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Comparison of Methods for Determining Screw Parameters of Finite Rigid Body Motion From Initial and Final Position Data

Five methods for determining screw parameters of finite rigid body motion, using position data of three noncollinear points, are compared on the basis of their efficiency, accuracy, and sensitivity to data error. It is found that the method based on Rodrigues' Formula (Bottema and Roth's method) is the most efficient. Angeles' method and Laub and Shiflett's method provide approximately the same level of accuracy, which is superior to that of the other methods. In terms of sensitivity, Bottema and Roth's method is preferable. On the basis of this study it is recommended that Bottema and Roth's method be used if uncertainty exists in the data, since it can provide a solution efficiently, accurately and it is the least sensitive to data error.

1 Introduction

Any general three-dimensional rigid body motion can be described as a screw motion, a combination of a rotation about a screw axis and a translation along the same axis. This concept is referred to in kinematics as Chasles' theorem [1][2]. The spatial displacement of a rigid body can thus be completely specified by a set of screw parameters: the screw axis, the magnitude of translation, and the magnitude of rotation. On the other hand, the coordinates of three noncollinear points fixed to a rigid body can uniquely determine the position and orientation of the rigid body and the motion of the body can be described by the initial and final position coordinates of three noncollinear points. Therefore, the screw parameters can be determined if the initial and final position coordinates of three noncollinear points are known. The problem of determining the screw parameters from initial and final position coordinates of three noncollinear points, is one form of the inverse problem in kinematics.

The screw representation of the spatial displacement of a rigid body has a number of applications in various fields. For instance, it may be used in robotics and automation to determine the position and orientation of the end-effector of a robot or the location of a component in an assembly line [3]; in computer graphics and computer aided design it may be used to specify a location or direction [4]; and in biomechanics it may be used to describe bone movements in cadaveric specimens or in living subjects [5], etc.

The problem of determining screw parameters from initial and final position data has resulted in a number of publications, and a number of algorithms for this purpose have been developed in recent years [5]–[10]. The most frequently used algorithms, using data of only three points, are as follows:

- *Angeles' method based on the invariant concept of a second-order tensor [7]*
- *Laub and Shiflett's method based on linear algebra and matrix perturbation theory [6]*
- *The methods based on the least square technique [5]*
- *The methods based on Rodrigues' formula [8], [9]*
- *Beggs' method based on solving a set of algebraic equations [10].*

These algorithms, although based on different concepts, yield equivalent results. For certain applications the computation of screw parameters must be repeatedly performed, for example, one such application is the end effector trajectory planning for continuous path motion in robotics. In this case, the computation is performed frequently and it is often done on-line. Therefore, it is essential that the algorithm be efficient for such applications.

Furthermore, due to the formulation and the accumulation of round-off errors, the algorithm might yield inexact results even though the initial data is precise. Hence, in the evaluation of different algorithms, accuracy should also be taken into consideration.

Finally, as it was discussed in [5] and [6], the computation of screw parameters is usually sensitive to inexact position data. In many of the applications, however, the position coordinates of the three noncollinear points, which are measured with respect to some reference frame, may incorporate a certain amount of error. Therefore, the sensitivity of the result of the computation to data uncertainty is also an important factor to consider.

In the present paper, the five algorithms listed above for screw parameter determination will be compared on the basis of their computational efficiency, computational accuracy, and sensitivity to data error. Methods for using the data of

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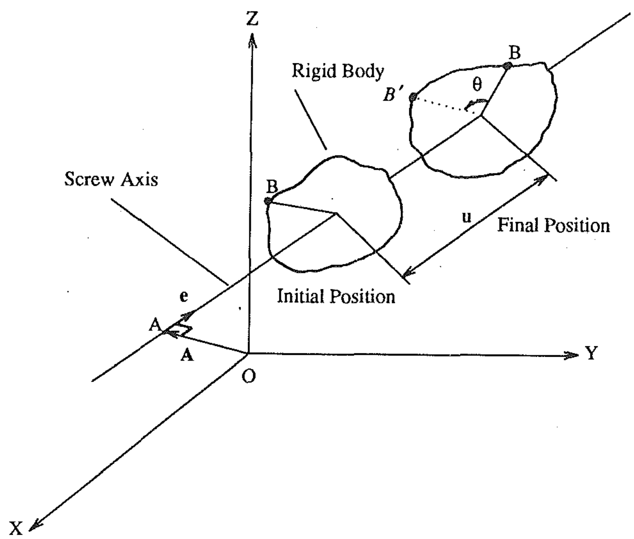

Fig. 1 Definition of screw parameters

more than three points will not be considered here. Methods in which only pure rotation is considered, such as Schut's Method [11], even though these methods can be modified to cover a more general case, will not be included in this paper. A parallel comparison of methods for determining screw parameters for infinitesimal displacement (velocity) is presented in a separate paper [12].

In the next section, the problem discussed in this paper is defined and in section 3 an outline of the five algorithms is presented. In section 4 the computational results as well as the comparison is provided and discussed. Finally, in section 5 the conclusions of our investigation is presented.

2 Problem Definition

The screw parameters can be divided into two groups. The first group of parameters defines the position and orientation of the screw axis; and the second group specifies the intensity of the screw motion: the magnitude of the rotation, θ, about the screw axis and the magnitude of the translation, u, along the screw axis.

Based on line geometry, the screw axis can be uniquely determined by its 6 *Plücker* coordinates, or alternatively, it can be defined by its 3 direction cosines expressed by a unit vector $\mathbf{e}$, together with a position vector $\mathbf{A}$ locating point A in the screw axis, see Fig. 1. Therefore, the screw motion can be defined by eight scalar parameters, the three components of $\mathbf{e}$, the three components of $\mathbf{A}$, and the magnitudes of θ and u. These eight scalar parameters can be combined into a column vector as $\mathbf{p}_a = [e_x\ e_y\ e_z\ A_x\ A_y\ A_z\ u\ \theta]^T$.

However, components of $\mathbf{e}$ must satisfy the constraint:

$$e_x^2 + e_y^2 + e_z^2 = 1 \tag{1}$$

Point A can be chosen such that vector $\mathbf{A}$ is perpendicular to the screw axis. In this case, a second constraint must also be applied:

$$\mathbf{e}^T \cdot \mathbf{A} = 0 \tag{2}$$

Therefore, of the eight scalar screw parameters, only 6 are independent, and the motion of the body can be defined by determining these six independent screw parameters from the initial and final position data of three noncollinear points of the body.

Spatial displacements can be conveniently described by a 4×4 transformation matrix $[T]$, so that

$$[P_f] = [T][P_i] \tag{3}$$

where, $[P_i]$ is a 4×3 matrix of the homogeneous coordinates of the initial position of the three points, $[P_f]$ is a 4×3 matrix of the homogeneous coordinates of the final position of the three points, and

$$[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} [R] & \mathbf{p} \\ 0 & 1 \end{bmatrix} \tag{4}$$

where $[R]$ is the 3×3 rotation matrix and $\mathbf{p}$ is the translation vector.

Matrix $[T]$ contains 12 scalar elements which are to be determined, however, the properties of the transformation matrix provide the following six constraints:

$$\mathbf{r}_j^T \mathbf{r}_j = 1 \qquad \text{for } j = 1,2,3 \tag{5a}$$

$$\mathbf{r}_i^T \mathbf{r}_j = 0 \qquad \text{for } i = 1,2,3\ j = 1,2,3 \text{ and } i \neq j \tag{5b}$$

where, $\mathbf{r}_i$, $\mathbf{r}_j$ are column vectors of $[R]$. Therefore, it is clear that the number of independent variables in $[T]$ is also six. Upon obtaining the transformation $[T]$, the required screw parameters can be readily calculated [5][13], therefore, the problem can be alternatively defined as determining the 6 independent elements in the transformation $[T]$ from the initial and final position data of the three noncollinear points.

3 Description of the Methods

Angeles' Method [7]

The method is based upon the invariant concepts of the second-moment tensor of three unit mass, noncollinear points of the rigid body about their centroid. This tensor is given by:

$$[\mathbf{I}] = \sum_{i=1}^{3} (\rho_i^2 [I] - \rho_i \rho_i^T) \tag{6a}$$

with

$$\rho_i = \mathbf{p}_i - \mathbf{c} \quad (i = 1,2,3)$$

$$\rho_i^2 = \rho_i^T \rho_i \quad (i = 1,2,3)$$

$$\mathbf{c} = \frac{1}{3} \sum_{i=1}^{3} \mathbf{p}_i \quad (i = 1,2,3) \tag{6b}$$

where, $[I]$ is a unit tensor (3 by 3 identity matrix), and $\mathbf{p}_i$ is the position vector of point $\mathbf{p}_i$.

From the invariants of tensor $[\mathbf{I}]$, three proper values, denoted by I_i $(i = 1,2,3)$ can be calculated and then the null space of tensor $[[\mathbf{I}] - I_i[I]]$ can be determined by applying Householder reflections H[14]

$$H[[\mathbf{I}] - I_i[I]] = [\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3] \tag{7}$$

with $\mathbf{a}_1^T = [a_{11}\ a_{12}\ a_{13}]$, $\mathbf{a}_2^T = [0\ a_{22}\ a_{23}]$, and $\mathbf{a}_3^T = [0\ 0\ 0]$.

There are two sets of such vectors corresponding to the initial and final positions, which can be used to constitute the columns of two matrices $[Q_1]$ and $[Q_2]$ [7], that in turn, constitute the rotation matrix $[R]$ by

$$[R] = [Q_2][Q_1]^T \tag{8}$$

Vector $\mathbf{e}$ and the rotation angle θ can be obtained from elements of [R] by applying the equivalent angle-axis concept in kinematics [15].

The algorithm for determining the location of the screw axis, point vector $\mathbf{A}$, uses Householder reflection to solve the redundant linear algebraic equations:

$$[D]\mathbf{A} = \mathbf{b} \tag{9}$$

where,

$$[D] = \begin{bmatrix} R \\ \mathbf{e}^T \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} [R]\mathbf{c} - \mathbf{c} + \mathbf{e}\mathbf{e}^T(\mathbf{c}' - \mathbf{c}) \\ 0 \end{bmatrix}$$

The magnitude of the translation, u, is obtained from:

$$u = \mathbf{e}^T(\mathbf{c}' - \mathbf{c}) \tag{10}$$

where, $\mathbf{c}$ and $\mathbf{c}'$ are the mean position vectors corresponding to the initial and final positions of the body as defined in equation (6b).

Laub and Shiflett's Method [6]

This algorithm is based on a linear algebraic approach, which offers the ability of handling inexact data through matrix perturbation theory. The method involves simple formulas and operations and provides a solution for the transformation $[T]$, which consists of a rotation matrix $[R]$ and a translation vector $\mathbf{p}$ as described in section 2.

Let $[P_i]$ and $[P_f]$ represent the initial and final position data matrix,

$$[P_i] = \begin{bmatrix} p_{i1x} & p_{i2x} & p_{i3x} \\ p_{i1y} & p_{i2y} & p_{i3y} \\ p_{i1z} & p_{i2z} & p_{i3z} \end{bmatrix} \tag{11a}$$

$$[P_f] = \begin{bmatrix} p_{f1x} & p_{f2x} & p_{f3x} \\ p_{f1y} & p_{f2y} & p_{f3y} \\ p_{f1z} & p_{f2z} & p_{f3z} \end{bmatrix} \tag{11b}$$

If $[P_i]$ is nonsingular, then the translating vector $\mathbf{p}$ and the rotation matrix [R] will be given by:

$$\mathbf{p} = \frac{1}{\mathbf{v}^T\mathbf{v}}\,[[Q] - (\det[Q])[Q]^{-T}]\mathbf{v} \tag{12}$$

and

$$[R] = [Q] - \mathbf{p}\mathbf{v}^T \tag{13}$$

with

$$[Q] = [P_f][P_i]^{-1},$$
$$\mathbf{v} = [P_i]^{-T}\mathbf{h},$$
$$\mathbf{h} = [1\ \ 1\ \ 1]^T,$$

and $[Q]^{-T}$ denotes the inverse tranpose of matrix $[Q]$.

From the resulting $\mathbf{p}$ and $[R]$, the required screw parameters, $\mathbf{p}_a$, can be calculated by using the algorithm described in the next method.

Spoor and Veldpaus' method [5]

This algorithm is based on the least square method which minimizes the function

$$f(\mathbf{p},[R]) = \frac{1}{3}\sum_{j=1}^{3}([R]\mathbf{p}_{ij} + \mathbf{p} - \mathbf{p}_{fj})^T([R]\mathbf{p}_{ij} + \mathbf{p} - \mathbf{p}_{fj}) \tag{14}$$

The application of the method involves a Lagrangian multiplier technique, which leads to the solution of an eigenvalue and eigenvector problem of matrix $[W]$, where,

$$[W] = [M]^T[M] \tag{15}$$

with

$$[M] = \frac{1}{3}\sum_{j=1}^{3}(\mathbf{p}_{ij}\mathbf{p}_{fj}^T) - \mathbf{c}\mathbf{c}'^T$$

$\mathbf{c}'$ and $\mathbf{c}$ are as defined before; $\mathbf{p}_{ij}$, and $\mathbf{p}_{fj}$ are the jth column vectors of $[P_i]$ and $[P_f]$, respectively.

$[W]$ is symmetric with eigenvalues $d_{11}^2 \ge d_{22}^2 \ge d_{33}^2 \ge 0$. The eigenvalues are the principal diagonal terms in a diagonal matrix $[D]$, while the eigenvectors are the corresponding columns of a 3 by 3 matrix $[V]$.

By denoting $[M][V] = [\mathbf{m}_1\ \mathbf{m}_2\ \mathbf{m}_3]$, the rotation matrix can be computed by:

$$[R] = \left[\frac{1}{d_{11}}\mathbf{m}_1\ \frac{1}{d_{22}}\mathbf{m}_2\ \frac{1}{d_{11}\cdot d_{22}}\mathbf{m}_1 \times \mathbf{m}_2\right][V]^T \tag{16}$$

The screw parameters are determined as follows:

$$\sin\theta = \frac{1}{2}\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2} \tag{17}$$

for $\sin\theta \ge \frac{\sqrt{2}}{2}$; and if $\cos\theta > \frac{\sqrt{2}}{2}$, θ is determined from:

$$\cos\theta = \frac{1}{2}(r_{11} + r_{22} + r_{33} - 1) \tag{18}$$

When $\theta < (3/4)\,\pi$, for a more accurate solution, $\mathbf{e}$ can be determined from:

$$\sin\theta \cdot \mathbf{e} = \frac{1}{2}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{19}$$

otherwise $\mathbf{e}$ is obtained by

$$\frac{1}{2}([R] + [R]^T) = \cos\theta[I] + (1 - \cos\theta)\mathbf{e}\mathbf{e}^T \tag{20}$$

The remaining screw parameters are calculated by using:

$$u = \mathbf{e}^T \cdot \mathbf{p} \tag{21}$$

where, $\mathbf{p} = \mathbf{p}_f - [R]\mathbf{p}_i$ is the translating vector, and

$$\mathbf{A} = -\frac{1}{2}\mathbf{e} \times (\mathbf{e} \times \mathbf{p}) + \frac{\sin\theta}{2(1 - \cos\theta)}\mathbf{e} \times \mathbf{p} \tag{22}$$

If $\theta = 0$, then the screw is undefined. In this case the motion is pure translation and the screw axis can be located anywhere in space with its orientation parallel to the direction of the translation of the rigid body.

The Method Based on Rodrigues' Formula (*Bottema and Roth's Method*)

This algorithm, unlike the others, does not need to form the rotation matrix [R], instead, it can directly provide the solutions of screw parameters. The main disadvantage of this method, mentioned in [6], is that three different solutions for the screw parameters may be obtained depending on the different order in which the points are considered, if data error exists. However, the numerical simulation of this study showed that the differences between the solutions are insignificant compared with the results obtained by other methods when using inexact data. This problem will be further discussed in the next section.

Based on pure vector manipulation, the magnitude of rotation, θ, and the unit direction vector, $\mathbf{e}$, are obtained using

$$\tan\left(\frac{\theta}{2}\right)\mathbf{e} = \frac{[(\mathbf{p}_{f3} - \mathbf{p}_{f2}) - (\mathbf{p}_{i3} - \mathbf{p}_{i2})] \times [(\mathbf{p}_{f1} - \mathbf{p}_{f2}) - (\mathbf{p}_{i1} - \mathbf{p}_{i2})]}{[(\mathbf{p}_{f3} - \mathbf{p}_{f2}) - (\mathbf{p}_{i3} - \mathbf{p}_{i2})] \cdot [(\mathbf{p}_{f1} - \mathbf{p}_{f2}) + (\mathbf{p}_{i1} - \mathbf{p}_{i2})]} \tag{23}$$

The position vector of the screw axis, **A**, is then determined from:

$$\mathbf{A} = \frac{1}{2}\,[\mathbf{p}_{i1} + \mathbf{p}_{f1} + (\mathbf{e}\times(\mathbf{p}_{f1} - \mathbf{p}_{i1})/\tan(\theta/2)) - \mathbf{e}\bullet(\mathbf{p}_{f1} + \mathbf{p}_{i1})\mathbf{e}] \tag{24}$$

and finally, the magnitude of the translation can be computed from:

$$u = \mathbf{e}\bullet(\mathbf{p}_{f1} - \mathbf{p}_{i1}) \tag{25}$$

or using equation (10) to reduce the computational sensitivity to data errors.

The algorithm described here is based on [8].

Beggs' Method [10]

This algorithm generates two matrices $[L]$ and $[L']$ from the initial and final position data.

$$[L] = \begin{bmatrix} (p_{i1x} - p_{i2x}) & (p_{i2x} - p_{i3x}) & (p_{i3x} - p_{i1x}) \\ (p_{i1y} - p_{i2y}) & (p_{i2y} - p_{i3y}) & (p_{i3y} - p_{i1y}) \\ (p_{i1z} - p_{i2z}) & (p_{i2z} - p_{i3z}) & (p_{i3z} - p_{i1z}) \end{bmatrix} \tag{26}$$

$$[L'] = \begin{bmatrix} (p_{f1x} - p_{f2x}) & (p_{f2x} - p_{f3x}) & (p_{f3x} - p_{f1x}) \\ (p_{f1y} - p_{f2y}) & (p_{f2y} - p_{f3y}) & (p_{f3y} - p_{f1y}) \\ (p_{f1z} - p_{f2z}) & (p_{f2z} - p_{f3z}) & (p_{f3z} - p_{f1z}) \end{bmatrix} \tag{27}$$

The elements r_{ij} of the rotation matrix $[R]$ can be obtained by solving the following algebraic equations:

$$(A^2 + B^2 + C^2)r_{i3}^2 + 2(AC_i + BD_i)r_{i3} + C_i^2 + D_i^2 - E^2 = 0 \tag{28}$$

$$r_{i1} = \frac{1}{E}\,[(l_{22}l'_{i1} - l_{21}l'_{i2}) + (l_{12}l_{32} - l_{22}l_{31})r_{i3}] \tag{29}$$

$$r_{i2} = \frac{1}{E}\,[(l_{11}l'_{i2} - l_{12}l'_{i1}) + (l_{12}l_{31} - l_{11}l_{32})r_{i3}] \tag{30}$$

with,

$$A = l_{21}l_{32} - l_{22}l_{31}$$
$$B = l_{12}l_{31} - l_{11}l_{32}$$
$$C_i = l'_{i1}l_{22} - l'_{i2}l_{21}$$
$$D_i = l'_{i2}l_{11} - l'_{i1}l_{12}$$
$$E = l_{11}l_{22} - l_{12}l_{21}$$

The solution of the above equations gives as many as eight matrices, but only one of them is a proper rotation matrix which satisfies equation (5). Therefore, the rotation matrix can thus be found.

The translation vector **p** can now be determined using equation (10) or from the coordinates of any one of the three points. In our computer implementation the following relation was used:

$$\mathbf{p}_{f1} = [T]\mathbf{p}_{i1} \tag{31}$$

Equation (31) can be solved for the translating vector **p**. From [R] and **p**, the required screw parameters can be obtained by using equation (17) to (22).

Limitations of the Methods

There are certain limitations of some of the methods described above, although they can yield equivalent results under normal circumstances. When using Bottema and Roth's method, it should be noticed that the order of the three points cannot always be chosen arbitrarily. In case the rigid body rotates about a line joining any two of the three points, if we choose these two points as $\mathbf{p}_2$ and $\mathbf{p}_3$, the term $[(\mathbf{p}_{f3} - \mathbf{p}_{f2}) - (\mathbf{p}_{i3} - \mathbf{p}_{i2})]$ in the denominator of equation (23) will be equal to zero and the equation becomes undefined. To ensure computational stability, one should choose the order of the three points such that the term $[(\mathbf{p}_{f3} - \mathbf{p}_{f2}) - (\mathbf{p}_{i3} - \mathbf{p}_{i2})] \gg 0$ or it should have the largest value among the three possible choices.

In both Angeles' method and Spoor and Veldpaus' method, the rotation matrix [R] is obtained from a matrix product as shown in equation (8) and equation (16), involving the concept of principal vectors of certain tensor quantities [5][7]. However, as only the orientations, instead of both the orientations and the directions, of the principal vectors are defined in both of the algorithms, application of these methods can possibly lead to a false result, that is, the resulting matrix [R], although it is orthogonal, is not a real rotation matrix of the rigid body motion. This problem can be avoided by employing additional constraints, such as the transformation equation relating to the initial and final positions of any one of the three point vectors of the rigid body. In this case, however, the computational efficiency will be affected as additional computation is required.

A similar but more serious problem can be encountered when using Beggs' method in which as many as eight orthogonal matrices may be found for some special configurations. This problem will be discussed in more detail in the next section and in the Appendix.

4 Computational Results and Comparison

A computer code was developed for the five algorithms using Fortran 77 language. The computation was performed using a MIPS – 1000 computing facility. The program first generates the initial position matrix $[P_i]$ of three points and six independent screw parameters by using a random number generator that can provide random numbers having uniform distribution, and then, by applying constraints $\mathbf{e}^T\bullet\mathbf{e} = 1$ and $\mathbf{e}^T\bullet\mathbf{A} = 0$, the remaining two parameters are determined. The set of screw parameters $\mathbf{p}_a$ is used to form a transformation matrix $[T]$, which in turn, determines the final position matrix $[P_f]$ of the three points by using equation (3).

To ensure that the three points are noncollinear, a subroutine is used to check the singularities of $[P_i]$ after they are generated by the random number generator.

Matrices $[P_i]$ and $[P_f]$ and vector $\mathbf{p}_a$ obtained using the random number generator are assumed to be exact values. A sequence of $[P_i]$ and $[P_f]$ are used by all five methods to determine the corresponding screw parameters $\mathbf{p}_{ai}$, $i = 1, \ldots 5$, indicating that the resulting screw parameters are obtained using one of the five methods.

In the following section, computational results for efficiency, accuracy, and sensitivity of the five methods are presented and discussed. The computational scheme is shown in Fig. 2.

Efficiency

The computation was performed repeatedly for 200 sets of $[P_i]$ and $[P_f]$ and the execution time for each method was recorded and then processed. The results are shown in Table 1.

It can be seen from the table that method No. 4 is the most efficient. This can be easily explained if one looks at the formulas used by this method, described in the previous section. This algorithm uses only simple arithmetic operations without matrix computation, furthermore, it approaches the solution for screw parameters directly, omiting operations required for forming the rotation matrix $[R]$ and solving it for the screw parameters.

Method No. 5 is in second place on the basis of computational efficiency. This method also involves only simple arithmetic operations, however, it is required to find the proper rotation matrix $[R]$, which is one of the main reasons which makes it less efficient than the previous method.

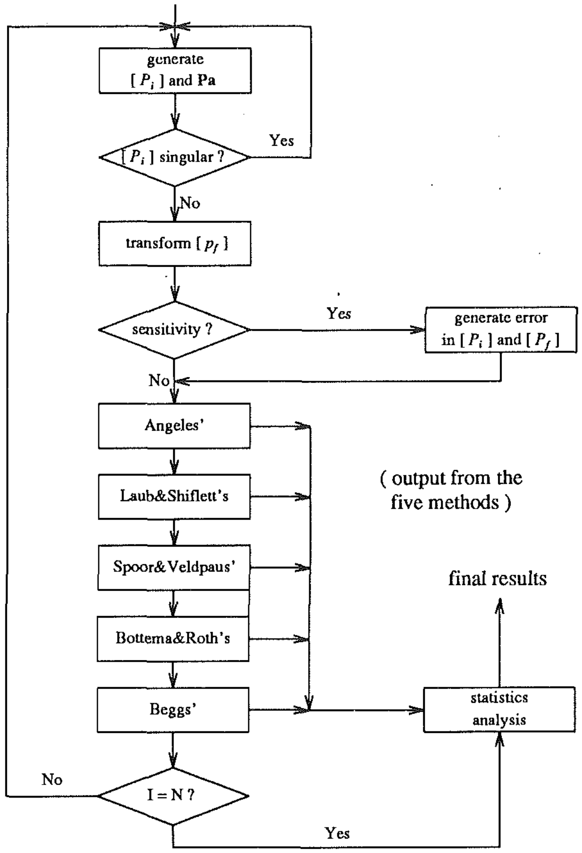

Fig. 2 Computational scheme

Table 1 Mean execution time and standard deviation

No.	Method	Execution Time (10^{-3}sec.)	Std. Deviation (10^{-3}sec.)
1	Angeles	8.748	0.162
2	Laub&Shiflett	4.680	0.104
3	Spoor&Veldpaus	5.360	0.453
4	Bottema&Roth	0.406	0.042
5	Beggs	1.674	0.107

Table 2 Comparison of accuracy for the five methods

Parameter	Relative Accuracy (in 10^{-6})				
	Angeles	Laub&Shiflett	Spoor&Veldpaus	Bottema&Roth	Beggs
e_x	-3.46	1.47	97.28	5.56	-111.93
e_y	-0.11	-0.32	7.41	0.95	348.25
A_x	-3.84	11.28	-130.97	3.58	12.98
A_y	1.03	2.08	46.95	-37.06	108.85
u	0.06	0.16	-6.88	-0.11	24.32
θ	-0.34	0.49	25.43	0.29	1.00

Table 3 Standard deviation of accuracy

Parameter	Standard deviation (in 10^{-6})				
	Angeles	Laub&Shiflett	Spoor&Veldpaus	Bottema&Roth	Beggs
e_x	18.76	12.36	1174.39	44.72	983.29
e_y	13.53	14.02	409.77	62.36	2669.05
A_x	24.20	91.07	1182.67	69.97	381.07
A_y	23.98	38.84	525.67	394.51	739.63
u	1.64	3.44	52.49	2.39	153.80
θ	2.42	4.78	317.21	3.38	21.94

Methods No. 1, No. 2, and No. 3 all involve matrix computations. In method No. 1, although an efficient means, Householder reflections, had been employed to reduce the floating-point operation required, the process to find $[Q_1]$ and $[Q_2]$ still needs quite a lot CPU time.

Method No. 2 is the third best in terms of efficiency. A statistical test [16] shows with 0.995 level of confidence that this method is more efficient than No. 3.

Accuracy

Accuracies of the methods are compared in terms of the relative error, which is defined as

$$\epsilon_{eij} = \frac{p_{aij} - p_{aj}}{|p_{aj}|} \times 10^{-6} \quad (i=1, \ldots, 5; j=1, \ldots, 8) \quad (32)$$

where, p_{aij} is the jth screw parameter obtained from method i, and p_{aj} is the jth screw parameter obtained using the random number generator, which is considered to be an exact value.

ϵ_{eij} are considered as random variables. Of the eight variables for each of the five methods, only six are independent due to the constraints of equation (1) and equation (2). The remaining two dependent variables can be arbitrarily chosen from the two sets of variables relating **e** and **A**, respectively. It is, therefore, assumed that the relative errors of e_x, e_y, A_x, A_y, u, and θ are the six independent variables, and the accuracy of the five methods are compared on a statistical basis of these variables.

200 sets of generated data, $[P_i]$ and $[P_f]$ are used to compute the screw parameters using the five methods, and then, based on equation (32), the relative errors are determined. The results are summarized in Table 2 and Table 3. It can be seen from the tables that results from Angeles' method, Laub and Shiflett's method and Bottema and Roth's method are remarkably more accurate than those of the other two methods. The tables also show that Angeles' method yields the smallest relative error in five of the eight screw parameters and the lowest standard deviations for all parameters, except for e_x. Therefore, it is reasonable to conclude that this method is the most accurate of the five methods.

Based on the estimation of mean value in statistics [16], the 0.99 confidence interval of the maximum relative error for Angeles' method is:

$$-10.072 \times 10^{-6} < \epsilon_{e_{\max}} < 2.392 \times 10^{-6}$$

From Table 2 and Table 3, it can be seen that the maximum relative error occurs when computing screw parameter A_x, the x component of position vector **A**.

Laub and Shiflett's method occupies the second place of the five methods in terms of accuracy. However, if accuracy of **e** is most important, this method can be considered to be superior to Angeles' method since the resulting e_x, e_y and thus e_z values obtained by this method are more accurate in general and the relative errors are more uniform and stable as shown in Tables 2 and 3. The 0.99 confidence interval for the maximum relative error for this method is

$$-12.1705 \times 10^{-6} < \epsilon_{e_{\max}} < 34.7305 \times 10^{-6}$$

The maximum relative error occurs when computing the screw parameter A_x.

Results from Beggs' method show that the relative error obtained when using this method is higher than that of others. One of the reasons is that the equations used in solving the rotation matrix $[R]$ become ill-conditioned when the orientation of the screw axis is close to the orientation of any of the coordinate axis, that is, when the values of any two of the components of vector **e** are much smaller than the value of the third component. In this case, the accuracy of the rotation matrix obtained using this method deteriorates and there is more than one matrix from the eight solutions which satisfy the orthogonality conditions given by equations (5a) and (5b). The conditions used in identifying the required rotation matrix, $[R]$, from the eight possible solutions in this case

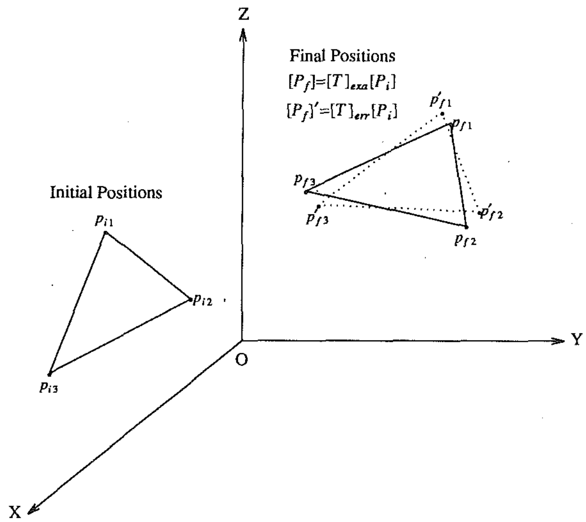

Fig. 3 Geomatrical description of a_4

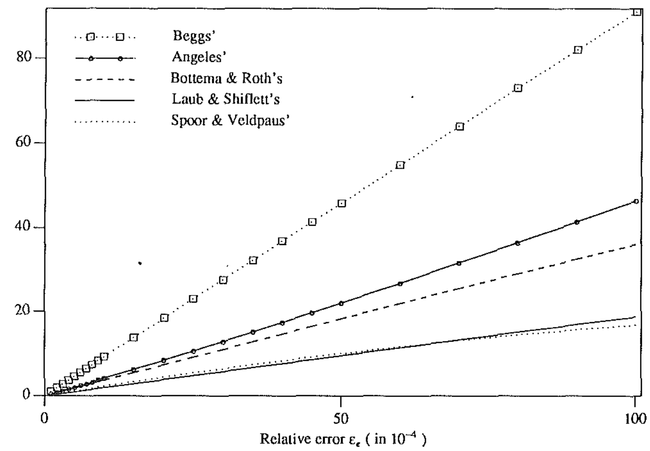

Fig. 4 Sensitivity in terms of a_1
a_1 (10^{-4} unit length)

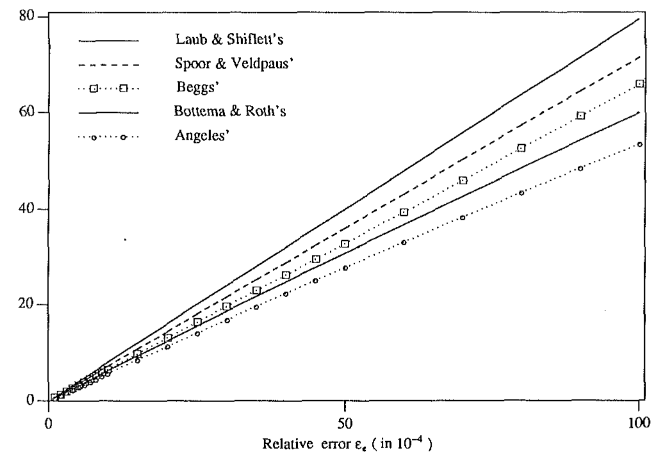

Fig. 5 Sensitivity in terms of a_2
a_2 (in 10^{-4} rad.)

become insufficient, which in turn, results in solutions for the screw parameters with large deviations. One such special configuration and the results obtained using this and other methods are given as an example in the Appendix.

Sensitivity

The term sensitivity in this paper represents the deteriorating level of accuracy due to the existence of data errors.

For comparison of the sensitivity of the methods due to inexact data, errors in the initial and final position matrix $[P_i]$ and $[P_f]$ are generated and then inexact values are used in the computation. The value of the relative error, ϵ_e, considered here ranges from 0.01 percent to 1.0 percent. Since the error may be either positive or negative, the sign of ϵ_e is also determined with the help of a random number generator.

As many of the error types in practical application can be formulated by or simplified to a uniform scaling transformation, the inexact position data under consideration are given by the following expressions:

$$[P_i]_\epsilon = (1+\epsilon_e)[P_i] \tag{33a}$$

$$[P_f]_\epsilon = (1+\epsilon_e)[P_f] \tag{33b}$$

When the variances of the errors are known, it is possible to derive analytical expressions for the variances and covariance matrices of the computed screw parameters from the five methods as discussed by Woltring et al. [17]. These expressions are usually nonlinear functions of the screw parameters, and numerical technique is necessary in order to compare the differences of these expressions for the different methods. Since we are mainly concerned about the general deviations between the screw parameters obtained from inexact data and the screw parameters obtained from error-free data, a more direct numerical approach is used in this paper to compare the sensitivities of the different methods.

The values of $[P_i]_\epsilon$ and $[P_f]_\epsilon$ are used as input by the five methods to compute the corresponding screw parameters, and the deviation from the results obtained from error-free data is taken as a criterion for comparing sensitivity. Four functions are used for this purpose, which are defined as

$$a_1 = \left[\left(\sum_{k=1}^{n} (u'^k - u^k)^2\right)\Big/ n\right]^{1/2} \tag{34}$$

$$a_2 = \left[\left(\sum_{k=1}^{n} (\theta'^k - \theta^k)^2\right)\Big/ n\right]^{1/2} \tag{35}$$

$$a_3 = \left[\left(\sum_{k=1}^{n} \|\mathbf{A}'^k - \mathbf{A}^k\|^2\right)/n\right]^{1/2} \tag{36}$$

$$a_4 = \left[\left(\sum_{k=1}^{n}\sum_{j=1}^{3} \|\mathbf{p}_{fj}'^k - \mathbf{p}_{fi}^k\|^2\right)\Big/ 3n\right]^{1/2} \tag{37}$$

where,

u'^k, u^k = translation along the screw axis obtained from the kth inexact and accurate data respectively;

θ'^k, θ^k = rotation angle in the screw motion obtained from the kth inexact and accurate data, respectively;

$\mathbf{A}'^k$, $\mathbf{A}^k$ = position vector of point A obtained from the kth inexact and accurate data, respectively;

$\mathbf{p}_{fj}'^k$, $\mathbf{p}_{fj}^k$ = position vector of final point j obtained from $[P_f]' = [T]_{err}$ $[P_i]$ and $[P_f] = [T]_{exa}$ $[P_i]$, respectively.

a_1, a_2, and a_3 are mainly concerned with the deviation of the resulting individual screw parameters, while a_4 is related to the general deviation of the final positions of the three points as shown in Fig. 3, where the deviations are caused by the sensitivity to data error. Since the differences between the components of $\mathbf{e}'^k$ and $\mathbf{e}^k$ are very small and not in the same order as that of other parameters, the computational sensitivity of $\mathbf{e}$ will not be discussed separately; however, its influence on the final positions of the three points are included implicitly in the general deviation index a_4.

For each given error, ϵ_e, 100 sets of $[P_i]_\epsilon$ and $[P_f]_\epsilon$, i.e., n = 100 in equations (34)-(37), are used to compute the screw

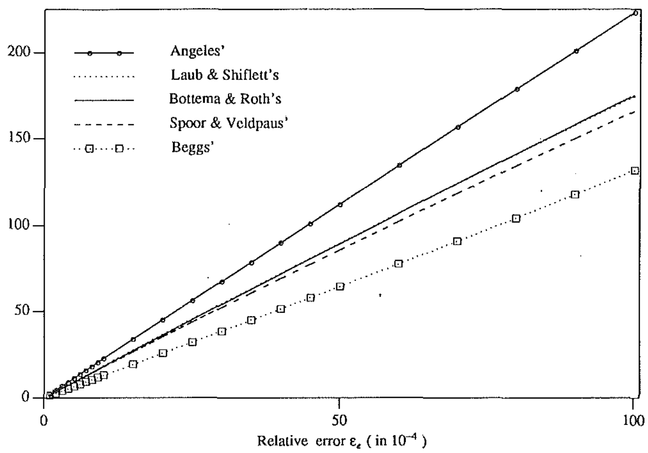

Fig. 6 Sensitivity in terms of a_3 a_3 (10^{-4} unit length)

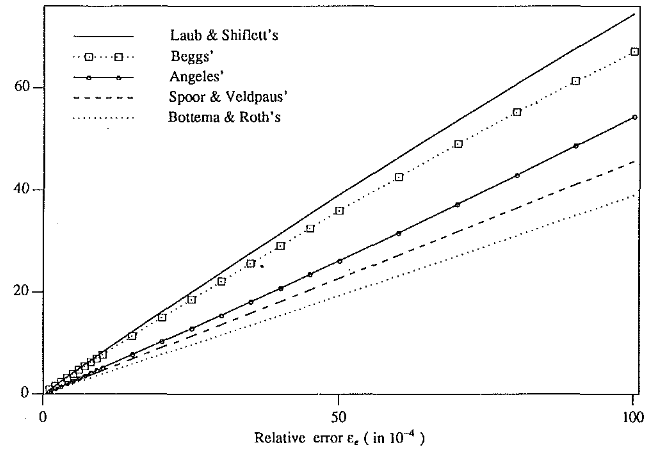

Fig. 7 Sensitivity in terms of a_4 a_4 (in 10^{-4} unit length)

parameters using the five methods, and the merit functions a_1 through a_4 are then determined using equations (34)–(37).

The computational results are shown in Fig. 4 to Fig. 7. It is found that all these merit functions, a_1, a_2, a_3, and a_4, are close to linear functions in terms of the data error for all five methods. This implies that all five methods can be used in computing screw parameters from inexact data, since the deviation of the results is likely to be predictable due to the linear relations between sensitivities and data errors.

Figure 4 shows that the influence of the relative error to sensitivity in terms of a_1 for the Laub and Shiflett method and the Spoor and Veldpaus method is almost the same, which is the lowest among the five methods. On the other hand, the influence of relative error to Beggs' method in terms of a_1 is the most significant of the five methods.

In Fig. 5, it can be seen that Angeles' method has the lowest sensitivity value in terms of a_2, while the Laub and Shiflett method has the highest.

In terms of a_3, Beggs' method is superior to the others. Laub and Shiflett's method, Bottema and Roth's method, and Spoor and Veldpaus' method have very close results as shown in Fig. 6.

In terms of a_4, which reflects the general deviation of the final positions of the three points, Bottema and Roth's method is the best of the five methods. This result indicates that when determining the screw parameters from inexact data, a screw motion determined by the Bottema and Roth method, in general, can yield final positions that are closer to the exact positions determined from accurate data $[P_i]$ and exact screw parameters.

5 Conclusions

Five methods for determining screw parameters have been compared in terms of their efficiency, accuracy, and sensitivity to inexact data. The Bottema and Roth method is the most efficient of the five methods since it involves only simple arithmetic operations of vectors. Angeles' method is the most accurate. The largest mean relative error of the results from this method is within the interval of -10.072×10^{-6} -2.392×10^{-6} with 0.99 confidence. Four merit functions are used in this paper to compare the sensitivities representing the deteriorating level of accuracy of the five methods due to the input of inexact data. It is found that all merit functions are close to linear functions of relative data error, which means that any of the methods can be used if uncertainty exists in the position data. In terms of sensitivity, the Bottema and Roth's method is, in general, less sensitive to data errors than the others.

It is recommended that the Bottema and Roth method be used when efficiency is especially important, but if accuracy is the dominating factor, the Laub and Shiflett method is recommended which provides an accurate solution with reasonable sensitivity and efficiency.

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APPENDIX

When using Beggs' method to determine the rotation matrix $[R]$ from the given initial and final position data, there may exist as many as eight possible solutions. Beggs [10] suggested that the required solution, the rotation matrix $[R]$, be identified by using the orthogonality conditions, equations (5a) and (5b). However, some special initial and final configurations exist for which more than one matrix, satisfying the orthogonality condition, equations (5a) and (5b), can be found from the eight possible solutions. One such configuration is given, as an example, as follows.

The coordinates of the initial and final positions of the three points are, respectively,

$$[P_i] = \begin{bmatrix} 3.51542 & 3.18322 & 1.66537 \\ 0.48235 & 0.69586 & 2.29896 \\ 3.43593 & 1.23011 & 0.97893 \end{bmatrix} \tag{A1}$$

and,

$$[P_f] = \begin{bmatrix} 3.44738 & 2.99723 & 0.78293 \\ 2.94288 & 2.91118 & 2.91222 \\ 5.02296 & 2.82798 & 2.64402 \end{bmatrix} \tag{A2}$$

The exact screw parameters, $\mathbf{p}_a$, generated by the random number generator, are

$$\mathbf{p}_a = [0.04864213\ 0.02074652\ 0.99860078\ 0.46231923\ 1.45288157 - 0.05270418\ 1.63255501\ 0.81610703]^T \tag{A3}$$

It can be seen that the first two components of vector $\mathbf{e}$, $e_x = 0.04864213$, and $e_y = 0.02074652$ are much smaller than $e_z = 0.99860078$ for this special configuration.

The rotation matrix, $[R]$, obtained by using Angeles' method, which is the same as the rotation matrix obtained by using Laub and Shiflett's method, is

$$[R] = \begin{bmatrix} 0.68581 & -0.72715 & 0.03041 \\ 0.72778 & 0.68520 & -0.02891 \\ 0.00018 & 0.04196 & 0.99912 \end{bmatrix} \tag{A4}$$

By using Beggs' method, the eight 3 by 3 matrices obtained are

$$[R_1] = \begin{bmatrix} 0.68335 & -0.72945 & 0.03056 \\ -0.72978 & -0.68119 & 0.05834 \\ -0.00047 & 0.04135 & 0.99916 \end{bmatrix}$$

$$[R_2] = \begin{bmatrix} 0.68579 & -0.72716 & 0.03041 \\ 0.72778 & 0.68520 & 0.02891 \\ 0.02240 & 0.06278 & 0.99779 \end{bmatrix}$$

$$[R_3] = \begin{bmatrix} 0.68336 & -0.72945 & 0.03056 \\ 0.72778 & 0.68520 & -0.02891 \\ 0.02240 & 0.06278 & 0.99779 \end{bmatrix}$$

$$[R_4] = \begin{bmatrix} 0.68579 & -0.72716 & 0.03041 \\ -0.72978 & -0.68119 & 0.05834 \\ -0.00047 & 0.04135 & 0.99916 \end{bmatrix}$$

$$[R_5] = \begin{bmatrix} 0.68335 & -0.72945 & 0.03056 \\ -0.72978 & -0.68119 & 0.05834 \\ 0.02240 & 0.06278 & 0.99779 \end{bmatrix}$$

$$[R_6] = \begin{bmatrix} 0.68579 & -0.72716 & 0.03041 \\ 0.72778 & 0.68520 & -0.02891 \\ -0.00047 & 0.04135 & 0.99916 \end{bmatrix}$$

$$[R_7] = \begin{bmatrix} 0.68335 & -0.72945 & 0.03056 \\ 0.72778 & 0.68520 & -0.02891 \\ -0.00047 & 0.04135 & 0.99916 \end{bmatrix}$$

$$[R_8] = \begin{bmatrix} 0.68579 & -0.72716 & 0.03041 \\ -0.72978 & -0.68119 & 0.05834 \\ 0.02240 & 0.06278 & 0.99779 \end{bmatrix} \tag{A5}$$

It can be easily demonstrated that four of the matrices, $[R_5]$, $[R_6]$, $[R_7]$, and $[R_8]$, satisfy the orthogonality conditions, equations (5a) and (5b), based on the same level of approximation. This ambiguous situation makes the identification of the required rotation matrix, $[R]$, by using only the orthogonality conditions, equations (5a) and (5b), difficult or even impossible. Furthermore, since the screw parameters are dependent on the rotation matrix, the solutions for screw parameters obtained by using these different matrices are, in general, different and some of the values for the screw parameters may diverge farther than others from the exact solution for the screw parameters. Thus, in this method, the insufficient conditions for identifying the required rotation matrix can lead to an improper selection of the rotation matrix which, in turn, results in deterioration in computational accuracy of the screw parameters.

Choosing $[R_6]$ as the rotation matrix, the resulting screw parameters, which are determined using the routines described in equations (17)-(22), are

$$\mathbf{p}_a = [0.04822037\ 0.02119716\ 0.99861175\ 0.46352512\ 1.45323098 - 0.05322962\ 1.63456631\ 0.81610012]^T \tag{A6}$$

while the screw parameters determined by Angeles' method using the same data are

$$\mathbf{p}_a = [0.04864204\ 0.02074642\ 0.99860078\ 0.46231875\ 1.45288169 - 0.05270398\ 1.63255465\ 0.81610692]^T \tag{A7}$$

It can be seen that the results obtained from Angeles' method are very close to the exact values given by equation (A3), whereas the results from Beggs' method may include as large as 2.172 percent relative error, which occurs when computing the y component of the direction vector $\mathbf{e}$. Since $[R_6]$ is the best approximation of the eight matrices to the exact rotation matrix $[R]$, any other matrix improperly selected as a rotation matrix would cause even larger relative errors.

To eliminate the ambiguity in selecting the required rotation matrix, additional conditions must be incorporated. One alternative is to minimize the differences between the given coordinates of the three final points and the computed vlaues of the coordinates obtained by using the selected rotation matrix $[R_k]$, the translation vector $\mathbf{p}$ [see equation (21)], and the given initial point coordinates. This additional condition assures the correct selection of the rotation matrix, $[R]$, thus it can increase the computational accuracy of the screw parameters; however, the computational efficiency of the method will decrease, since satisfying this additional condition requires a number of additional matrix computations requiring considerable execution time.