm{UB}(X)$, then $x_1 \in \mathrm{UB}(\mathrm{pr}_D(X))$ and $x_2 \in \mathrm{UB}(\mathrm{pr}_E(X))$. Hence, there are $z_1 \in \mathrm{pr}_D(Z)$ and $z_2 \in \mathrm{pr}_E(Z)$ so that $(z_1, z_2) \sqsubseteq (x_1, x_2)$, which implies that $z \sqsubseteq (x_1, x_2)$, for some $z \in Z$. □

If X is not compatible, only the empty set is complete for X. It follows that the category **IDOMC** of indexed domains with completeness test and continuous maps is Cartesian.

Proposition 4.4. **ESFP** is a proper full subcategory of **IDOMC**.

Proof. Let (D, δ) be an effective SFP domain and note that

$$\begin{aligned}\delta(\Delta_j) \subseteq \mathrm{UB}(\delta(\Delta_i)) \wedge \delta(\Delta_j) &\text{ is complete for } \delta(\Delta_i) \\ &\Leftrightarrow \mathcal{U}(\delta(\Delta_i)) \subseteq \delta(\Delta_j) \wedge \delta(\Delta_j) \subseteq \mathrm{UB}(\delta(\Delta_i)).\end{aligned}$$

Since D is effectively strongly algebraic and hence also effectively given, the right hand side of this equivalence is recursive in i and j. Thus, D has a completeness test. □

Now, we can state our version of Smyth's conjecture.

Theorem 4.5. If D and $[D \to D]$ are indexed domains such that D has a completeness test, then D is effectively strongly algebraic.

By Smyth's result D is an SFP domain and hence, by Lemma 4.2(3), it is effectively strongly algebraic.

The second important result in Smyth's paper says that **SFP** is the largest Cartesian closed full subcategory of **DOM**. For the proof he needed the next result.

Lemma 4.6. Let **K** be a full subcategory of the category **CPO** of cpo's and continuous maps. Then the following three statements hold:

1 If **K** has a terminal object T, then T is the one-point cpo.
2 If **K** has a terminal object and the product $A \times_{\mathbf{K}} B$ of objects A and B exists, then $A \times_{\mathbf{K}} B$ is isomorphic in **CPO** to the usual product $A \times B$.
3 If **K** has a terminal object and all products of pairs, and the exponent E^D of objects D and E exists, then E^D is isomorphic in **CPO** to the usual function space $[D \to E]$.

Now note that if D is an indexed domain with a completeness test and D is isomorphic to a cpo E, then also E is an indexed domain with a completeness test. With Theorem 4.5 we therefore obtain the following analogue of Smyth's result.

Theorem 4.7. **ESFP** is the largest Cartesian closed full subcategory of **IDOMC**.

5. The constructive case

In the rest of this paper we deal with the question whether a similar statement is true with respect to **CSFP** and a suitable category of constructive domains. We shall see that such a statement holds, but only under an additional effectivity assumption. Note that now, since we are working in a category with computable maps as morphisms, one has to consider the space of all computable maps instead of the space of all continuous maps, i.e., the assumptions about the domain used in the verification of the requirements for a strongly algebraic domain are weaker than in the classical case.

Let X be a finite set of compact elements. As long as we do not know that $\mathcal{U}(X)$ is finite, we cannot use the completeness test to find out whether it is complete for X. To the contrary, in the case that $\mathcal{U}(X)$ is not complete we must have an effective witness for this. As we will see it is sufficient here to consider only sets X of cardinality two.

Definition 5.1. We say that an indexed domain (D, δ) *provides incompleteness realizers* if for any two compact elements x_1, x_2 of D the set

$$\mathrm{NC}_{\{x_1,x_2\}} = \{\, i \mid \neg(\exists j \in \omega)\delta_j \in \mathcal{U}(\{x_1, x_2\}) \wedge \delta_j \sqsubset \delta_i \,\}$$

is r.e.

Here, $\delta_j \sqsubset \delta_i$ means that $\delta_j \sqsubseteq \delta_i$, but $\delta_j \neq \delta_i$.

Note that this condition is less restrictive than it might seem. We will use it to show that for the domains we are interested in, $\mathcal{U}(\{x_1, x_2\})$ is always finite. Since these domains are also effectively given, $\mathrm{NC}_{\{x_1,x_2\}}$ is even recursive.

Lemma 5.2. Let (D, δ) be an effectively given domain and $x_1, x_2 \in D^0$ such that $\mathcal{U}(\{x_1, x_2\})$ is complete for $\{x_1, x_2\}$ and $\mathrm{NC}_{\{x_1,x_2\}}$ is r.e. Then $\{\, i \mid \delta_i \in \mathcal{U}(\{x_1, x_2\}) \,\}$ is recursive.

Proof. We have for $i \in \omega$ that

$$\delta_i \notin \mathcal{U}(\{x_1, x_2\}) \Leftrightarrow x_1 \not\sqsubseteq \delta_i \vee x_2 \not\sqsubseteq \delta_i \vee [(\exists j \in \omega)\delta_j \sqsubset \delta_i \wedge x_1 \sqsubseteq \delta_j \wedge x_2 \sqsubseteq \delta_j]$$

and

$$\delta_i \in \mathcal{U}(\{x_1, x_2\}) \Leftrightarrow x_1 \sqsubseteq \delta_i \wedge x_2 \sqsubseteq \delta_i \wedge i \in \mathrm{NC}_{\{x_1,x_2\}} .$$

Note that in the last line the right hand side follows from the left one by the minimality of δ_i and the converse implication holds because of the completeness of $\mathcal{U}(\{x_1, x_2\})$. □

In the remainder of this paper let **IDOMCI** be the category with indexed domains that have a completeness test and provide incompleteness realizers as objects and continuous maps as morphisms. Moreover, let **CDOMCI** be the category of constructive domains obtained from domains in **IDOMCI** and computable maps. Obviously, both categories are Cartesian. The domain product is the categorical product. Note that for two indexed domains D and E and compact elements $(x_1, y_1), (x_2, y_2)$ of $D \times E$, $\mathrm{NC}^{D\times E}_{\{(x_1,y_1),(x_2,y_2)\}} = \langle \mathrm{NC}^D_{\{x_1,x_2\}}, \omega\rangle \cup \langle \omega, \mathrm{NC}^E_{\{y_1,y_2\}}\rangle$.

The proof of the analogue of Theorem 4.7 we are going to present consists of four main steps. First we show that in the Cartesian closed full subcategories we will consider the space $[D \to_c D]$ can be taken as exponential object. Then we show that for any finite set X of compact elements of a domain in such a category, $\mathcal{U}(X)$ is finite and complete for X and $\mathcal{U}^*(X)$ is finite as well.

5.1. *Exponents*

In addition to the above effectivity assumptions on domains we require that also the categorical setting we are using satisfies certain effectivity conditions. But, whereas in the definition of effective SFP domains we used rather strong, though absolutely natural, requirements, we shall now employ only very weak conditions.

Definition 5.3. Let $\mathbf{K}$ be a category and for any two objects A and B, $\alpha^{A,B}$ be a partial indexing of the morphism set $\mathbf{K}[A, B]$. Then $(\mathbf{K}, (\alpha^{A,B})_{A,B\in\mathrm{Ob}_\mathbf{K}})$ is called *weakly indexed.*

Definition 5.4. Let $(\mathbf{K}, (\alpha^{A,B})_{A,B\in\mathrm{Ob}_\mathbf{K}}), (\mathbf{K}', (\beta^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}'}})$ be weakly indexed categories.

1 The categorical product $(A\times B, \mathrm{pr}_A, \mathrm{pr}_B)$ of two objects A and B of $\mathbf{K}$ is *constructive* if for any object C of $\mathbf{K}$ there is a function $\mathrm{prod}_C \in P^{(2)}$ such that for all $a \in \mathrm{dom}(\alpha^{C,A})$ and all $b \in \mathrm{dom}(\alpha^{C,B})$, $\mathrm{prod}_C(a, b)\downarrow \in \mathrm{dom}(\alpha^{C,A\times B})$ and $\alpha^{C,A\times B}_{\mathrm{prod}_C(a,b)}$ is the unique morphism in $\mathbf{K}[C, A \times B]$ with

$$\alpha^{C,A}_a = \mathrm{pr}_A \circ \alpha^{C,A\times B}_{\mathrm{prod}_C(a,b)} \quad \text{and} \quad \alpha^{C,B}_b = \mathrm{pr}_B \circ \alpha^{C,A\times B}_{\mathrm{prod}_C(a,b)}.$$

2 $(\mathbf{K}, (\alpha^{A,B})_{A,B\in\mathrm{Ob}_\mathbf{K}})$ is an *effectively full* subcategory of $(\mathbf{K}', (\beta^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}'}})$ if $\mathbf{K}$ is a full subcategory of $\mathbf{K}'$ and for all objects A, B of $\mathbf{K}$, $\alpha^{A,B} \equiv \beta^{A,B}$.

Definition 5.5. A weakly indexed category $\mathbf{K}$ is *constructively Cartesian closed* if $\mathbf{K}$ contains a terminal object and for every pair of objects there is a constructive categorical product and a categorical exponent.

We now have to verify that **CSFP** and **CDOMCI** are weakly effective categories. Let (D, δ) and (E, ε) be both effective SFP domains or both domains that have a completeness test and provide incompleteness realizers. If $F\colon D \to E$ is computable, then we call any r.e. index of graph_F an *index* of the restriction of F to D_c. This defines partial indexings $\psi^{D,E}$ and $\rho^{D,E}$, respectively, of $\mathbf{CSFP}[D, E]$ and $\mathbf{CDOMCI}[D, E]$.

Proposition 5.6.

1 The category **CDOMCI** is weakly indexed.
2 The category **CSFP** is a weakly indexed constructively Cartesian closed effectively full subcategory of **CDOMCI**.

For a constructive SFP domain (D, δ) such that $g \in R^{(1)}$ witnesses the effectivity of $\mathcal{U}$, the effectivity conditions that objects in **CDOMCI** have to satisfy are easily verified. In this case NC_X is the set of all indices i such that δ_i is either in $\mathcal{U}(X)$ or not among the upper bounds of X. Moreover, for the completeness test one only has to see whether δ_a is an upper bound of $\delta(\Delta_i)$, for every $a \in \Delta_j$, and whether for every $b \in \Delta_{g(i)}$ there is some $e \in \Delta_j$ with $\delta_b = \delta_e$, i.e., whether $\mathcal{U}(\delta(\Delta_i)) \subseteq \delta(\Delta_j))$.

In the framework of weakly effective categories Lemma 4.6 can be strengthened.

Lemma 5.7. Let $(\mathbf{K}, (\alpha^{A,B})_{A,B \in \mathrm{Ob}_{\mathbf{K}}})$ be a weakly indexed effectively full subcategory of **CDOMCI**. Then the following three statements hold:

1 If $\mathbf{K}$ has a terminal object T, then T is the one-point domain.
2 Let $\mathbf{K}$ have a terminal object. If the product $A \times_{\mathbf{K}} B$ of objects A and B exists and is constructive, then $A \times_{\mathbf{K}} B$ is isomorphic in **CDOMCI** to the usual product $A \times B$.
3 Let $\mathbf{K}$ have a terminal object and constructive products for all pairs. If the exponent B^A of objects A and B exists in $\mathbf{K}$, then the function space $[A \to_c B]$ is an object of **CDOMCI** and isomorphic to B^A.

Proof. The proof is a refinement of Smyth's proof of Lemma 4.6 (cf. (Smyth 1983, Lemma 5)). Statement (1) follows by the same argument. As is readily verified, the functions constructed there are computable.

(2). Let (A, δ^A), (B, δ^B) be objects of $\mathbf{K}$ and $F\colon A \times_{\mathbf{K}} B \to A \times B$ be the computable map with

$$\mathrm{pr}_A^{\mathbf{K}} = \mathrm{pr}_A \circ F \quad \text{and} \quad \mathrm{pr}_B^{\mathbf{K}} = \mathrm{pr}_B \circ F.$$

Smyth shows that there is a continuous map $G\colon A \times B \to A \times_{\mathbf{K}} B$ such that $G \circ F$ and $F \circ G$, respectively, are the identity maps on $A \times_{\mathbf{K}} B$ and $A \times B$. It remains to show that G is also computable.

For $(x, y) \in A \times B$, $G(x, y) = H_{(x,y)}(\bot)$, where $H_{(x,y)} \in \mathbf{K}[T, A \times_{\mathbf{K}} B]$ is the unique morphism such that

$$\lambda z.x = \mathrm{pr}_A^{\mathbf{K}} \circ H_{(x,y)} \quad \text{and} \quad \lambda z.y = \mathrm{pr}_B^{\mathbf{K}} \circ H_{(x,y)}.$$

Let $f \in R^{(1)}$ with $W_{f(i)} = \{\, \langle m, n\rangle \mid m, n \in \omega \wedge \delta^A_n \sqsubseteq \delta^A_i \,\}$. Then $f(i)$ is an index of the map $\lambda z.\delta^A_i \in \mathbf{K}[T, A]$ with respect to **CDOMCI**. Since $\mathbf{K}$ is an effectively full subcategory of **CDOMCI**, there is also a function $f' \in R^{(1)}$ so that $f'(i)$ is an index of this map with respect to $\mathbf{K}$, i.e., $\alpha^{T,A}_{f'(i)} = \lambda z.\delta^A_i$. Similarly, there is a function $g' \in R^{(1)}$ such that $\alpha^{T,B}_{g'(j)} = \lambda z.\delta^B_j$. Hence,

$$H_{(\delta^A_i, \delta^B_j)} = \alpha^{T, A\times_{\mathbf{K}} B}_{\mathrm{prod}_T(f'(i), g'(j))}.$$

Using again that $\mathbf{K}$ is an effectively full subcategory it follows that there is also a function $h \in R^{(2)}$ so that $h(i,j)$ is an index of $H_{(\delta^A_i, \delta^B_j)}$ with respect to **CDOMCI**, i.e.,

$$W_{h(i,j)} = \{\, \langle a, b\rangle \mid a, b \in \omega \wedge \delta^{A\times_{\mathbf{K}} B}_b \sqsubseteq H_{(\delta^A_i, \delta^B_j)}(\bot) \,\}.$$

Thus

$$\delta^{A\times_{\mathbf{K}} B}_b \sqsubseteq G(\delta^A_i, \delta^B_j) \Leftrightarrow (\exists a \in \omega) \langle a, b\rangle \in W_{h(i,j)},$$

which shows that G is computable.

(3). Let (A, δ^A), (B, δ^B) be objects of $\mathbf{K}$ such that the exponent (B^A, δ^{B^A}) exists in $\mathbf{K}$. For $y \in B^A$ and $x \in A$ set

$$\mathcal{F}(y)(x) = \mathrm{eval}_{A,B}(y, x).$$

Then

$$\delta^B_n \sqsubseteq \mathcal{F}(y)(\delta^A_m) \Leftrightarrow \delta^B_n \sqsubseteq \mathrm{eval}_{A,B}(y, \delta^A_m) \Leftrightarrow (\exists a \in \omega) \delta^{B^A}_a \sqsubseteq y \wedge \delta^B_n \sqsubseteq \mathrm{eval}_{A,B}(\delta^{B^A}_a, \delta^A_m).$$

Since B^A is also an object of **CDOMCI**, we have that y is constructive, i.e., $\{\, i \mid \delta^{B^A}_i \sqsubseteq y \,\}$ is r.e. Therefore the right hand side in the last line is r.e. in m, n. It follows that $\mathcal{F}(y) \in [A \to_c B]$, which means that $\mathcal{F}(y) \in \mathbf{K}[A, B]$, as $\mathbf{K}$ is a full subcategory of **CDOMCI**.

For any $F \in \mathbf{K}[A, B]$ there is a unique $F' \in \mathbf{K}[T \times A, B]$ with $F(x) = F'(\bot, x)$ and for any such F' there is, by the universal property of B^A, a unique $L_F \in \mathbf{K}[T, B^A]$ with $L_F(\bot) = F'$. It follows that for all $x \in A$

$$\mathrm{eval}_{A,B}(L_F(\bot), x) = F'(\bot, x) = F(x).$$

Define $\mathcal{G}\colon \mathbf{K}[A, B] \to B^A$ by $\mathcal{G}(F) = L_F(\bot)$. Then we obtain as in (Smyth 1983) that $\mathcal{G}$ and $\mathcal{F}$ are inverse to each other and order preserving.

Next, note that since A, B, B^A are objects in **CDOMCI**, there are domains $\overline{A}$, $\overline{B}$, $\overline{B^A}$ in **IDOMCI** such that $A = \overline{A}_c$, $B = \overline{B}_c$ and $B^A = (\overline{B^A})_c$, and observe that $(\overline{B^A})^0 \subseteq B^A$. Let $K_{A,B}$ be the ideal completion of $\mathcal{F}((B^A)^0)$ with respect to the partial order in $\mathbf{K}[A, B]$. Note here that $\mathbf{K}[A, B]$ is order-isomorphic to a subset of $K_{A,B}$. Without restriction we identify both sets. Now, let $\kappa = \mathcal{F} \circ \delta^{B^A}$. Then $(K_{A,B}, \kappa)$ is an indexed domain. Since $B^A = (\overline{B^A})_c$, we moreover have that

$$\begin{aligned} H \in \mathbf{K}[A,B] &\Leftrightarrow \mathcal{G}(H) \in B^A \\ &\Leftrightarrow \{\, i \mid \delta_i^{B^A} \sqsubseteq \mathcal{G}(H) \,\} \text{ r.e.} \\ &\Leftrightarrow \{\, i \mid \mathcal{F}(\delta_i^{B^A}) \sqsubseteq H \,\} \text{ r.e.} \\ &\Leftrightarrow \{\, i \mid \kappa_i \sqsubseteq H \,\} \text{ r.e.} \\ &\Leftrightarrow H \in (K_{A,B})_c. \end{aligned}$$

Both, $\mathcal{F}$ and $\mathcal{G}$ have unique continuous extensions $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$, respectively, to $\overline{B^A}$ and $K_{A,B}$. As is easily verified, they are inverse to each other. As a consequence we obtain that $(K_{A,B}, \kappa)$ has a completeness test and provides incompleteness verifiers. Thus, it is an object in **IDOMCI** and hence $(\mathbf{K}[A,B], \kappa)$ is an object in **CDOMCI**. It remains to show that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are computable. For $m, n \in \omega$ we have that

$$\kappa_n \sqsubseteq \overline{\mathcal{F}}(\delta_m^{B^A}) \Leftrightarrow \mathcal{F}(\delta_n^{B^A}) \sqsubseteq \mathcal{F}(\delta_m^{B^A}) \Leftrightarrow \delta_n^{B^A} \sqsubseteq \delta_m^{B^A}$$

and

$$\delta_n^{B^A} \sqsubseteq \overline{\mathcal{G}}(\kappa_m) \Leftrightarrow \delta_n^{B^A} \sqsubseteq \mathcal{G}(\mathcal{F}(\delta_m^{B^A})) \Leftrightarrow \delta_n^{B^A} \sqsubseteq \delta_m^{B^A}.$$

Since B^A has a completeness test and is therefore effectively given, it follows that both maps are indeed computable. □

5.2. *Completeness*

For the remainder of this paper let $(\mathbf{K}, (\alpha^{D,E})_{D,E \in \mathrm{Ob}_{\mathbf{K}}})$ be a constructively Cartesian closed effectively full subcategory of **CDOMCI**. Moreover, assume that (D, δ) is an object of $\mathbf{K}$ with exponent $([D \to_c D], \nu)$. In this and the next two sections we will show that such a D must be a constructive domain that is obtained from an effectively strongly algebraic domain. As above we have that since (D, δ) is in **CDOMCI**, it is obtained from an indexed domain $(\overline{D}, \delta)$ in **IDOMCI**.

Lemma 5.8. $\mathcal{U}(\{x_1, x_2\})$ is complete for $\{x_1, x_2\}$, for all $x_1, x_2 \in D^0$.

Proof. The proof is a modification of Smyth's proof of his analogous result. Assume to the contrary, that there are elements $x_1, x_2 \in D^0$ such that $\mathcal{U}(\{x_1, x_2\})$ is not complete for $\{x_1, x_2\}$. Moreover, let $C = \{\, i \in \mathrm{NC}_{\{x_1,x_2\}} \mid \delta_i \in \mathrm{UB}(\{x_1, x_2\}) \,\}$. Then C is r.e. As in (Smyth 1983, Lemma 1) a function $g \in R^{(1)}$ with $\mathrm{range}(g) \subseteq C$ can be constructed such that $(\delta_{g(i)})_{i \in \omega}$ is strictly decreasing with respect to the domain order and for any $a \in \omega$

$$(\forall i \in \omega)\delta_a \sqsubseteq \delta_{g(i)} \Rightarrow a \notin C \qquad \text{(and hence } \delta_a \notin \mathrm{UB}(\{x_1, x_2\})).$$

Let $\sigma \in R^{(1)}$ be an increasing function with $n \le \sigma(n)$, for all n, and let the continuous map $\tilde{\sigma}\colon D \to D$ be defined by

$$\tilde{\sigma}(x) = \begin{cases} \bot & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_1 & \text{if } x_1 \sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_2 & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \sqsubseteq x, \\ \delta_{g(0)} & \text{if } x \in \mathrm{UB}(\{x_1, x_2\}) \wedge x \not\sqsubseteq \delta_{g(0)}, \\ \delta_{g(\sigma(n))} & \text{where } n \text{ is the greatest } k \text{ such that } x \sqsubseteq \delta_{g(k)}, \\ & \text{if } x \in \mathrm{UB}(\{x_1, x_2\}) \wedge x \sqsubseteq \delta_{g(0)}. \end{cases}$$

Note that by what we have just seen there must be some $m \in \omega$ with $x \not\sqsubseteq \delta_{g(m)}$ in the last case. Since

$$\begin{aligned} \delta_j \sqsubseteq \tilde{\sigma}(\delta_i) \Leftrightarrow & [x_1 \not\sqsubseteq \delta_i \wedge x_2 \not\sqsubseteq \delta_i \wedge \delta_j \sqsubseteq \bot] \\ & \vee [x_1 \sqsubseteq \delta_i \wedge x_2 \not\sqsubseteq \delta_i \wedge \delta_j \sqsubseteq x_1] \\ & \vee [x_1 \not\sqsubseteq \delta_i \wedge x_2 \sqsubseteq \delta_i \wedge \delta_j \sqsubseteq x_2] \\ & \vee [\delta_i \in \mathrm{UB}(\{x_1, x_2\}) \wedge \delta_i \not\sqsubseteq \delta_{g(0)} \wedge \delta_j \sqsubseteq \delta_{g(0)}] \\ & \vee [\delta_i \in \mathrm{UB}(\{x_1, x_2\}) \wedge \delta_i \sqsubseteq \delta_{g(0)} \wedge (\exists n \in \omega) \delta_i \not\sqsubseteq \delta_{g(n+1)} \wedge \delta_j \sqsubseteq \delta_{g(\sigma(n))}], \end{aligned}$$

$\tilde{\sigma}$ is computable.

For each such map $\tilde{\sigma}$ we can find an increasing sequence $\tilde{\sigma}_0 \sqsubseteq \tilde{\sigma}_1 \sqsubseteq \cdots$ which has $\tilde{\sigma}$ as its least upper bound. Define $\sigma_n \in R^{(1)}$ by

$$\sigma_n(i) = \begin{cases} \sigma(i) & \text{if } i < n, \\ \sigma(i+1) & \text{if } i \geq n. \end{cases}$$

Now, let $\iota(n) = n$, then $\tilde{\iota} \in [D \to_c D]$. Since $[D \to_c D]$ is a constructive domain, the set of all compact functions below $\tilde{\iota}$ must be directed. We will show that this is not the case and thus derive a contradiction.

By construction $(x_\nu \searrow x_\nu) \sqsubseteq \tilde{\iota}$, for $\nu = 1, 2$. Hence, there is some compact function $F \in [D \to_c D]$ with $(x_\nu \searrow x_\nu) \sqsubseteq F \sqsubseteq \tilde{\iota}$ $(\nu = 1, 2)$. As we will verify now, $F \sqsubseteq \tilde{\iota} \circ F$.

In case that both $x_1 \not\sqsubseteq x$ and $x_2 \not\sqsubseteq x$, $x_1 \sqsubseteq x$ but $x_2 \not\sqsubseteq x$, or $x_1 \not\sqsubseteq x$ but $x_2 \sqsubseteq x$, we have that $F(x) = \bot$, $F(x) = x_1$ and $F(x) = x_2$, respectively, i.e., $F(x) = \tilde{\iota}(F(x))$. If $x_1, x_2 \sqsubseteq x$, but $x \not\sqsubseteq \delta_{g(0)}$, $F(x) \sqsubseteq \delta_{g(0)}$. Hence $\tilde{\iota}(F(x)) = \delta_{g(m)}$, where m is the greatest k with $F(x) \sqsubseteq \delta_{g(k)}$. It follows that $F(x) \sqsubseteq \tilde{\iota}(F(x))$, similarly in the remaining case.

By the remark above we obtain that $F \sqsubseteq \bigsqcup_n \tilde{\iota}_n \circ F$. Because F is compact, there is some $\bar{n}$ such that $F \sqsubseteq \tilde{\iota}_{\bar{n}} \circ F$. Thus $F(\delta_{g(\bar{n})}) \sqsubseteq \tilde{\iota}_{\bar{n}}(F(\delta_{g(\bar{n})}))$. Since $x_1, x_2 \sqsubseteq F(\delta_{g(\bar{n})}) \sqsubseteq \delta_{g(\bar{n})}$, we have for $m = \max\{\, k \mid F(\delta_{g(\bar{n})}) \sqsubseteq \delta_{g(k)} \,\}$ that $m \geq \bar{n}$. Hence $\tilde{\iota}_{\bar{n}}(F(\delta_{g(\bar{n})})) = \delta_{g(m+1)}$, which implies that $F(\delta_{g(\bar{n})}) \sqsubseteq \delta_{g(m+1)}$. This is impossible by the definition of m. Consequently, the set of compact functions below $\tilde{\iota}$ cannot be directed. □

5.3. *Minimal upper bounds*

In the next step we will show that $\mathcal{U}(\{x_1, x_2\})$ is finite. Here, the proofs in (Smyth 1983; Jung 1989) proceed in such a way that under the assumption that $\mathcal{U}(\{x_1, x_2\})$ is infinite it is shown that $[\overline{D} \to \overline{D}]$ has uncountably many compact elements, which implies that it is not ω-algebraic. This construction is of no use in our constructive setting. Thus, another contradiction is derived.

The general idea is to show that if $\mathcal{U}(\{x_1, x_2\})$ is infinite then an effective enumeration of all computable functions on $\mathcal{U}(\{x_1, x_2\})$ can be constructed and to use a simple diagonalization argument to show that this is impossible.

Lemma 5.9. Let $x_1, x_2 \in D^0$. If $\mathcal{U}(\{x_1, x_2\})$ is not finite, then it has no compatible subset with at least two elements.

Proof. The proof uses an idea of Jung (Jung 1989, Lemma 2.13). Assume there exist distinct $y_1, y_2 \in \mathcal{U}(\{x_1, x_2\})$ and $z \in \mathrm{UB}(\{x_1, x_2\})$ with $y_1, y_2 \sqsubseteq z$. Define $G\colon \overline{D} \to \overline{D}$ by

$$G(x) = \begin{cases} \bot & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_1 & \text{if } x_1 \sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_2 & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \sqsubseteq x, \\ y_1 & \text{if } x_1 \sqsubseteq x \wedge x_2 \sqsubseteq x. \end{cases}$$

Then G is computable. Moreover, it is a minimal upper bound of $(x_1 \searrow x_1)$ and $(x_2 \searrow x_2)$. Hence, it is also compact.

By Lemma 5.2 the set A of all indices i such that $\delta(\Delta_i)$ is contained in $\mathcal{U}(\{x_1, x_2\})$ is r.e. For any such i define a map $F_i\colon \overline{D} \to \overline{D}$ as follows

$$F_i(x) = \begin{cases} \bot & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_1 & \text{if } x_1 \sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_2 & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \sqsubseteq x, \\ y_2 & \text{if } x \in \mathcal{U}(\{x_1, x_2\}) \setminus \delta(\Delta_i), \\ z & \text{otherwise.} \end{cases}$$

Then also F_i is computable and there is some function $h \in R^{(1)}$ with $\mathrm{graph}_{F_i} = W_{h(i)}$. Let F be the least upper bound in $[\overline{D} \to \overline{D}]$ of all F_i. Then $\mathrm{graph}_F = \{\, \langle m, n\rangle \mid (\exists i \in A) \langle m, n\rangle \in \mathrm{graph}_{F_i} \}$, which shows that $F \in [\overline{D} \to_c \overline{D}]$. Moreover, we have for $x \in \mathcal{U}(\{x_1, x_2\})$ that $F(x) = z$. Thus, $G \sqsubseteq F$. On the other hand, for no $i \in A$, $G \sqsubseteq F_i$. Since A is infinite, this contradicts the compactness of G. □

Lemma 5.10. Let x_1 and x_2 be compact elements of D such that $\mathcal{U}(\{x_1, x_2\})$ is not finite. Moreover, let $F\colon \mathcal{U}(\{x_1, x_2\}) \to \mathcal{U}(\{x_1, x_2\})$. Then there is a map $\hat{F} \in \mathcal{U}(\{(x_1 \searrow x_1), (x_2 \searrow x_2)\})$ which coincides with F on $\mathcal{U}(\{x_1, x_2\})$. Moreover, if there is some $p \in P^{(1)}$ so that for $i \in \omega$ with $\delta_i \in \mathcal{U}(\{x_1, x_2\})$, $p(i){\downarrow}$ and $F(\delta_i) = \delta_{p(i)}$, then $\hat{F}$ is computable.

Proof. Define $\hat{F}$ by

$$\hat{F}(x) = \begin{cases} \bot & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_1 & \text{if } x_1 \sqsubseteq x \wedge x_2 \not\sqsubseteq x, \\ x_2 & \text{if } x_1 \not\sqsubseteq x \wedge x_2 \sqsubseteq x, \\ F(\text{the uniquely determined } y \in \mathcal{U}(\{x_1, x_2\}) \text{ with } y \sqsubseteq x) & \text{if } x_1 \sqsubseteq x \wedge x_2 \sqsubseteq x. \end{cases}$$

Note that in the last case there is always some $y \in \mathcal{U}(\{x_1, x_2\})$ with $y \sqsubseteq x$, by Lemma 5.8, and there is at most one such y, by Lemma 5.9. As is readily verified, $\hat{F}$ is monotone and commutes with existing least upper bounds. Observe that for $x \sqsubseteq x'$ and $y, y' \in \mathcal{U}(\{x_1, x_2\})$ with $y \sqsubseteq x$ and $y' \sqsubseteq x'$ we have that $y = y'$. The remaining properties follow quite easily. □

As a minimal upper bound of finitely many compact elements, $\hat{F}$ is also compact.

With the preceding results we can now derive that $\mathcal{U}(\{x_1, x_2\})$ must be finite, for all compact elements x_1, x_2 of D.

Lemma 5.11. $\mathcal{U}(\{x_1, x_2\})$ is finite, for all $x_1, x_2 \in D^0$.

Proof. Assume to the contrary that there are elements $x_1, x_2 \in D^0$ so that $\mathcal{U}(\{x_1, x_2\})$ is infinite. Then D^0 is infinite as well. Since the equality between compact elements is decidable with respect to δ, we can thus construct a total indexing δ' of D^0 which is equivalent to δ and one-to-one. Without restriction we therefore assume that δ is one-to-one.

By Lemma 5.2 there are functions $h, k \in R^{(1)}$, respectively, which enumerate the sets

$$\{\, i \mid \delta_i \in \mathcal{U}(\{x_1, x_2\}) \,\} \qquad \text{and} \qquad \{\, i \mid \nu_i \in \mathcal{U}(\{(x_1 \searrow x_1), (x_2 \searrow x_2)\}) \,\}.$$

Here, h can be taken as one-to-one, because $\mathcal{U}(\{x_1, x_2\})$ is infinite. Since the application map $\mathrm{eval}_{D,D}$ is a morphism in our category $\mathbf{K}$, we obtain that the set

$$E = \{\, \langle a, i, j \rangle \mid \delta_{h(j)} \in \mathcal{U}(\{x_1, x_2\}) \wedge \delta_{h(j)} \sqsubseteq \mathrm{eval}_{D,D}(\nu_{k(a)}, \delta_{h(i)}) \,\}$$

is r.e. Because of the completeness of $\mathcal{U}(\{x_1, x_2\})$ and as the function h is one-to-one, we obtain with Lemma 5.9 that for any $a, i \in \omega$ there is exactly one $j \in \omega$ such that $\langle a, i, j \rangle \in E$. For $a, i \in \omega$ define $g(a, i)$ to be this uniquely determined index j. It follows that $g \in R^{(2)}$. Let $f \in R^{(1)}$ with $\varphi_{f(a)}(i) = g(a, i)$. Then $\varphi_{f(a)} \in R^{(1)}$, for any $a \in \omega$. Now, note that any function $t \in R^{(1)}$ determines a map $\hat{F}$ in $\mathcal{U}(\{(x_1 \searrow x_1), (x_2 \searrow x_2)\})$ such that for any $a \in \omega$ with $\hat{F} = \nu_{k(a)}$ we have that $\varphi_{f(a)} = t$. Set $F(\delta_{h(i)}) = \delta_{h(t(i))}$. Then F is well defined, as δ is one-to-one, and maps $\mathcal{U}(\{x_1, x_2\})$ into itself. Define $\hat{F}$ as in Lemma 5.10. This shows that $\lambda a.\varphi_{f(a)}$ is an effective enumeration of $R^{(1)}$. Such enumerations do not exist.

In order to see this, let $s(n) = g(n, n) + 1$. Then $s \in R^{(1)}$. Hence, there is an index m with $\varphi_{f(m)} = s$. It follows that

$$g(m, m) + 1 = s(m) = \varphi_{f(m)}(m) = g(m, m),$$

which is impossible, since g is a total function. □

Up to now we have proved completeness and finiteness of $\mathcal{U}(X)$ only for sets X of cardinality two. The following lemma of Plotkin says that it was sufficient to do so. Define to this end that a subset S of a partial order has *property M* if $\mathcal{U}(S)$ is finite and complete for S.

Lemma 5.12. If every pair of elements of a partial order has property M then each of its finite subsets has property M.

A proof of this lemma can be found in (Smyth 1983).

Proposition 5.13. Any finite set X of compact elements of D has property *M*.

Because D has a completeness test, we obtain as in Lemma 4.2(3) that $\mathcal{U}$ is effective.

5.4. *Iterated minimal upper bounds*

It remains to show that also $\mathcal{U}^*(X)$ is finite. In the proofs given in Jung (Jung 1989) and Smyth (Smyth 1983), respectively, either the Heine-Borel Theorem or König's Lemma is used. Both results are not valid in recursive mathematics (Beeson 1985). Here, another approach is presented.

For $z \in D^0$ define $G_z : D \to D$ by

$$G_z(x) = \begin{cases} x & \text{if } x \sqsubseteq z, \\ z & \text{otherwise.} \end{cases}$$

Then G_z is computable.

Lemma 5.14. For $z, z' \in D^0$, $G_z \sqsubseteq G_{z'}$ if and only if $z \sqsubseteq z'$ and for all $x \in D$

$$x \sqsubseteq z' \Rightarrow z \sqsubseteq x \vee x \sqsubseteq z.$$

Now, for any $n \in \omega$, assume that $\mathcal{U}^{n+1}(X)$ and $\mathcal{U}^n(X)$ do not coincide. By Proposition 5.13, $\mathcal{U}^{n+1}(X)$ is finite. Let z_n be maximal in $\mathcal{U}^{n+1}(X) \setminus \mathcal{U}^n(X)$ and set $G_n = G_{z_n}$. Then $z_m \neq z_n$, for all $m, n \in \omega$ with $m \neq n$.

Lemma 5.15. For all $m, n \in \omega$ with $m \neq n$, G_m and G_n are incomparable with respect to the domain order.

Proof. Without restriction let $m < n$. Assume first that $G_n \sqsubset G_m$. Then $z_n \sqsubset z_m$ and for all $x \in D$ with $x \sqsubseteq z_m$, either $x \sqsubseteq z_n$ or $z_n \sqsubset x$. Because of the maximality of z_n we obtain from the last property that $z \sqsubseteq z_n$, for all $z \in \mathcal{U}^{n+1}(X) \setminus \mathcal{U}^n(X)$ with $z \sqsubseteq z_m$. Now, let $z \in \mathcal{U}^n(X)$ with $z \sqsubseteq z_m$. Since $z_n \in \mathcal{U}^{n+1}(X) \setminus \mathcal{U}^n(X)$, there is some $y \in \mathcal{U}^n(X) \setminus \mathcal{U}^{n-1}(X)$ with $y \sqsubseteq z_n$. Then, both $y, z \sqsubseteq z_m$. By completeness there is hence some $\bar{z} \in \mathcal{U}(\{y, z\})$ with $\bar{z} \sqsubseteq z_m$. It follows that $\bar{z} \in \mathcal{U}^{n+1}(X) \setminus \mathcal{U}^n(X)$ and therefore, by what we have just seen, that $z \sqsubseteq \bar{z} \sqsubseteq z_n$. Since $z_m \in \mathcal{U}^{m+1}(X)$ and $\mathcal{U}^{m+1}(X) \subseteq \mathcal{U}^n(X)$, we obtain that $z_m \sqsubseteq z_n$. Thus $z_m = z_n$, in contradiction to the choice of the z_a.

Assume next that $G_m \sqsubset G_n$. Then $z_m \sqsubset z_n$ and for all $x \in D$ with $x \sqsubseteq z_n$, either $x \sqsubseteq z_m$ or $z_m \sqsubset x$. As above it follows for all $z \in \mathcal{U}^{m+1}(X)$ with $z \sqsubseteq z_n$ that $z \sqsubseteq z_m$. Now, let $m + 1 \leq i \leq n$ and assume that $z \sqsubseteq z_m$, for all $z \in \mathcal{U}^i(X)$ with $z \sqsubseteq z_n$. Moreover, let $Z \subseteq \mathcal{U}^i(X)$ and $\bar{z} \in \mathcal{U}(Z)$ with $\bar{z} \sqsubseteq z_n$. Then either $\bar{z} \sqsubseteq z_m$ or $z_m \sqsubset \bar{z}$. Since $z_m \in \mathrm{UB}(Z)$, by our hypothesis, the last case contradicts the minimality of $\bar{z}$. Therefore $\bar{z} \sqsubseteq z_m$. This shows that for all $z \in \mathcal{U}^{i+1}(X)$ with $z \sqsubseteq z_n$ also $z \sqsubseteq z_m$. By induction we thus obtain for all $z \in \mathcal{U}^{n+1}(X)$ with $z \sqsubseteq z_n$ that $z \sqsubseteq z_m$. But as we have already seen, $z_m \sqsubset z_n$. $\square$

Set $X_n = \{ x \in X \mid x \sqsubseteq z_n \}$, for $n \in \omega$. Then $G_n \in \mathrm{UB}(\{ (u \searrow u) \mid u \in X_n \})$. Since

$[D \rightarrow_c D]$ is also an object of our category $\mathbf{K}$, it follows from Proposition 5.13 that there is an $F_n \in \mathcal{U}(\{ (u \searrow u) \mid u \in X_n \})$ below G_n.

Lemma 5.16. Let $m, n \in \omega$ so that $m \neq n$, but $X_m = X_n$. Then there is no $F \in \mathcal{U}(\{ (u \searrow u) \mid u \in X_m \})$ with $F \sqsubseteq G_m, G_n$.

Proof. Assume there is some $F \in \mathcal{U}(\{ (u \searrow u) \mid u \in X_m \})$ with $F \sqsubseteq G_m, G_n$. Then we can show, by induction on all levels $i \leq m+1$, that $F(x) = x$, for all $x \in \mathcal{U}^{m+1}(X_m)$ with $x \sqsubseteq z_m$. At level 0 this is trivial. For the induction step, suppose that $F(x) = x$, for all $x \in \mathcal{U}^i(X_m)$ with $x \sqsubseteq z_m$. If $z \in \mathcal{U}^{i+1}(X_m) \setminus \mathcal{U}^i(X_m)$ with $x \sqsubseteq z_m$, then $z \in \mathcal{U}(Y)$, for some $Y \subseteq \mathcal{U}^i(X_m)$. By induction hypothesis and monotonicity of F, $y = F(y) \sqsubseteq F(z)$ for each $y \in Y$, so that $F(z) \in \mathrm{UB}(Y)$. But $F(z) \sqsubseteq z$ since $F \sqsubseteq G_m$, so by minimality of z, $F(z) = z$.

Since $z_m \in \mathcal{U}^{m+1}(X_m)$ we have that $z_m = F(z_m) \sqsubseteq G_n(z_m) \sqsubseteq z_n$. In the same way it follows that also $z_n \sqsubseteq z_m$. Hence $z_m = z_n$, which is impossible by the choice of the z_a. □

As a consequence of this lemma we obtain that there exist two different maps $F_m, F_n \in \mathcal{U}(\{ (u \searrow u) \mid u \in X_m \})$ with $F_m \sqsubseteq G_m$ and $F_n \sqsubseteq G_n$.

Proposition 5.17. For any finite set X of compact elements of D, $\mathcal{U}^*(X)$ is finite.

Proof. Assume to the contrary that there is some finite subset X of D^0 so that $\mathcal{U}^*(X)$ is infinite. Then $\mathcal{U}^n(X)$ is a proper subset of $\mathcal{U}^{n+1}(X)$, for all $n \in \omega$. Thus, a family $(z_n)_{n\in\omega}$ of compact elements can be chosen as described above. Since X is finite, $(X_n)_{n\in\omega}$ is an enumeration of finitely many objects. It follows that there is a subfamily $(z_{m_\nu})_{\nu\in\omega}$ and a subset M of X such that $X_{m_\nu} = M$, for all $\nu \in \omega$. As we have just seen, this implies that $\mathcal{U}(\{ (u \searrow u) \mid u \in M \})$ is infinite, which is impossible by Proposition 5.13. □

5.5. *The result*

From what we have shown in the preceding sections it follows that if $\mathbf{K}$ is a weakly indexed constructively Cartesian closed effectively full subcategory of **CDOMCI**, then any of its objects must be a constructive domain that is derived from an effectively strongly algebraic domain. This gives us our constructive analogue of Smyth's second result.

Theorem 5.18. CSFP is the largest constructively Cartesian closed weakly indexed effectively full subcategory of **CDOMCI**.

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References

Abramsky, S. and Jung, A. (1994) Domain theory. In: Abramsky, S., Gabbay, D. M. and Maibaum, T. S. E. (editors), *Handbook of Logic in Computer Science* **3**, 1–168. Clarendon Press.

Amadio, R. (1991) Bifinite domains: Stable case. In: Pitt, D. H., Curien, P.-L., Abramsky, S., Pitts, A. M., Poigné, A. and Rydehard, D. E. (editors), *Category Theory and Computer Science, Lecture Notes in Computer Science* **530**, 16–33. Springer-Verlag.

Amadio, R. M. and Curien, P.-L. (1998) *Domains and Lambda-Calculi*. Cambridge University Press.

Beeson, M. J. (1985) *Foundations of Constructive Mathematics*. Springer-Verlag.

Gunter, C. A. (1992) *Semantics of Programming Languages*. MIT Press.

Gunter, C. A. and Scott, D. S. (1990) Semantic Domains. In: van Leeuwen, J. (editor), *Handbook of Theoretical Computer Science* **B**, 633–674. Elsevier.

Jung, A. (1988) New results on hierarchies of domains. In: Main, M., Melton, A., Mislove, M. and Schmidt, D. (editors), *Mathematical Foundations of Programming Language Semantics, Lecture Notes in Computer Science* **298**, 303–310. Springer-Verlag.

Jung, A. (1989) *Cartesian Closed Categories of Domains*. Centrum voor Wiskunde en Informatica, Amsterdam.

Jung, A. (1990) The classification of continuous domains. In: *Proc. Logic in Computer Science*, 35–40. IEEE Computer Society Press.

Kanda, A. (1979a) Fully effective solution of recursive domain equations. In: Bečvář, J. (editor), *Mathematical Foundations of Computer Science, Lecture Notes in Computer Science* **74**, 326–336. Springer-Verlag.

Kanda, A. (1979b) Effective solutions of recursive domain equations. Ph. D. thesis, University of Warwick.

Plotkin, G. D. (1976) A powerdomain construction. *SIAM J. on Computing* **5**, 452–488.

Smyth, M. B. (1980) Computability in categories. In: de Bakker, J. and van Leeuwen, J. (editors), *Automata, Languages and Programming, Proc. Seventh Colloq., Lecture Notes in Computer Science* **85**, 609–620. Springer-Verlag.

Smyth, M. B. (1983) The largest cartesian closed category of domains. *Theoretical Compututer Science* **27**, 109–119.

Smyth, M. B. and Plotkin, G. D. (1982) The category-theoretic solution of recursive domain equations. *SIAM J. on Computing* **11**, 761–783.

Stoltenberg-Hansen, V., Lindström, I. and Griffor, E. R. (1994) *Mathematical Theory of Domains*, Cambridge University Press.

Vickers, S. (2001) Strongly algebraic = SFP (topically). *Mathematical Structures in Computer Science* **11**, 717–742.

Zhang, G.-Q. (1996) The largest cartesian closed category of stable domains. *Theoretical Computer Science* **166**, 203–219.

Zhang, G.-Q., Jiang, Y. and Chen, Y. (2002) Stable bifinite domains: the finite antichain condition (manuscript). Available at `http://newton.eecs.cwru.edu/~gqz/publications.html`.