Z. U. RAHMAN
Graduate Assistant.

W. H. BUSSELL
Professor.

Department of Mechanical Engineering, Auburn University, Auburn, Ala.

An Iterative Method for Analyzing Oscillating Cam Follower Motion

This paper describes a method of analyzing the kinematic characteristics of cams, with profiles represented by an eight-term Fourier series and with oscillating-type followers. Two kinds of cam follower systems, viz., roller follower and flat-face follower systems, are used. In general the method applies to any cam whose profile can be expressed in terms of its radius being a function of the rotation angle [$r = f(\theta)$].

Introduction

The analysis of cam follower motion directly from the cam profile is usually done graphically, the numerical analysis being so complex that it has been rejected as generally useless except for a few special cases. However, difficulty in incorporating the effects of elasticity and inertia in follower response with the indirect methods provides some incentive for the development of a method for performing the analysis directly from cam profile specifications.

The usual way of doing this is by assuming the displacement-time diagram. This diagram is either constructed numerically or with the help of standard mathematical curves. The shape of the cam profile, velocity, and acceleration curves are then obtained from this diagram. The classical way of doing this utilizes graphical analysis, which is cumbersome as well as time consuming. A more direct method, preferably programmable on a digital computer, is therefore a worthwhile goal.

A number of authors have investigated the graphical analysis method. The more notable of these are Franklin de Ronde Furman [1],[1] J. Hirschhorn [2], and S. Molian [3]. In the field of analytical methods Harold A. Rothbart [4], S. Lindroth [5], and R. R. Griffin [6] have done significant work.

The method of analysis presented here consists of representing the rise and fall curves of a dwell-rise-dwell cam by $r = f(\theta)$, where r is the magnitude of the radius vector, measured from the cam center to the profile, and θ is the cam rotation angle. For the purpose of this presentation, $f(\theta)$ has been taken to be an eight-term Fourier series in θ. The constants for the two curves are obtained by applying the boundary conditions and solving the resulting equations simultaneously. Once the profile has been defined, the cam is rotated at small intervals, the point of cam follower contact determined iteratively, and the displacement, velocity, and acceleration computed. Two types of oscillating followers have been used, i.e., roller follower and the flat-face follower. The knife-edge case, though included in the original work [7], has not been dealt with here. Two reasons justify for this. First, the knife-edge has more of an academic value than its usefulness in a practical application. Secondly, the roller follower case can be reduced to the knife-edge case by reducing the roller radius to zero. A computer program, written separately for each case, was interfaced with a digital plotter routine. The final output consists of graphs of displacement, velocity, and acceleration against cam rotation angle.

Development of Cam Profile

Consider the cam radius vector at the beginning of the lower dwell period (Fig. 1). Let this be the rotating reference line to measure cam rotation angles from the fixed reference line OX. Let r be the magnitude of the radius vector at any cam angle θ.

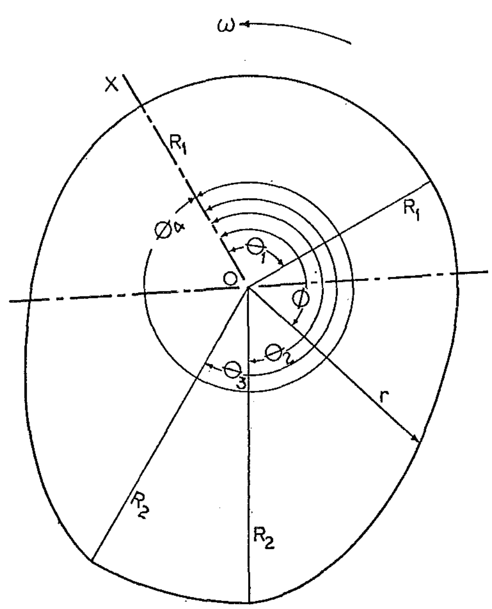

Fig. 1 Cam profile showing radii at various cam angles

[1] Numbers in brackets designate References at end of paper.

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As θ varies from 0 to θ_4 the radius vector describes the following four portions of the cam profile:

$$0 \text{ to } \theta_1,\ r = R_1 \text{ (Lower Dwell)} \tag{1}$$

$$\begin{aligned}\theta_1 \text{ to } \theta_2,\ r = f(\theta) = {} & a_1 + a_2 \cos\theta + a_3 \cos 2\theta \\ & a_4 \cos 3\theta + a_5 \cos 4\theta + \\ & a_6 \sin\theta + a_7 \sin 2\theta + \\ & a_8 \sin 3\theta \qquad \text{(Rise)}\end{aligned} \tag{2}$$

$$\theta_2 \text{ to } \theta_3,\ r = R_2 \text{ (Upper Dwell)} \tag{3}$$

$$\begin{aligned}\theta_3 \text{ to } \theta_4,\ r = f(\theta) = {} & b_1 + b_2 \cos\theta + b_3 \cos 2\theta \\ & b_4 \cos 3\theta + b_5 \cos 4\theta + \\ & b_6 \sin\theta + b_7 \sin 2\theta + \\ & b_8 \sin 3\theta \qquad \text{(Fall)}\end{aligned} \tag{4}$$

To find the values of constants for the rise curve, the following boundary conditions are applied to equation (2):

At $\theta = \theta_1$:

$$r = R_1$$

$$\frac{dr}{d\theta} = 0$$

$$\frac{d^2r}{d\theta^2} = 0$$

$$\frac{d^3r}{d\theta^3} = 0$$

At $\theta = \theta_2$:

$$r = R_2$$

$$\frac{dr}{d\theta} = 0$$

$$\frac{d^2r}{d\theta^2} = 0$$

$$\frac{d^3r}{d\theta^3} = 0$$

The resulting eight equations are solved simultaneously by the matrix inversion method to give the values of constants a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8. The values of constants for the fall curve are obtained in the same manner. Once these constants are known, equations (1), (2), (3), and (4) fully describe the cam profile.

Having defined the cam profile, it is now intended to find the equations of motion for the roller and the flat-face type of oscillating followers. It is assumed that the cam rotates with a constant angular velocity, i.e., $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$.

Roller Follower. Consider the cam and the roller follower system at the beginning of the lower dwell period. Let O_cX and O_cX' be the fixed and the rotating axes, respectively, coincident at this starting position. Let the cam rotate through an angle θ (Fig. 2). The roller center O_f swings in an arc centered on 0. The point of contact no longer lies on the fixed axis O_cX. It moves in such a way that the contact angle α leads the cam rotation angle θ in the earlier part of the rise period and lags behind in the latter part. This lead or lag is dependent on the radius of the roller follower. In fact, if the roller has zero radius there will be no lead and α will always lag behind θ. The difference is maximum during the upper dwell period, and is referred to as the forward shift of upper dwell period.

Shift $= \phi_1 - \phi_2$

where

$$\phi_1 = \cos^{-1}\left[\frac{E^2 + (R_1 + R_f)^2 - D^2}{2E(R_1 + R_f)}\right] \tag{7}$$

$$\phi_2 = \cos^{-1}\left[\frac{E^2 + (R_2 + R_f)^2 - D^2}{2E(R_2 + R_f)}\right] \tag{8}$$

The angles of rotation at which the upper dwell period begins and ends must, therefore, be corrected for this shift. The corrected upper dwell angles will thus be:

$$\theta_2 + \text{Shift}$$

and

$$\theta_3 + \text{Shift}.$$

The values of shift for intermediate positions of the follower on rise and fall curves must be determined to define the point of contact. This is accomplished by computing α with the help of an iterative process. Consider T to be the point of contact for a moment. Construct a roller follower at T. Draw a tangent AB to the cam profile at T. Angle θ is successively decreased or increased by very small amounts, depending on whether α lags behind or leads θ, and the value of distance x is computed. This value is compared with the known follower arm length D. The iterative process ceases as the value of x equals D, of course within a specified tolerance.

$$x = (R_f^2 + \overline{OT}^2 - 2R_f\overline{OT}\cos\lambda)^{1/2}$$

where

Nomenclature

- a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8 = constants in Fourier series for period of rise
- b_1, b_2, b_3, b_4; b_5, b_6, b_7, b_8 = constants in Fourier series for period of fall
- D = length of follower arm, in.
- E = distance between cam center and follower arm pivot, in.
- r = magnitude of radius vector to cam profile at cam rotation angle θ
- r_α = magnitude of radius vector to point of cam follower contact
- r', r'', r''' = first, second, and third derivatives, respectively, of r with respect to θ
- r'_α, r''_α, r'''_α = first, second, and third derivatives, respectively, of r_α with respect to α
- $\dot{r}$, $\ddot{r}$ = first and second derivatives of r with respect to time. (In general, dot over letters implies derivatives with respect to time.)
- R_1 = radius of base circle
- R_2 = radius of nose circle
- R_f = radius of roller follower
- α = angle from rotating axis to the point of contact
- β = angle $0T0_c$ (roller follower case)
- γ = angle between the tangent at T and radius vector
- δ = angle $L0_fP$ (roller follower case)
- θ = cam rotation angle
- λ = angle $0T0_f$ (roller follower case)
- ψ = angle between follower arm and center to center line
- $\dot{\psi}$ = angular velocity of the follower arm
- $\ddot{\psi}$ = angular acceleration of the follower arm
- ω = input angular velocity of cam

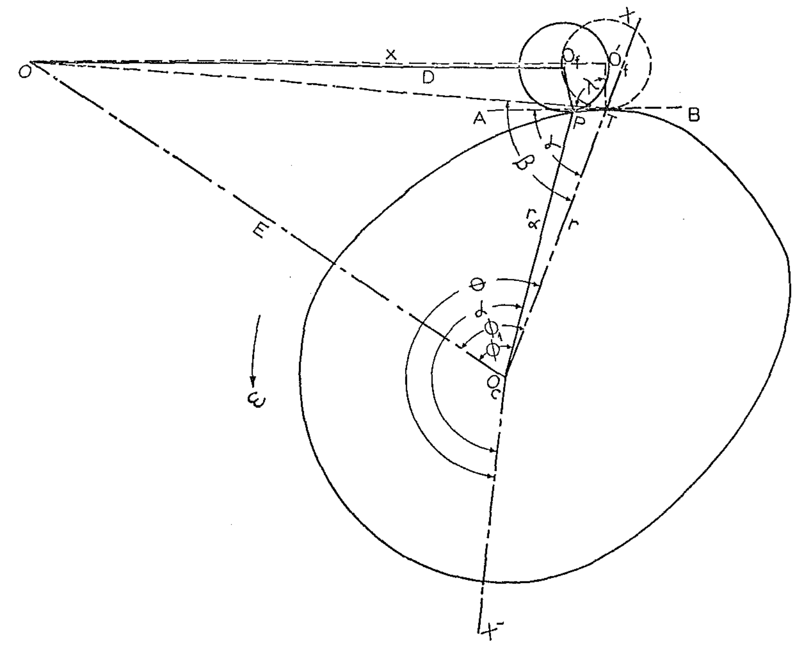

Fig. 2 Cam and roller follower system

$$\overline{OT} = (r^2 + E^2 - 2rE \cos \phi_1)^{1/2}$$

$$\lambda = \left(\frac{\pi}{2} + \gamma\right) - \beta$$

$$\beta = \cos^{-1}\left[\frac{r^2 + \overline{OT}^2 - E^2}{2r\overline{OT}}\right]$$

From differential calculus,

$$\gamma = \tan^{-1}\left[\frac{r}{\frac{dr}{d\theta}}\right] = \tan^{-1}\left[\frac{r}{\acute{r}}\right]$$

The cam profile is expressed by

$$r = f(\theta)$$

and

$$\acute{r} = \acute{f}(\theta).$$

It may be kept in mind that the values of r and $\acute{r}$ change with each successive iteration until at the end θ becomes α, r becomes $r_\alpha = f(\alpha)$, and $\acute{r}$ becomes $\acute{r}_\alpha = \acute{f}(\alpha)$. Once α is computed, angular displacement, velocity, and acceleration can be calculated. (See Appendix.)

Analysis of Follower Motion

Flat-Face Follower. The flat-face follower system differs from the roller follower system in that the length of its follower arm is not fixed. The point of contact is the tangent point between the cam curve and the follower arm. In this case the contact angle α leads the cam rotation angle θ in the earlier part of the rise period and lags behind in the latter part. The reverse is true for the fall period. The shift is not necessarily maximum during the upper dwell period. However, the dwell period does shift forward in this case also. It is now a function of cam radii R_1 and R_2 and distance E only.

$$\text{Shift} = \phi_1 - \phi_2$$

where

$$\phi_1 = \cos^{-1}\left[\frac{R_1}{E}\right] \tag{9}$$

$$\phi_2 = \cos^{-1}\left[\frac{R_2}{E}\right] \tag{10}$$

The contact angle is again determined by an iterative process. Let P be the contact point (Fig. 3). Point T is on the profile at

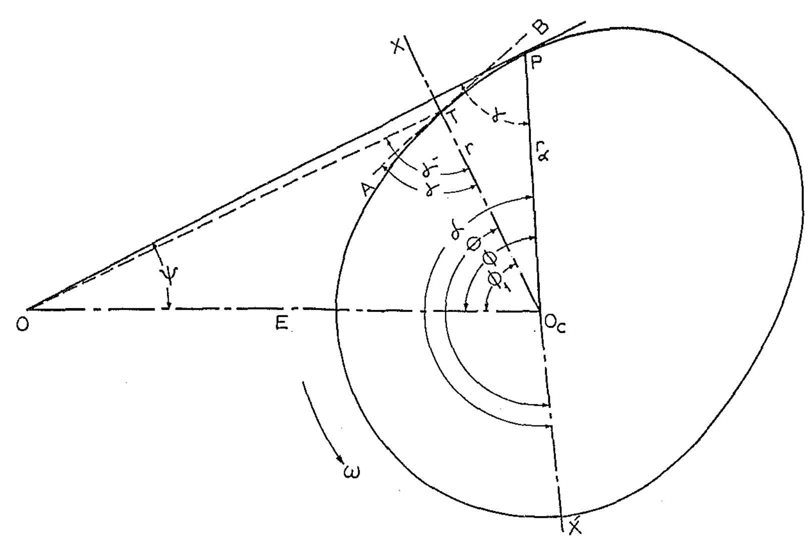

Fig. 3 Cam and flat-face follower system

θ angle of rotation, and' AB is a tangent to profile curve at T. Angle $\gamma(O_cTA)$ and $\gamma(O_cTO)$ have different values at this point. However, as θ approaches α, their difference tends to diminish. Thus θ is increased or decreased, depending on whether γ is less than or greater than $\acute{\gamma}$,until they have the same value, again within a specified tolerance. The two angles are expressed as

$$\gamma = \tan^{-1}\left[\frac{r}{\acute{r}}\right]$$

and

$$\acute{\gamma} = \cos^{-1}\left[\frac{r^2 + \overline{OT}^2 - E^2}{2r\overline{OT}}\right]$$

where

$$\overline{OT} = (E^2 + r^2 - 2Er\cos\phi_1)^{1/2}$$

and ϕ_1 is defined by equation (9).

Once α is computed, r_α is given by $r_\alpha = f(\alpha)$ where $f(\alpha)$ is the eight-term Fourier series in α. Equations of motion can now be written. These are given in the Appendix.

Results and Conclusion

Equations of angular displacement, velocity, and acceleration were programmed in the computer with parameters $R_1 = 2''$, $R_2 = 2.75''$, $D = 7''$, $E = 7''$, $\phi_1 = 60$ deg, $\theta_2 = 180$ deg, $\theta_3 = 210$ deg, $\theta_4 = 360$ deg, and cam angular velocity of 10 radians per second. This program was interfaced with a digital plotter which gave graphs of each drawn against the cam rotation angle (Figs. 4 and 5). Values of displacement, velocity, and acceleration are different in each case of follower system. However, the results obtained do not greatly differ from those obtained by Stoddart [8] and Duddley [9] for their 3-4-5 polynomial or cycloidal cams, respectively.

The values of contact angles during the rise and the fall period were approximated within 0.00005 radian, thus assuring a very high degree of accuracy for all practical purposes. Any number of cams with different parameters may be analyzed by writing only one program. A digital computer is a "must" to use this method. The calculations involved would be impossible to handle manually. However, if such a facility is readily accessible, the method would certainly prove superior to conventional graphical methods in areas of accuracy, economy, and convenience.

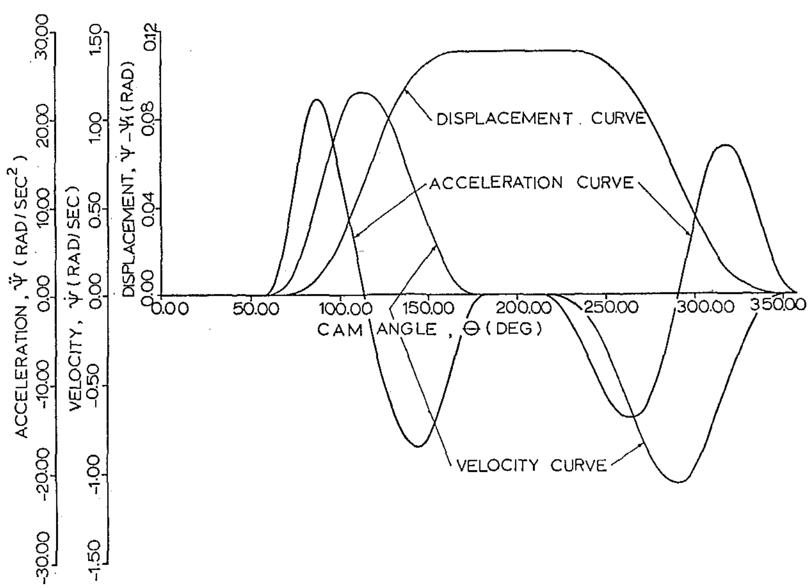

Fig. 4 Follower motion curves—roller follower

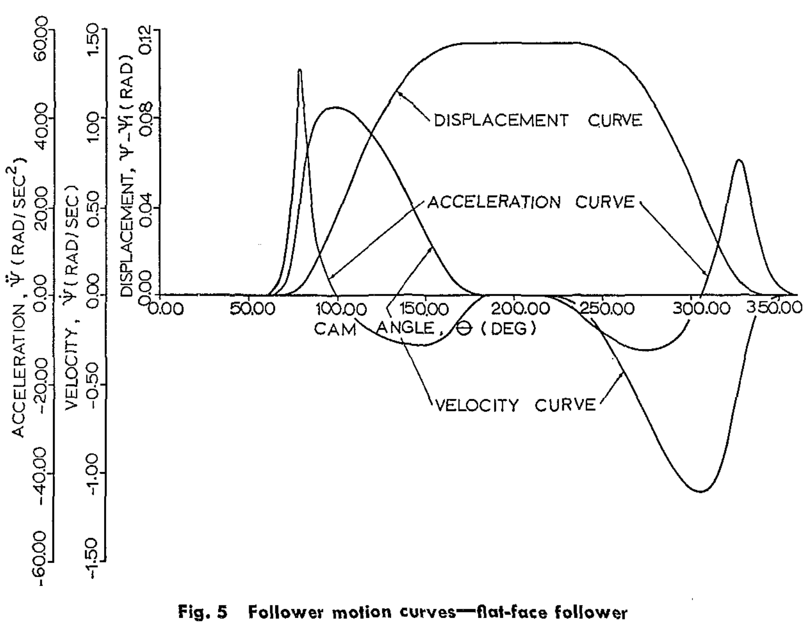

Fig. 5 Follower motion curves—flat-face follower

The method may be readily adapted to the design of cams for which the profile can be expressed as $r = f(\theta)$. Since the shift in upper dwell period is a function of initially known parameters, cams for actual dwell periods may be designed by allowing for this shift at the very start. To manufacture a cam, data may be obtained directly by computing the values of r at sufficiently small intervals of rotation angle. When transferred to a punched tape milling machine, these data can produce a profile of the desired accuracy.

The author recommends that cams, whose profiles can be expressed *in terms of radius, a function of cam rotation angle, be* analyzed for functions other than Fourier series. This may open an entirely new avenue to cam design. In conclusion, the method presented here is sufficiently general to apply to most cam design problems and has definite advantages over the conventional graphical methods.

APPENDIX

Development of Equations of Motion for Roller Follower Case and Flat-Face Follower Case

Roller Follower Case. Consider Fig. 6.

$$R_f \sin \delta + r_\alpha \cos \phi = E - D \cos \psi \tag{11}$$

$$R_f \cos \delta + r_\alpha \sin \phi = D \sin \psi \tag{12}$$

$$\delta + \phi = \pi - \gamma \tag{13}$$

Squaring and adding equations (11) and (12), substituting for $(\delta + \phi)$, and solving for ψ yields

$$\psi = \cos^{-1}\left[\frac{E^2 + D^2 - R_f^2 - r_\alpha^2 - 2R_f r_\alpha \sin \gamma}{2ED}\right] \tag{14}$$

Thus displacement $= \psi - \psi_1$, where

$$\psi_1 = \cos^{-1}\left[\frac{E^2 + D^2 - (R_1 + R_f)^2}{2ED}\right]$$

Differentiating this expression with respect to time once yields angular velocity, and differentiating again gives angular acceleration.

$$\dot\psi = \frac{r_\alpha \dot r_\alpha + R_f(\dot r_\alpha \sin \gamma + r_\alpha \cos \gamma\dot\gamma)}{ED \sin \psi} \tag{15}$$

$$\ddot\psi = \frac{\sin \psi(I + R_f(J + K + L)) - M}{ED \sin^2 \psi} \tag{16}$$

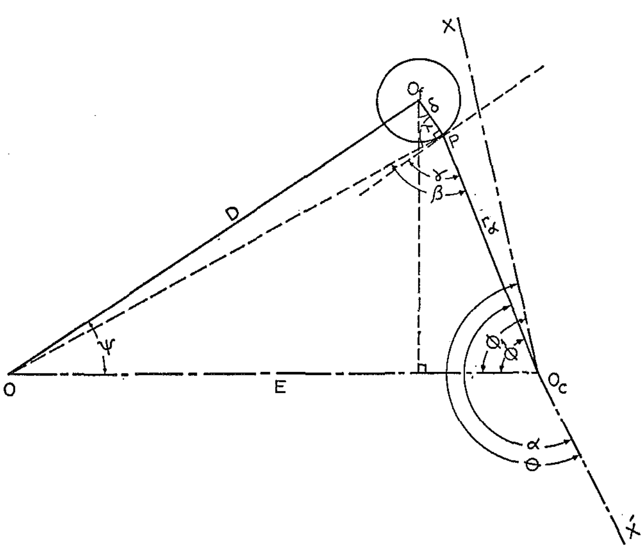

Fig. 6 Roller follower case showing angles β, γ, δ, λ at the contact point

where

$$I = r_\alpha \ddot r_\alpha + \dot r_\alpha^2$$

$$J = \dot r_\alpha \cos \gamma\dot\gamma + \ddot r_\alpha \sin \gamma$$

$$K = r_\alpha(\cos \gamma\ddot\gamma - \sin \gamma\dot\gamma^2)$$

$$L = \dot r_\alpha \cos \gamma\dot\gamma$$

and

$$M = \cos \psi\dot\psi[r_\alpha \dot r_\alpha + R_f(\dot r_\alpha \sin \gamma + r_\alpha \cos \gamma\dot\gamma)]$$

In equations (15) and (16) $\dot\gamma$, $\ddot\gamma$, $\dot r_\alpha$, and $\ddot r_\alpha$ were undefined. They are now defined as follows:

Angle γ at point P (Fig. 6) is given by

$$\gamma = \tan^{-1}\left[\frac{r_\alpha}{r'_\alpha}\right] \tag{17}$$

$$\dot\gamma = Q\dot\alpha \tag{18}$$

where

$$Q = \frac{(r'^2_\alpha - r_\alpha r''_\alpha) \cos^2 \gamma}{r'^2_\alpha} \tag{19}$$

and

$$\ddot\gamma = Q\ddot\alpha + \dot\alpha\dot Q \tag{20}$$

where

$$\dot Q = \dot\alpha \times \left[\frac{2(r_\alpha r'_{\alpha\alpha} r''^2 - r_\alpha r'''_\alpha - r'^3_\alpha r''_\alpha) \cos \gamma - Q(r'^2_\alpha - r_\alpha r''_\alpha) \sin 2\gamma}{r'^2_\alpha}\right] \tag{21}$$

where r'_α, r''_α, and r'''_α are first, second, and third derivatives, respectively, of r_α with respect to α.

$\dot\alpha$ and $\ddot\alpha$ are determined as follows:

$$D^2 = R_f^2 + \overline{OP}^2 - 2R_f\overline{OP} \cos \lambda \tag{22}$$

where

$$\overline{OP}^2 = E^2 + r_\alpha^2 - 2Er_\alpha \cos \Theta$$

$\Theta = \phi_1 - (\theta - \alpha)$ where ϕ_1 has been defined in equation (7).

$$\lambda = \frac{\pi}{2} + \gamma - \beta$$

and

$$\beta = \cos^{-1}\left[\frac{r_\alpha^2 + \overline{OP}^2 - E^2}{2r_\alpha\overline{OP}}\right]$$

Differentiating equation (22) with respect to time results in a linear equation in $\dot\alpha$ which yields

$$\dot\alpha = \frac{\overline{OP}\, B_r - R_f(B_f \cos \lambda - K_r\, \overline{OP} \sin \lambda)}{\overline{OP}\, F_r - R_f[F_r \cos \lambda - \overline{OP} \sin \lambda(Q - N_r)]} \tag{23}$$

where

$$B_r = \frac{Er_\alpha\dot\theta \sin \Theta}{\overline{OP}}$$

$$F_r = \frac{r_\alpha r'_\alpha - E(r'_\alpha \cos \Theta - r_\alpha \sin \Theta)}{\overline{OP}}$$

$$K_r = \frac{E^2\dot\theta(E - r_\alpha \cos \Theta) \sin \Theta}{\overline{OP}^3 \sin \beta}$$

and

$$N_r = \frac{\overline{OP^2}\, r'_\alpha(1 - 4r_\alpha^2) + Er_\alpha(r'_\alpha \cos \Theta - r_\alpha \sin \Theta)(2\overline{OP^2} - 1)}{2r_\alpha^2 \overline{OP^3} \sin \beta}$$

Differentiating equation (23) with respect to time yields $\ddot{\alpha}$ directly.

Having computed $\dot{\alpha}$ and $\ddot{\alpha}$, $\dot{r}_\alpha$ and $\ddot{r}_\alpha$ can now be obtained from

$$r'_\alpha = r'_\alpha \dot{\alpha} \tag{24}$$

$$r_\alpha = r_\alpha \ddot{\alpha} + \ddot{r}_\alpha \dot{\alpha}^2 \tag{25}$$

Thus in equations (15) and (16) every variable has been defined. Therefore angular velocity and acceleration can be computed.

Flat-Face Follower Case. Consider Fig. 3 again, with angle γ now shifted to point P.

$$\overline{OP} \sin \psi = r_\alpha \sin \phi \tag{26}$$

$$\overline{OP} \cos \psi = E - r_\alpha \cos \phi \tag{27}$$

and

$$\phi = \pi - (\gamma + \psi)$$

Dividing equation (26) by (27) and substituting the value of ϕ and simplifying, gives

$$\psi = \sin^{-1}\left[\frac{r_\alpha \sin \gamma}{E}\right] \tag{28}$$

Thus displacement $= \psi - \psi_1$ where $\psi_1 = \sin^{-1}\left(\frac{R_1}{E}\right)$ and γ is as defined in equation (17).

Differentiating equation (28) with respect to time gives

$$\dot{\psi} = \frac{r_\alpha \dot{\gamma} \cos \gamma + \dot{r}_\alpha \sin \gamma}{E \cos \psi} \tag{29}$$

and

$$\ddot{\psi} = \frac{\cos \psi\{(r_\alpha \ddot{\alpha} + 2\dot{r}_\alpha \dot{\alpha}) \cos \gamma - (r_\alpha \dot{\alpha}^2 - \ddot{r}_\alpha) \sin \gamma\} + \dot{\psi} \sin \psi(r_\alpha \dot{\alpha} \cos \gamma + \dot{r}_\alpha \sin \gamma)}{E \cos^2 \psi} \tag{30}$$

In equations (29) and (30), $\dot{\gamma}$ and $\ddot{\gamma}$ are the same as defined in equations (18) and (19). However, $\dot{\alpha}$ and $\ddot{\alpha}$ acquire different values in this case. These are obtained as follows:

In triangle OO_cP (Fig. 3),

$$E^2 = \overline{OP^2} + r_\alpha^2 - 2\overline{OP} r_\alpha \cos \gamma$$

This is a quadratic in $\overline{OP}$.

$$\overline{OP} = r_\alpha \cos \gamma + [r_\alpha^2 \cos^2 \gamma - (r_\alpha^2 - E^2)]^{1/2} \tag{31}$$

Only the positive sign is considered since negative $\overline{OP}$ does not signify anything.

Also

$$\overline{OP} = (r_\alpha^2 + E^2 - 2r_\alpha E \cos \Theta)^{1/2} \tag{32}$$

where $\Theta = \phi_1 - (\theta - \alpha)$ and ϕ_1 has been defined in equation (9).

From equations (31) and (32),

$$r_\alpha \cos \gamma + [r_\alpha^2 \cos^2 \gamma - (r_\alpha^2 - E^2)]^{1/2} - [r_\alpha^2 + E^2 - 2Er_\alpha \cos \Theta]^{1/2} = 0 \tag{33}$$

Differentiating equation (33) with respect to time and solving for $\dot{\alpha}$ gives

$$\dot{\alpha} = \frac{W_f}{X_f - Y_f - Z_f} \tag{34}$$

where

$$W_f = \frac{Er_\alpha \dot{\theta} \sin \Theta}{(r_\alpha^2 + E^2 - 2Er_\alpha \cos \Theta)^{1/2}}$$

$$X_f = \frac{E[r'_\alpha \cos \Theta - r_\alpha \sin \Theta] - r_\alpha r'_\alpha}{(r_\alpha^2 + E^2 - 2Er_\alpha \cos \Theta)^{1/2}}$$

$$Y_f = r'_\alpha \cos \gamma - r_\alpha Q \sin \gamma$$

where Q is as defined in equation (19),
and

$$Z_f = \frac{r_\alpha r'_\alpha \cos 2\gamma - r_\alpha^2 Q \sin 2\gamma - r_\alpha r'_\alpha}{2(r_\alpha^2 \cos^2 \gamma - r_\alpha^2 + E^2)^{1/2}}$$

$\ddot{\alpha}$ is directly obtained by taking the time derivative of equation (34). Using these values of $\dot{\alpha}$ and $\ddot{\alpha}$ in equations (24) and (25), $\dot{r}_\alpha$ and $\ddot{r}_\alpha$ are defined for this case. Thus all variables in equations (29) and (30) are defined. Hence values of angular velocity and acceleration can be computed.

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