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THE DUAL HOROSPHERICAL RADON TRANSFORM FOR POLYNOMIALS

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Abstract. Let $X = G/K$ be a semisimple symmetric space of noncompact type. A horosphere in X is an orbit of a maximal unipotent subgroup of G. The set Hor X of all horospheres is a homogeneous space of G. The horospherical Radom transform suggested by I. M. Gelfand and M. I. Graev in 1959 takes any function φ on X to a function on Hor X obtained by integrating φ over horospheres. We explicitly describe the dual transform in terms of its action on polynomial functions on Hor X.

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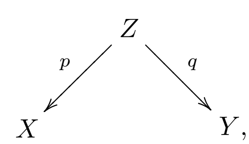
1. Introduction

A Radon transform is generally associated to a double fibration



where one may assume without loss of generality that the maps p and q are surjective and Z is embedded into $X \times Y$ via $z \mapsto (p(z), q(z))$. Let some measures be chosen on X, Y, Z and on the fibers of p and q so that

$$\int_X \left(\int_{p^{-1}(x)} f(u)\, du \right) dx = \int_Z f(z)\, dz = \int_Y \left(\int_{q^{-1}(y)} f(v)\, dv \right) dy. \qquad (1)$$

Then the Radon transform $\mathcal{R}$ is the linear map assigning to a function φ on X the function on Y defined by

$$(\mathcal{R}\varphi)(y) = \int_{q^{-1}(y)} (p^*\varphi)(v)\, dv,$$

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where we have set $p^*\varphi := \varphi \circ p$. In a dual fashion, one defines a linear transform $\mathcal{R}^*$ from functions on Y to functions on X via

$$(\mathcal{R}^*\psi)(x) = \int_{p^{-1}(x)} (q^*\psi)(u)\, du.$$

It is dual to $\mathcal{R}$. Indeed, formally,

$$\begin{aligned}(\mathcal{R}\varphi,\, \psi) &:= \int_Y \left(\int_{q^{-1}(y)} (p^*\varphi)(v)\, dv\right)\psi(y)\, dy = \int_Y \left(\int_{q^{-1}(y)} (p^*\varphi)(q^*\psi)(v)\, dv\right) dy \\ &= \int_Z (p^*\varphi)(z)(q^*\psi)(z)\, dz = \int_X \varphi(x)\left(\int_{p^{-1}(x)} (q^*\psi)(u)\, du\right) dx \\ &=: (\varphi,\, \mathcal{R}^*\psi).\end{aligned}$$

In particular, if $X = G/K$ and $Y = G/H$ are homogeneous spaces of a Lie group G, one can take $Z = G/(K \cap H)$, where p and q are G-equivariant maps sending $e(K \cap H)$ to eK and eH, respectively. Assume that there exist G-invariant measures on X, Y and Z. If such measures are fixed, one can uniquely define measures on the fibers of p and q so that condition (1) holds. (Here, speaking about a measure on a smooth manifold, we mean a measure defined by a differential form of top degree.) In this setting we can consider the transforms $\mathcal{R}$ and $\mathcal{R}^*$, and if $\dim X = \dim Y$, one can hope that they are invertible. The basic example is provided by the classical Radon transform, which acts on functions on the Euclidean space $\mathbb{E}^n = (\mathbb{R}^n \rtimes \mathrm{SO}_n)/\mathrm{SO}_n$ and maps them to functions on the space $H\mathbb{E}^n = (\mathbb{R}^n \rtimes \mathrm{SO}_n)/(\mathbb{R}^{n-1} \rtimes \mathrm{O}_{n-1})$ of hyperplanes in $\mathbb{E}^n$.

For a semisimple Riemannian symmetric space $X = G/K$ of noncompact type, one can consider the horospherical Radon transform as proposed by I. M. Gelfand and M. I. Graev in [GG59]. Namely, generalizing the classical notion of a horosphere in Lobachevsky space, one can define a horosphere in X as an orbit of a maximal unipotent subgroup of G. The group G naturally acts on the set $\mathrm{Hor}\, X$ of all horospheres. This action is transitive, so we can identify $\mathrm{Hor}\, X$ with some quotient space G/S (see Section 5 for details). It turns out that $\dim X = \dim \mathrm{Hor}\, X$. Moreover, the groups G, K, S and $K \cap S = M$ are unimodular, so there exist G-invariant measures on X, $\mathrm{Hor}\, X$ and $Z = G/M$. The Radon transform $\mathcal{R}$ associated to the double fibration

$$\begin{array}{ccc} & G/M & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X = G/K & & G/S = \mathrm{Hor}\, X \end{array}$$

is called the *horospherical Radon transform*.

The space $Z = G/M$ can be interpreted as the set of pairs $(x, \mathcal{H}) \in X \times \mathrm{Hor}\, X$ with $x \in \mathcal{H}$, so that p and q are just the natural projections. The fiber $q^{-1}(\mathcal{H})$ with $\mathcal{H} \in \mathrm{Hor}\, X$ is then identified with the horosphere $\mathcal{H}$, and the fiber $p^{-1}(x)$ with $x \in X$ is identified with the submanifold $\mathrm{Hor}_x X \subseteq \mathrm{Hor}\, X$ of all horospheres passing through x. Note that, in contrast to the horospheres, all submanifolds $\mathrm{Hor}_x X$ are compact, since $\operatorname{Hor}_x X$ is the orbit of the stabilizer of x in G, which is conjugate to K.

In this paper, we describe the *dual horospherical Radon transform* $\mathcal{R}^*$ in terms of its action on polynomial functions. Here a differentiable function φ on a homogeneous space $Y = G/H$ of a Lie group G is called *polynomial*, if the linear span of the functions $g\varphi$ with $g \in G$ is finite dimensional. The polynomial functions constitute an algebra denoted by $\mathbb{R}[Y]$.

For $X = G/K$ as above, the algebra $\mathbb{R}[X]$ is finitely generated and X is naturally identified with a connected component of the corresponding affine real algebraic variety (the real spectrum of $\mathbb{R}[X]$). The natural linear representation of G in $\mathbb{R}[X]$ decomposes into a sum of mutually non-isomorphic absolutely irreducible finite-dimensional representations whose highest weights λ form a semigroup Λ. Let $\mathbb{R}[X]_\lambda$ be the irreducible component of $\mathbb{R}[X]$ with highest weight λ, so

$$\mathbb{R}[X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[X]_\lambda. \tag{2}$$

Denote by φ_λ the highest weight function in $\mathbb{R}[X]_\lambda$ normalized by the condition

$$\varphi_\lambda(o) = 1,$$

where $o = eK$ is the base point of X. Then the subgroup S is the intersection of the stabilizers of all the φ_λ's. Its unipotent radical U is a maximal unipotent subgroup of G.

The algebra $\mathbb{R}[\operatorname{Hor} X]$ is also finitely generated. The manifold $\operatorname{Hor} X$ is naturally identified with a connected component of a quasi-affine algebraic variety, which is a Zariski open subset in the real spectrum of $\mathbb{R}[\operatorname{Hor} X]$. The natural linear representation of G in $\mathbb{R}[\operatorname{Hor} X]$ is isomorphic to the representation of G in $\mathbb{R}[X]$. Let $\mathbb{R}[\operatorname{Hor} X]_\lambda$ be the irreducible component of $\mathbb{R}[\operatorname{Hor} X]$ with highest weight λ, so that

$$\mathbb{R}[\operatorname{Hor} X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[\operatorname{Hor} X]_\lambda. \tag{3}$$

Denote by ψ_λ the highest weight function in $\mathbb{R}[\operatorname{Hor} X]_\lambda$ normalized by the condition

$$\psi_\lambda(sUo) = 1,$$

where s is the symmetry with respect to o and $sUo = (sUs^{-1})o$ is regarded as a point of $\operatorname{Hor} X$.

The decomposition (2) defines a filtration of the algebra $\mathbb{R}[X]$ (see Section 4 for the precise definition). Let $\operatorname{gr} \mathbb{R}[X]$ be the associated graded algebra. There is a canonical G-equivariant algebra isomorphism

$$\Gamma\colon \operatorname{gr} \mathbb{R}[X] \to \mathbb{R}[\operatorname{Hor} X]$$

mapping each φ_λ to ψ_λ. As a G-module, $\operatorname{gr} \mathbb{R}[X]$ is canonically identified with $\mathbb{R}[X]$, so we can view Γ as a G-module isomorphism from $\mathbb{R}[X]$ to $\mathbb{R}[\operatorname{Hor} X]$.

The horospherical Radon transform $\mathcal{R}$ is not defined for polynomial functions on X but its dual transform $\mathcal{R}^*$ is defined for polynomial functions on $\operatorname{Hor} X$, since it reduces to integrating along compact submanifolds. Moreover, it follows from the definition of polynomial functions, that the dual transform maps polynomial functions on $\operatorname{Hor} X$ to polynomial functions on X. Obviously, it is G-equivariant. Thus, we have G-equivariant linear maps

$$\mathbb{R}[X] \xrightarrow{\Gamma} \mathbb{R}[\operatorname{Hor} X] \xrightarrow{\mathcal{R}^*} \mathbb{R}[X].$$

Their composition $\mathcal{R}^* \circ \Gamma$ is a G-equivariant linear operator on $\mathbb{R}[X]$, so

$$(\mathcal{R}^* \circ \Gamma)(\varphi) = c_\lambda \varphi \quad \forall \varphi \in \mathbb{R}[X]_\lambda,$$

where the c_λ's are constants. To give a complete description of $\mathcal{R}^*$, it is therefore sufficient to find these constants. Our main result is the following theorem.

Theorem 1. *We have $c_\lambda = \mathbf{c}(\lambda + \rho)$, where $\mathbf{c}$ is the Harish-Chandra $\mathbf{c}$-function and ρ is the half-sum of the positive roots of X (counted with multiplicities).*

The Harish-Chandra **c**-function governs the asymptotic behavior of the zonal spherical functions on X. A product formula for the **c**-function was found by S. G. Gindikin and F. I. Karpelevich [GK62]: for a rank-one Riemannian symmetric space of noncompact type, the **c**-function is a ratio of gamma functions involving only the root multiplicities; in the general case, it is the product of the **c**-functions for the rank-one symmetric spaces defined by the indivisible roots of the space. Thus, if the root structure of the symmetric space is known, the product formula makes the **c**-function, and hence our description of the dual horospherical Radon transform, explicitly computable.

For the convenience of the reader, we collect some crucial facts about the **c**-function in an appendix to this paper.

The following basic notation will be used in the paper without further comments.

- Lie groups are denoted by capital Latin letters, and their Lie algebras by the corresponding small Gothic letters.
- The dual space of a vector space V is denoted by V^*.
- The complexification of a real vector space V is denoted by $V(\mathbb{C})$.
- The centralizer (resp. the normalizer) of a subgroup H in a group G is denoted by $Z_G(H)$ (resp. $N_G(H)$).
- The centralizer (resp. the normalizer) of a subalgebra $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ is denoted by $\mathfrak{z}_\mathfrak{g}(\mathfrak{h})$ (resp. by $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$).
- If a group G acts on a set X, we denote by X^G the subset of fixed points of G in X.

2. Groups, spaces, and functions

For any connected semisimple Lie group G admitting a faithful (finite-dimensional) linear representation, there is a connected complex algebraic group defined over $\mathbb{R}$ such that G is the connected component of the group of its real points. Among all such algebraic groups, there is a unique one such that all the others are its quotients. It is called the *complex hull* of G and is denoted by $G(\mathbb{C})$. The group of real points of $G(\mathbb{C})$ is denoted by $G(\mathbb{R})$. If $G(\mathbb{C})$ is simply connected, then $G = G(\mathbb{R})$.

The restrictions of polynomial functions on the algebraic group $G(\mathbb{R})$ to G are called *polynomial functions* on G. They are precisely those differentiable functions

φ for which the linear span of the functions $g\varphi$, $g \in G$, is finite-dimensional (see e. g. [CSM95], § II.8). (Here G is supposed to act on itself by left multiplications.) The polynomial functions on G constitute an algebra which we denote by $\mathbb{R}[G]$ and which is naturally isomorphic to $\mathbb{R}[G(\mathbb{R})]$.

For any subgroup $H \subseteq G$, we denote by $H(\mathbb{C})$ (resp. $H(\mathbb{R})$) its Zariski closure in $G(\mathbb{C})$ (resp. $G(\mathbb{R})$). If H is Zariski closed in G, i. e., $H = H(\mathbb{R}) \cap G$, then H is a subgroup of finite index in $H(\mathbb{R})$; if H is a semidirect product of a connected unipotent group and a compact group, then $H = H(\mathbb{R})$.

For a homogeneous space $Y = G/H$ with H Zariski closed in G, set $Y(\mathbb{C}) = G(\mathbb{C})/H(\mathbb{C})$. This is an algebraic variety defined over $\mathbb{R}$, and Y is naturally identified with a connected component of the variety $Y(\mathbb{R})$ of real points of $Y(\mathbb{C})$. We call $Y(\mathbb{C})$ the *complex hull* of Y.

If H is reductive, then also $H(\mathbb{C})$ is reductive and the variety $Y(\mathbb{C})$ is affine, the algebra $\mathbb{C}[Y(\mathbb{C})]$ being naturally isomorphic to the algebra $\mathbb{C}[G(\mathbb{C})]^{H(\mathbb{C})}$ of $H(\mathbb{C})$-right-invariant polynomial functions on $G(\mathbb{C})$ (see e. g. [VP89], Section 4.7 and Theorem 4.10). Accordingly, the algebra $\mathbb{R}[Y(\mathbb{R})]$ is naturally isomorphic to $\mathbb{R}[G(\mathbb{R})]^{H(\mathbb{R})}$.

In general, the functions on $Y(\mathbb{R})$ arising from $H(\mathbb{R})$-right-invariant polynomial functions on $G(\mathbb{R})$ are called polynomial functions on $Y(\mathbb{R})$, and their restrictions to Y are called *polynomial functions* on Y. They are precisely those differentiable functions φ for which the linear span of the functions $g\varphi$, $g \in G$, is finite-dimensional. They form an algebra which we denote by $\mathbb{R}[Y]$.

In the following we consider a semisimple Riemannian symmetric space $X = G/K$ of noncompact type. This means that G is a connected semisimple Lie group without compact factors and K is a maximal compact subgroup of G. We do not assume that the center of G is trivial, so the action of G on X may be non-effective. We do, however, require that G have a faithful linear representation. According to the above, the space X is then a connected component of the affine algebraic variety $X(\mathbb{R})$.

In order to describe the algebra $\mathbb{R}[X]$ explicitly, one can realize X as a closed orbit in a representation space V of G. The polynomial functions on X will be exactly the restrictions to X of ordinary polynomials on V.

Example 2. The Lobachevsky plane $L^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$ can be realized as the orbit of the matrix

$$I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathfrak{sl}_2(\mathbb{R}).$$

under the adjoint representation of $\mathrm{SL}_2(\mathbb{R})$. Comparing this model with the Poincaré model on the upper half-plane

$$H^2 = \{z = x + yi \in \mathbb{C} \colon y > 0\},$$

we see that the point $i \in H^2$ corresponds to the matrix I, so the point

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \circ i = a(ai + b) = x + yi \in H^2$$

(where $x = ab$, $y = a^2$) corresponds to the matrix

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1}b & -a^2 - b^2 \\ a^{-2} & -a^{-1}b \end{bmatrix}.$$

It follows that

$$\mathbb{R}[H^2] = \mathbb{R}[xy^{-1},\, y^{-1},\, x^2y^{-1} + y].$$

Note that the functions

$$\varphi = xy^{-1}, \quad \psi = y^{-1}, \quad \chi = x^2y^{-1} + y$$

are subject to the relation

$$\psi\chi - \varphi^2 = 1.$$

3. Subgroups and subalgebras

We recall some facts about the structure of Riemannian symmetric spaces of noncompact type (see [Hel01] for details). Let $X = G/K$ be as above and θ be the Cartan involution of G with respect to K, so $K = G^\theta$. Let $\mathfrak{a}$ be a Cartan subalgebra for X, i. e., a maximal abelian subalgebra in the (-1)-eigenspace of $d\theta$. Its dimension r is called the *rank* of X. Under any representation of G, the elements of $\mathfrak{a}$ are simultaneously diagonalizable. The group $A = \exp\mathfrak{a}$ is a maximal connected abelian subgroup of G such that $\theta(a) = a^{-1}$ for all $a \in A$. It is isomorphic to $(\mathbb{R}^*_+)^r$. Its Zariski closure $A(\mathbb{R})$ in $G(\mathbb{R})$ is a split algebraic torus which is isomorphic to $(\mathbb{R}^*)^r$. Let $\boldsymbol{X}(A)$ denote the (additively written) group of real characters of the torus $A(\mathbb{R})$. It is a free abelian group of rank r. We identify each character χ with its differential $d\chi \in \mathfrak{a}^*$.

The root decomposition of $\mathfrak{g}$ with respect to A (or with respect to $A(\mathbb{R})$, which is the same) is of the form

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 = \mathfrak{z}_\mathfrak{g}(\mathfrak{a})$. If $\mathfrak{m} := \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$, then $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a}$.

The set $\Delta \subset \boldsymbol{X}(A)$ is the *root system* of X (or the *restricted root system* of G) with respect to A and $\mathfrak{g}_\alpha$ is the *root subspace* corresponding to α. The dimension of $\mathfrak{g}_\alpha$ is called the *multiplicity* of the root α and is denoted by m_α. By the identification of a character with its differential, we will consider Δ as a subset of $\mathfrak{a}^*$. Choose a system Δ^+ of *positive roots* in Δ. Suppose $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ is the corresponding system of *simple roots*. Then

$$C = \{x \in \mathfrak{a} \colon \alpha_i(x) \geq 0 \text{ for } i = 1, \ldots, r\}$$

is called the *Weyl chamber* with respect to Δ^+. The subspace

$$\mathfrak{u} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$$

is a *maximal unipotent subalgebra* of $\mathfrak{g}$.

Set

$$G_0 = Z_G(A), \quad M = Z_K(A).$$

Then $G_0 = M \times A$ and the Lie algebras of G_0 and M are $\mathfrak{g}_0$ and $\mathfrak{m}$, respectively. Clearly, G_0 is Zariski closed in G.

The group $U = \exp \mathfrak{u}$ is a *maximal unipotent subgroup* of G. It is normalized by A and the map

$$U \times A \times K \to G, \quad (u, a, k) \mapsto uak,$$

is a diffeomorphism. The decomposition $G = UAK$ (or $G = KAU$) is called the *Iwasawa decomposition* of G. Since every root subspace is G_0-invariant, G_0 normalizes U, so

$$P := UG_0 = U \rtimes G_0$$

is a subgroup of G. Moreover, $P = N_G(U)$ (see e. g. [War72], Proposition 1.2.3.4), so P is Zariski closed in G.

We say that a Zariski closed subgroup of G is *parabolic*, if its Zariski closure in $G(\mathbb{C})$ is a parabolic subgroup of $G(\mathbb{C})$. Then P is a *minimal parabolic subgroup* of G. The subgroup

$$S := UM = U \rtimes M$$

of P is normal in P and P/S is isomorphic to A. It follows from the Iwasawa decomposition that $K \cap S = M$.

4. Representations

For later use we collect some well-known facts about finite-dimensional representations of G.

The natural linear representation of G in $\mathbb{R}[X]$ decomposes into a sum of mutually non-isomorphic absolutely irreducible finite-dimensional representations called (finite-dimensional) *spherical representations* (see e. g. [GW98], Chap. 12).

Theorem 3 (see [Hel84], § V.4)**.** *An irreducible finite-dimensional representation of G on a real vector space V is spherical if and only if the following equivalent conditions hold*:

(1) $V^K \neq \{0\}$,
(2) $V^S \neq \{0\}$.

If these conditions hold, then $\dim V^K = \dim V^S = 1$ *and the subspace V^S is invariant under P.*

For a spherical representation, the group P acts on V^S via multiplication by some character of P vanishing on S. The restriction of this character to A is called the *highest weight* of the representation. A spherical representation is uniquely determined by its highest weight. The highest weights of all irreducible spherical representations constitute a subsemigroup $\Lambda \subset \boldsymbol{X}(A)$.

An explicit description of Λ as a subset of $\mathfrak{a}^*$ can be given as follows. Let $\Delta_\star$ denote the set of roots $\alpha \in \Delta$ such that $2\alpha \notin \Delta$. Then $\Delta_\star$ is a root system in $\mathfrak{a}^*$, and a system of simple roots corresponding to $\Delta_\star^+ := \Delta^+ \cap \Delta_*$ can be obtained from the system $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots in Δ^+ by setting, for $j = 1, \ldots, r$,

$$\beta_j := \begin{cases} \alpha_j & \text{if } 2\alpha_j \notin \Delta^+, \\ 2\alpha_j & \text{if } 2\alpha_j \in \Delta^+. \end{cases}$$

Let $\omega_1, \ldots, \omega_r \in \mathfrak{a}^*$ be defined by

$$\frac{(\omega_j, \beta_i)}{(\beta_i, \beta_i)} = \delta_{ij}, \tag{4}$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathfrak{a}^*$ induced by an invariant scalar product in $\mathfrak{g}$.

Proposition 4 (see [Hel94], Proposition 4.23)**.** *The semigroup Λ is freely generated by $\omega_1, \ldots, \omega_r$.* □

For $\lambda \in \Lambda$, let $\mathbb{R}[X]_\lambda$ denote the irreducible component of $\mathbb{R}[X]$ with highest weight λ. Then we have

$$\mathbb{R}[X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[X]_\lambda.$$

We denote by φ_λ^S the highest weight function of $\mathbb{R}[X]_\lambda$ normalized by the condition

$$\varphi_\lambda^S(o) = 1.$$

(Since $Po = PKo = Go = X$, we have $\varphi_\lambda^S(o) \neq 0$.) Obviously,

$$\varphi_\lambda^S \varphi_\mu^S = \varphi_{\lambda+\mu}^S.$$

In general, the multiplication in $\mathbb{R}[X]$ has the property

$$\mathbb{R}[X]_\lambda \mathbb{R}[X]_\mu \subset \bigoplus_{\nu \leq \lambda+\mu} \mathbb{R}[X]_\nu,$$

where "$\leq$" is the ordering in the group $\boldsymbol{X}(A)$ defined by the subsemigroup generated by the simple roots. In other words, the subspaces

$$\mathbb{R}[X]_{\leq\lambda} = \bigoplus_{\mu \leq \lambda} \mathbb{R}[X]_\mu$$

constitute a $\boldsymbol{X}(A)$-filtration of the algebra $\mathbb{R}[X]$ with respect to the ordering "$\leq$".

The functions $\varphi \in \mathbb{R}[X]_\lambda$ vanishing at o constitute a K-invariant subspace of codimension 1. The K-invariant complement of it is a 1-dimensional subspace, on which K acts trivially. Let φ_λ^K denote the function of this subspace normalized by the condition $\varphi_\lambda^K(o) = 1$. It is called the *zonal spherical function of weight* λ.

Lemma 5 (see e. g. [Hel84], p. 537)**.** *For any finite-dimensional irreducible representation of G on a real vector space V there is a positive definite scalar product $(\cdot\,|\,\cdot)$ on V such that*

$$(gx \,|\, \theta(g)y) = (x \,|\, y) \quad \textit{for all } g \in G \textit{ and } x, y \in V.$$

This scalar product is unique up to a scalar multiple.

The scalar product given by Lemma 5 is called *G-skew-invariant.* Note that it is K-invariant.

With respect to the G-skew-invariant scalar product on $\mathbb{R}[X]_\lambda$ the zonal spherical function φ_λ^K is orthogonal to the subspace of functions vanishing at $o \in X$. Let α_λ denote the angle between φ_λ^K and φ_λ^S. Then the projection of φ_λ^K to $\mathbb{R}[X]_\lambda^S = \mathbb{R}\varphi_\lambda^S$ is equal to $(\cos^2 \alpha_\lambda)\varphi_\lambda^S$ (see Figure 1). In particular, we see that φ_λ^K and φ_λ^S are not orthogonal.

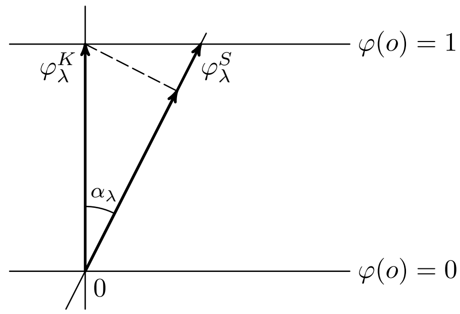

FIGURE 1.

The weight decomposition of φ^K_λ is of the form

$$\varphi^K_\lambda = (\cos^2 \alpha_\lambda)\, \varphi^S_\lambda + \sum_{\mu<\lambda} \varphi_{\lambda,\mu},$$

where $\varphi_{\lambda,\mu}$ is some weight vector of weight μ in $\mathbb{R}[X]_\lambda$. This gives the asymptotic behavior of φ^K_λ on $(\exp(-C^o))o$, where C^o is the interior of the Weyl chamber C in $\mathfrak{a}$. More precisely, for $\xi \in C^o$ we have

$$\varphi^K_\lambda((\exp(-t\xi))o) = ((\exp t\xi)\, \varphi^K_\lambda)(o) \underset{t\to+\infty}{\sim} (\cos^2 \alpha_\lambda) e^{t\lambda(\xi)}.$$

But it is known (see [Hel84], § IV.6) that the same asymptotics is described in terms of the Harish-Chandra **c**-function:

$$\varphi^K_\lambda((\exp(-t\xi))o) \underset{t\to+\infty}{\sim} \mathbf{c}(\lambda+\rho) e^{t\lambda(\xi)},$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta+} m_\alpha \alpha$ is the half-sum of positive roots. This shows that

$$\cos^2 \alpha_\lambda = \mathbf{c}(\lambda+\rho).$$

5. HOROSPHERES

Definition 6. A *horosphere* in X is an orbit of a maximal unipotent subgroup of G.

Since all maximal unipotent subgroups are conjugate to U, any horosphere is of the form gUx ($g \in G$, $x \in X$). Moreover, since $X = Po$ and P normalizes U, any horosphere can be represented in the form gUo ($g \in G$). In other words, the set $\mathrm{Hor}\, X$ of all horospheres is a homogeneous space of G. The following lemma shows that the G-set $\mathrm{Hor}\, X$ is identified with G/S if we take the horosphere Uo as the base point for $\mathrm{Hor}\, X$.

Lemma 7 (see also [Hel94], Theorem 1.1, p. 77)**.** *The stabilizer of the horosphere Uo is the algebraic subgroup*

$$S = UM = U \rtimes M.$$

Proof. Obviously, S stabilizes Uo. Hence the stabilizer of Uo can be written as $\widetilde{S} = U\widetilde{M}$, where

$$M \subset \widetilde{M} \subset K.$$

Since U is a maximal unipotent subgroup in G (and hence in $\widetilde{S}$), it contains the unipotent radical $\widetilde{U}$ of $\widetilde{S}$. The reductive group $\widetilde{S}/\widetilde{U}$ can be decomposed as

$$\widetilde{S}/\widetilde{U} = (U/\widetilde{U})\widetilde{M},$$

so the manifold $(\widetilde{S}/\widetilde{U})/(U/\widetilde{U})$ is compact. But then the Iwasawa decomposition for $\widetilde{S}/\widetilde{U}$ shows that the real rank of $\widetilde{S}/\widetilde{U}$ equals 0, that is $\widetilde{S}/\widetilde{U}$ is compact. This is possible only if $U = \widetilde{U}$. Hence,

$$\widetilde{S} \subset N(U) = P = S \rtimes A.$$

It follows from the Iwasawa decomposition $G = AUK$ that $\widetilde{S} \cap A = \{e\}$. Thus $\widetilde{S} = S$. $\square$

It follows from [VP72] that the G-module structure of $\mathbb{R}[\operatorname{Hor} X]$ is exactly the same as that of $\mathbb{R}[X]$, but in contrast to the case of $\mathbb{R}[X]$, the decomposition of $\mathbb{R}[\operatorname{Hor} X]$ into the sum of irreducible components is a $\boldsymbol{X}(A)$-grading.

Let $\mathbb{R}[\operatorname{Hor} X]_\lambda$ be the irreducible component of $\mathbb{R}[\operatorname{Hor} X]$ with highest weight λ, so

$$\mathbb{R}[\operatorname{Hor} X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[\operatorname{Hor} X]_\lambda.$$

Denote by ψ_λ^S and ψ_λ^K the highest weight function and the K-invariant function in $\mathbb{R}[\operatorname{Hor} X]_\lambda$ normalized by

$$\psi_\lambda^S(sUo) = \psi_\lambda^K(sUo) = 1.$$

To see that this is possible, note that the horosphere sUo is stabilized by $sSs^{-1} = \theta(S)$. Hence the subspace V_0 of functions in $\mathbb{R}[\operatorname{Hor} X]_\lambda$ vanishing at sUo is $\theta(S)$-invariant. Its orthogonal complement is therefore S-invariant and must coincide with $\mathbb{R}[\operatorname{Hor} X]_\lambda^S$. This implies $\psi_\lambda^S(sUo) \neq 0$, so we can normalize ψ_λ^S as asserted. Since $\mathbb{R}[\operatorname{Hor} X]_\lambda^K$ and $\mathbb{R}[\operatorname{Hor} X]_\lambda^S$ are not orthogonal in $\mathbb{R}[\operatorname{Hor} X]_\lambda \cong \mathbb{R}[X]_\lambda$, we have $\mathbb{R}[\operatorname{Hor} X]_\lambda^K \cap V_0 = \{0\}$ and we can normalize also ψ_λ^K as asserted. Notice that the horospheres passing through o form a single K-orbit and therefore ψ_λ^K takes the value 1 at each of them.

Consider the $\boldsymbol{X}(A)$-graded algebra $\operatorname{gr} \mathbb{R}[X]$ associated with the $\boldsymbol{X}(A)$-filtration of $\mathbb{R}[X]$ defined in Section 4. As a G-module, $\operatorname{gr} \mathbb{R}[X]$ can be identified with $\mathbb{R}[X]$, but when we multiply elements $\varphi_\lambda \in \mathbb{R}[X]_\lambda$ and $\varphi_\mu \in \mathbb{R}[X]_\mu$ in $\operatorname{gr} \mathbb{R}[X]$, only the highest term in their product in $\mathbb{R}[X]$ survives. Moreover, there is a unique G-equivariant linear isomorphism

$$\Gamma \colon \mathbb{R}[X] = \operatorname{gr} \mathbb{R}[X] \to \mathbb{R}[\operatorname{Hor} X]$$

such that $\Gamma(\varphi_\lambda^S) = \psi_\lambda^S$.

Proposition 8. *The map Γ is an isomorphism of the algebra $\operatorname{gr} \mathbb{R}[X]$ onto the algebra $\mathbb{R}[\operatorname{Hor} X]$.*

Proof. For any semisimple complex algebraic group, the tensor product of the irreducible representations with highest weights λ and μ contains a unique irreducible component with highest weight $\lambda+\mu$. It follows that, if we identify the irreducible components of $\mathbb{R}[X]$ with the corresponding irreducible components of $\mathbb{R}[\operatorname{Hor} X]$ via Γ, then the product of functions $\varphi_\lambda \in \mathbb{R}[X]_\lambda$ and $\varphi_\mu \in \mathbb{R}[X]_\mu$ in $\operatorname{gr}\mathbb{R}[X]$ will differ from their product in $\mathbb{R}[\operatorname{Hor} X]$ by some factor $a_{\lambda\mu}$ depending only on λ and μ. Taking $\varphi_\lambda = \varphi_\lambda^S$ and $\varphi_\mu = \varphi_\mu^S$, we conclude that $a_{\lambda\mu} = 1$. □

Remark 9. The definition of Γ makes use of the choice of the base point o in X and of the maximal unipotent subgroup U of G, but it is easy to see that all such pairs (o, U) are G-equivalent. It follows that Γ is in fact canonically defined.

6. Proof of the Main Theorem

Consider the double fibration

$$\begin{array}{ccc} & G/M & \\ {}^{p}\swarrow & & \searrow^{q} \\ X = G/K & & G/S = \operatorname{Hor} X \end{array}$$

Since all the groups involved are unimodular, there are invariant measures on the homogeneous spaces X, $\operatorname{Hor} X$, G/M and on the fibers of p and q, which are the images under the action of G of K/M and S/M, respectively. Let us normalize these measures so that:

(1) the volume of K/M is 1;
(2) the measure on G/M is the product of the measures on K/M and X;
(3) the measure on G/M is the product of the measures on S/M and $\operatorname{Hor} X$.

(This leaves two free parameters.)

Consider the dual horospherical Radon transform

$$\mathcal{R}^* : \mathbb{R}[\operatorname{Hor} X] \to \mathbb{R}[X].$$

Combining it with the map Γ defined in Section 5, we obtain a G-equivariant linear isomorphism

$$\mathcal{R}^* \circ \Gamma : \mathbb{R}[X] \to \mathbb{R}[X].$$

In view of absolute irreducibility, Schur's lemma shows that $\mathcal{R}^* \circ \Gamma$ acts on each $\mathbb{R}[X]_\lambda$ by scalar multiplication. The scalars are given by the following theorem:

Theorem 10. *For* $\varphi \in \mathbb{R}[X]_\lambda$,

$$(\mathcal{R}^* \circ \Gamma)(\varphi) = \mathbf{c}(\lambda+\rho)\,\varphi,$$

where $\mathbf{c}$ *is the Harish-Chandra* $\mathbf{c}$*-function.*

Proof. We test the map at the zonal spherical function $\varphi_\lambda^K \in \mathbb{R}[X]_\lambda$. The map Γ takes it to $c_\lambda \psi_\lambda^K$ for some $c_\lambda \in \mathbb{R}$. Since the function ψ_λ^K has value 1 at the horospheres passing through o, the map $\mathcal{R}^*$ takes it to φ_λ^K. Thus we have

$$(\mathcal{R}^* \circ \Gamma)(\varphi_\lambda^K) = c_\lambda \varphi_\lambda^K.$$

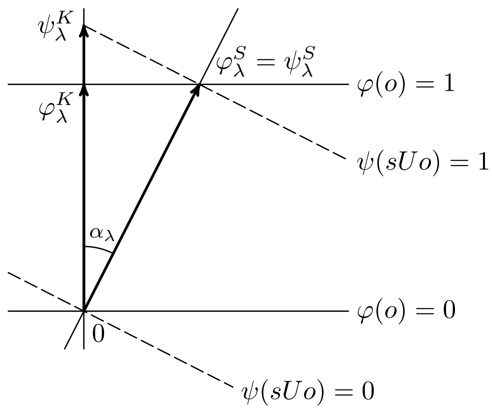

FIGURE 2.

Identifying $\mathbb{R}[X]_\lambda$ and $\mathbb{R}[\mathrm{Hor}\, X]_\lambda$ via Γ, we now find $c_\lambda = \cos^2 \alpha_\lambda$ (see Figure 2), and this proves the claim. □

APPENDIX A. THE c-FUNCTION

Because of the Iwasawa decomposition $G = KAU$, every $g \in G$ can be written as $g = k \exp H(g) u$ for a uniquely determined $H(g) \in \mathfrak{a}$. Let $\bar{U} := \theta(U)$, and let $d\bar{u}$ denote the invariant measure on $\bar{U}$ normalized by the condition

$$\int_{\bar{U}} e^{-2\rho(H(\bar{u}))}\, d\bar{u} = 1.$$

The **c**-function was defined by Harish-Chandra [HC58] as the integral

$$\mathbf{c}(\lambda) := \int_{\bar{U}} e^{-(\lambda+\rho)(H(\bar{u}))}\, d\bar{u},$$

which absolutely converges for all $\lambda \in \mathfrak{a}(\mathbb{C})^*$ satisfying $\mathrm{Re}(\lambda, \alpha) > 0$ for all $\alpha \in \Delta^+$. The computation of the integral gives the so-called Gindikin–Karpelevich product formula [GK62] (see also [Hel84], Section IV.6.4, or [GV88], p. 179):

$$\mathbf{c}(\lambda) = \kappa \prod_{\alpha \in \Delta^{++}} \frac{2^{-\lambda_\alpha}\, \Gamma(\lambda_\alpha)}{\Gamma\left(\frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}\right)}, \tag{5}$$

where Δ^{++} denotes the set of indivisible roots in Δ^+, $\lambda_\alpha := (\lambda, \alpha)/(\alpha, \alpha)$, and the constant κ is chosen so that $\mathbf{c}(\rho) = 1$. This formula provides the explicit meromorphic continuation of $\mathbf{c}$ to the entire space $\mathfrak{a}(\mathbb{C})^*$.

Formula (5) simplifies in the case of a reduced root system (i. e., when $\Delta^{++} = \Delta^+$), since the duplication formula

$$\Gamma(2z) = 2^{2z-1} \sqrt{\pi}\, \Gamma(z) \Gamma(z + 1/2) \tag{6}$$

for the gamma function yields

$$\mathbf{c}(\lambda) = \kappa \prod_{\alpha\in\Delta^+} \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha + m_\alpha/2)}, \quad \text{with} \quad \kappa = \prod_{\alpha\in\Delta^+} \frac{\Gamma(\rho_\alpha + m_\alpha/2)}{\Gamma(\rho_\alpha)}. \tag{7}$$

If, moreover, all the multiplicities m_α are even (which is equivalent to the property that all Cartan subalgebras of $\mathfrak{g}$ are conjugate), then the functional equation $z\Gamma(z) = \Gamma(z+1)$ implies

$$\mathbf{c}(\lambda) = \prod_{\alpha\in\Delta^+} \frac{\rho_\alpha(\rho_\alpha+1)\dots(\rho_\alpha+m_\alpha/2-1)}{\lambda_\alpha(\lambda_\alpha+1)\dots(\lambda_\alpha+m_\alpha/2-1)}.$$

Finally, suppose that the group G admits a complex structure. In this case the root system is reduced and $m_\alpha = 2$ for every root α, and (7) reduces to

$$\mathbf{c}(\lambda) = \prod_{\alpha\in\Delta^+} \frac{\rho_\alpha}{\lambda_\alpha}.$$

Example 11. For n-dimensional Lobachevsky space, there is only one positive root α with $m_\alpha = n-1$, so $\rho = (n-1)\alpha/2$. Let $\lambda = l\alpha$. Then formula (7) gives

$$\mathbf{c}(\lambda) = \frac{\Gamma(n-1)\Gamma(l)}{\Gamma(\frac{n-1}{2})\Gamma(l+\frac{n-1}{2})}.$$

The semigroup Λ is generated by α. For $\lambda = l\alpha$ ($l \in \mathbb{N}$), we obtain

$$c_\lambda = \mathbf{c}(\lambda+\rho) = \frac{\Gamma(n-1)\Gamma(l+\frac{n-1}{2})}{\Gamma(\frac{n-1}{2})\Gamma(l+n-1)} = \begin{cases} \dfrac{(n+l-1)(n+l)\cdots(n+2l-3)}{2^{2l-1}\,\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+l-2)}, & n \text{ even}, \\ \dfrac{(\frac{n-1}{2}+1)(\frac{n-1}{2}+2)\cdots(\frac{n-1}{2}+l-1)}{2n(n+1)\cdots(n+l-2)}, & n \text{ odd}. \end{cases}$$

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