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CONFIGURATIONS OF BALLS IN EUCLIDEAN SPACE THAT BROWNIAN MOTION CANNOT AVOID

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Abstract. We consider a collection of balls in Euclidean space and the problem of determining if Brownian motion has a positive probability of avoiding all the balls indefinitely.

1. Introduction

We consider a region Ω that is formed by removing a countable collection of *non-overlapping* closed balls from $\mathbf{R}^d$. We write $B(c,r)$ for the open ball in $\mathbf{R}^d$ with centre c and radius r; we write $\overline{B}(c,r)$ for the closed ball and $S(c,r)$ for the sphere of the same centre and radius. Then

$$\Omega = \mathbf{R}^d \setminus \bigcup_{n=1}^{\infty} \overline{B}(c_n, r_n),$$

and we assume, for convenience, that 0 lies in Ω. We say that such a collection of balls is *avoidable* if there is a positive probability that Brownian motion in $\mathbf{R}^d$, starting from 0, never hits any of the balls. Thus the collection of balls is avoidable if the balls do not have full harmonic measure with respect to the domain Ω, or if infinity has positive harmonic measure with respect to Ω. We address the problem of obtaining a geometric characterization of avoidable configurations of balls.

The genesis of this problem is to be found in the paper of Ortega-Cerdà and Seip [2]. Motivated by a question of Akeroyd [1], the analogous problem in the setting of the unit disk was solved when the centres of the removed disks form a regular 'uniformly dense sequence'.

In the plane, a single disk hides infinity from the origin. This reflects the fact that Brownian motion in the plane is recurrent and that the sphere $S(c,r)$ has full harmonic measure with respect to $\mathbf{R}^2 \setminus \overline{B}(c,r)$. For this reason, our results are set in Euclidean space of dimension three or more, in which Brownian motion is transient. It is helpful to bear in mind that, in dimension three or more, $(r/|c|)^{d-2}$ is the harmonic measure at 0 of the sphere $S(c,r)$ with respect to the domain $\Omega =$

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$\mathbf{R}^d \setminus \overline{B}(c,r)$. In fact, the harmonic measure of this sphere at x is $u(x) = (r/|x-c|)^{d-2}$. We begin with a straightforward result that has a Borel–Cantelli feel to it.

Proposition 1. *We suppose that $d \geq 3$. If*

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{|c_n|} \right)^{d-2} < \infty, \tag{1.1}$$

then the collection of balls $\{\overline{B}(c_n, r_n)\}_{n\geq 1}$ is avoidable.

In order to exclude situations in which a number of small balls packed very close together can contribute significantly to the sum in (1.1) but contribute relatively little to the overall harmonic measure, we now require a separation condition on the balls:

(S) there is a positive number ε such that $|c_n - c_m| \geq 2\varepsilon$ for $n \neq m$.

Theorem 1. *We suppose that $d \geq 3$. We assume the separation condition* (S) *and that there is a number M such that*

$$r_n^{d-2} |c_n|^2 \leq M \text{ for } n \geq 1. \tag{1.2}$$

If the collection of balls $\{\overline{B}(c_n, r_n)\}_{n\geq 1}$ is avoidable, then

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{|c_n|} \right)^{d-2} < \infty. \tag{1.3}$$

The solid angle subtended by the sphere $S(c,r)$ at 0 is comparable to $(r/|c|)^{d-1}$. The appropriate translation of Akeroyd's question to the present setting is whether there is an unavoidable sequence of balls for which the sum $\sum_n (r_n/|c_n|)^{d-1}$ is finite. If so, it is possible to hide infinity from the origin from the point of view of harmonic measure even though geometrically there is a clear line of sight to infinity except for a set of directions on the sphere S^d of arbitrarily small $(d-1)$-dimensional measure. Consider m^{d-1} balls of radius ρ_m, with $\rho_m^{d-2} = 1/m^2$, arranged evenly on the sphere $S(0,m)$, this for each integer m greater than some large m_0. These balls will be non-intersecting and separated, and (1.2) will hold since $\rho_m^{d-2} m^2 = 1$. But (1.3) does not hold: in fact

$$\sum_n \left(\frac{r_n}{|c_n|} \right)^{d-2} = \sum_{m=m_0}^{\infty} m^{d-1} \left(\frac{\rho_m^{d-2}}{m^{d-2}} \right) = \sum_{m=m_0}^{\infty} \frac{1}{m}.$$

By Theorem 1, the collection of balls is unavoidable. Yet, $\sum_n (r_n/|c_n|)^{d-1}$ is finite and can be made arbitrarily small by increasing m_0.

We will now consider a more regular configuration of balls. We say that the balls are *regularly located* if (i) the separation condition (S) is satisfied, (ii) the balls are uniformly dense, in that there is a positive R such that any ball $B(x,R)$ contains at least one centre c_n, (iii) the radius of any ball depends only on the distance from the ball's centre to the origin, with $r_n = \phi(|c_n|)$ where ϕ is a decreasing positive function.

Theorem 2. *We suppose that $d \geq 3$ and that the balls $B(c_n, r_n)$, $n \geq 1$, are regularly located. Then the collection of balls is avoidable if and only if*

$$\int^{\infty} r\phi(r)^{d-2}\, dr < \infty.$$

Theorem 1 is a partial converse to Proposition 1 in that if the radii of the balls decrease sufficiently rapidly then the collection of balls is avoidable only if (1.1) holds. Theorem 2 will be proved by showing that condition (1.2) is automatically satisfied if the collection of balls is both regularly located and avoidable. Hence these results do not give rise to a configuration of separated balls that is both avoidable and for which $\sum \left(r_n/|c_n|\right)^{d-2}$ is divergent: in fact, the possibility that condition (1.2) in Theorem 1 is redundant has not been ruled out as yet. We address this gap in our final result.

Theorem 3. *Suppose that f is any increasing unbounded function on $[0, \infty)$. Then there is a separated and avoidable collection of balls $\overline{B}(c_n, r_n)$, $n \geq 1$, for which*

$$r_n^{d-2}|c_n|^2 \leq f(|c_n|) \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{r_n}{|c_n|}\right)^{d-2} = \infty.$$

We will write $\omega(x, E; D)$ to denote the harmonic measure at x of a Borel set E on the boundary of a region D with respect to D.

2. Proof of Proposition 1

We suppose that (1.1) holds and choose N so large that

$$\sum_{n=N+1}^{\infty} \left(\frac{r_n}{|c_n|}\right)^{d-2} < \frac{1}{2}.$$

We write Ω_N for $\mathbf{R}^d \setminus \bigcup_{n>N} \overline{B}(c_n, r_n)$. For $n > N$, the harmonic measure of the sphere $S(c_n, r_n)$ at 0 with respect to Ω_N is less than its harmonic measure with respect to the larger domain $\mathbf{R}^d \setminus \overline{B}(c_n, r_n)$, which is $(r_n/|c_n|)^{d-2}$. Thus the combined harmonic measure at 0 with respect to Ω_N of the spheres $S(c_n, r_n)$, $n > N$, is at most $1/2$. As a consequence, Brownian motion in $\mathbf{R}^d$ starting from 0 has a positive probability (at least $1/2$) of avoiding the set $E = \bigcup_{n>N} S(c_n, r_n)$ indefinitely.

We write $u(x)$ for the harmonic measure $\omega(x, E; \Omega_N)$, so that $u(0) < 1/2$. The set of points x in Ω_N at which $u(x) < 1/2$ is unbounded. In fact, suppose that it was the case that $u \geq 1/2$ on $S(0, R) \cap \Omega_N$. We could then apply the maximum principle to the harmonic function u in $B(0, R) \cap \Omega_N$, noting that $u = 1$ on the boundary of Ω_N inside $B(0, R)$ (that is, on $E \cap B(0, R)$), and deduce that $u \geq 1/2$ in $B(0, R) \cap \Omega_N$.

We now write F for the bounded set $\bigcup_{n\le N} S(c_n, r_n)$, and choose R so that $F \subset B(0,R)$. Then, for $|x| > R$,

$$\omega(x, F; \Omega) \le \omega\Big(x, S(0,R); \mathbf{R}^d \setminus \overline{B}(0,R)\Big) = \left(\frac{R}{|x|}\right)^{d-2}.$$

It follows that, as $|x| \to \infty$ in Ω, the harmonic measure $\omega(x, F; \Omega)$ tends to 0. Thus we may be sure that there is a point x_0 in Ω for which both $\omega(x_0, F; \Omega) < 1/2$ and $\omega(x_0, E; \Omega) \le \omega(x_0, E; \Omega_N) < 1/2$. The finite boundary of Ω, that is $E \cup F = \bigcup_{n=1}^{\infty} S(c_n, r_n)$, does not have full harmonic measure at x_0. By the maximum principle, it does not have full harmonic measure at 0 either, and so the balls $\overline{B}(c_n, r_n)$, $n \ge 1$, do not hide infinity from the origin.

3. Proof of Theorem 1

Let us suppose that (1.2) holds and that $\sum_{n=1}^{\infty} \big(r_n/|c_n|\big)^{d-2}$ is divergent. We wish to show that Brownian motion starting from 0 will never escape to infinity in Ω.

We set $I_m = \{n \in \mathbf{N} \colon \varepsilon 2^{m-1} < |c_n| \le \varepsilon 2^m\}$. We note that there is a k_0 between 1 and 4 inclusive for which

$$\sum_{j=0}^{\infty} \sum_{n \in I_{k_0+4j}} \left(\frac{r_n}{|c_n|}\right)^{d-2} = \infty.$$

We ignore all balls whose index does not lie in I_{k_0+4j} for some j: with fewer balls to avoid, it is easier for Brownian motion starting from 0 to escape to infinity in this new domain Ω. The balls that remain lie more or less in annuli whose inner radius is half the outer radius but arranged so that the annuli are far apart, in that the inner radius of each annulus is 16 times that of the previous annulus.

Following the argument of Ortega-Cerdà and Seip [2, p. 909], we write m_j for k_0+4j, R_j for $\varepsilon 2^{m_j-1}$, S_j for $S(0,R_j)$ and set P_j to be the probability that Brownian motion in Ω starting from 0 hits $S_j \cap \Omega$. We need to show that $P_j \to 0$ as $j \to \infty$.

We let Q_j be the supremum of the probabilities that Brownian motion with starting point on $S_j \cap \Omega$ hits $S_{j+1} \cap \Omega$. Then

$$P_{j+1} \le Q_j\, P_j$$

and so

$$P_{n+1} \le P_1 \prod_{j=1}^{n} Q_j.$$

If $0 < a_j < 1$ and $\sum_{j=1}^{\infty} (1-a_j)$ is divergent, then the infinite product $\prod_{j=1}^{\infty} a_j = 0$. Theorem 1 therefore follows from the next lemma.

Lemma 1. *We set C to be $1 + 4^{d+3} M \varepsilon^{-d}$. Then, for all sufficiently large j,*

$$1 - Q_j \;\ge\; \frac{1}{2^{d-1}\, C} \sum_{n \in I_{m_j}} \left(\frac{r_n}{|c_n|}\right)^{d-2}. \tag{3.1}$$

Proof. We write

$$\Omega_j = B(0, R_{j+1}) \setminus \bigcup_{n\in I_{m_j}} \overline{B}(c_n, r_n).$$

Then $Q_j \le \hat{Q}_j$ where $\hat{Q}_j$ is the supremum of the probabilities that a Brownian motion in Ω_j with starting point on $S_j \cap \Omega_j$ hits $S_{j+1} \cap \Omega_j$. Lemma 1 may be proved, therefore, by showing that

$$\inf_{x\in S_j\cap\Omega_j} \omega\Big(x, \bigcup_{n\in I_{m_j}} S(c_n, r_n)\,; \Omega_j\Big) \ge \frac{1}{2^{d-1}\,C} \sum_{n\in I_{m_j}} \left(\frac{r_n}{|c_n|}\right)^{d-2}. \tag{3.2}$$

We consider

$$u(x) = \sum_{n\in I_{m_j}} \left(\frac{r_n}{|x - c_n|}\right)^{d-2} \quad \text{for } x \in \Omega_j,$$

so that u is harmonic in Ω_j. We suppose that $x \in S(c_m, r_m)$ for some $m \in I_{m_j}$. Then $r_m/|x - c_m| = 1$. We now show that the assumption that $r_n^{d-2}|c_n|^2 \le M$, for each n, leads to

$$\sum_{\substack{n\neq m\\ n\in I_{m_j}}} \frac{r_n^{d-2}}{|x - c_n|^{d-2}} \le 4^{d+3} M \varepsilon^{-d}. \tag{3.3}$$

By (1.2), we may assume that $r_n < \varepsilon$ for $n \in I_{m_j}$, once j is sufficiently large. The separation condition (S) implies that there are at most $4^d 2^{kd}$ balls whose centres lie at a distance of more than $2^k\varepsilon$ but less than $2^{k+1}\varepsilon$ from x, for $k \ge 1$. Each putative ball in this annulus contributes at most

$$\frac{M}{|c_n|^2}\frac{1}{|x - c_n|^{d-2}} \le \frac{M}{R_j^2}\,\frac{1}{(2^k\varepsilon)^{d-2}} = \frac{M4^k}{\varepsilon^{d-2}}\,\frac{1}{R_j^2\,2^{kd}}$$

to the sum in (3.3). Since $m_j + 1$ annuli centred at x will cover all balls $B(c_n, r_n)$ with $n \in I_{m_j}$, we find that

$$\sum_{\substack{n\neq m\\ n\in I_{m_j}}} \frac{r_n^{d-2}}{|x - c_n|^{d-2}} \le \sum_{k=1}^{m_j+1} 4^d\, 2^{kd}\, \frac{M4^k}{\varepsilon^{d-2}}\,\frac{1}{R_j^2\,2^{kd}} = \frac{4^d\, M}{R_j^2\,\varepsilon^{d-2}} \sum_{k=1}^{m_j+1} 4^k \le \frac{4^{d+2}M}{R_j^2\,\varepsilon^{d-2}}\, 4^{m_j}.$$

Since $R_j^2 = \varepsilon^2 4^{m_j-1}$, the estimate (3.3) follows. We have shown that the harmonic function u satisfies

$$u(x) \le 1 + 4^{d+3} M\varepsilon^{-d} = C \quad \text{for} \quad x \in \bigcup_{n\in I_{m_j}} S(c_n, r_n). \tag{3.4}$$

We now need an estimate for the size of u on the sphere S_{j+1}. If $|c_n| \le 2R_j$ and $|x| = R_{j+1}$, then

$$|x - c_n| \ge R_{j+1} - 2R_j = (16 - 2)R_j \ge 2^3 R_j.$$

Thus, for $x \in S_{j+1}$,

$$
\begin{aligned}
u(x) &= \sum_{n\in I_{m_j}} \left(\frac{r_n}{|x-c_n|}\right)^{d-2} \le \frac{1}{2^{3(d-2)}} \sum_{n\in I_{m_j}} \left(\frac{r_n}{R_j}\right)^{d-2} \\
&\le \frac{1}{4^{d-2}} \sum_{n\in I_{m_j}} \left(\frac{r_n}{|c_n|}\right)^{d-2} \le \frac{1}{2^{d-1}} \sum_{n\in I_{m_j}} \left(\frac{r_n}{|c_n|}\right)^{d-2}.
\end{aligned} \tag{3.5}
$$

It follows from (3.4), (3.5) and the maximum principle that, for $x \in \Omega_j$,

$$C\,\omega\Bigl(x, \bigcup_{n\in I_{m_j}} S(c_n, r_n); \Omega_j\Bigr) \ge u(x) - \frac{1}{2^{d-1}} \sum_{n\in I_{m_j}} \left(\frac{r_n}{|c_n|}\right)^{d-2}.$$

Finally, we use this inequality with $x \in S_j \cap \Omega_j$. For such x, we have $|x-c_n| \le |x| + |c_n| \le 2|c_n|$, and so

$$u(x) \ge \frac{1}{2^{d-2}} \sum_{n\in I_{m_j}} \left(\frac{r_n}{|c_n|}\right)^{d-2}$$

The estimate (3.2) follows immediately. This completes the proof of the lemma, and hence of Theorem 1. □

Remark. If the centres of the balls lie on a $(d-1)$-dimensional hyperplane, then the conclusion of Theorem 1 still holds with the assumption (1.2) replaced by the weaker assumption $r_n^{d-2}|c_n| \le M$. Working through the proof of Lemma 1, it is still possible to conclude that u is bounded on the boundary of the balls with index in I_{m_j} by a constant that is independent of j, as in (3.4). (In fact, there are at most $4^{d-1}2^{k(d-1)}$ balls 'whose centres lie at a distance of more than $2^k\varepsilon$ but less than $2^{k+1}\varepsilon$ from x, for $k \ge 1$'.) The remainder of the proof of Lemma 1 is as before.

If the centres of the balls lie on a $(d-2)$-dimensional hyperplane then one may replace (1.2) by the weaker assumption $r_n^{d-2}\log|c_n| \le M$ and still retain the conclusion of Theorem 1. For example, suppose that in $\mathbf{R}^3$ we put a ball of radius r_n at the point $(n,0,0)$, for $n \ge 2$. Under the assumption that $r_n \le M/\log n$, this string of beads in $\mathbf{R}^3$ is avoidable if and only if $\sum r_n/n$ is finite.

If the centres of the balls lie on a $(d-3)$-dimensional hyperplane, then it suffices to assume that the radii of the balls are uniformly bounded in order for Theorem 1 to hold. In this case, however, there can be at most about m^{d-4} balls whose distance from the origin is about m. Assuming that the radii of the balls are bounded by R, say, it follows that

$$\sum_n \left(\frac{r_n}{|c_n|}\right)^{d-2} \le R^{d-2} \sum_n \frac{1}{|c_n|^{d-2}} \le CR^{d-2} \sum_m m^{d-4} \frac{1}{m^{d-2}}$$

which is finite. A collection of balls of uniformly bounded radius whose centres lie on a $(d-3)$-dimensional hyperplane will always be avoidable.

4. Proof of Theorem 2

To begin with we note that, in the case of a regularly located configuration of balls, the sum $\sum_n (r_n/|c_n|)^{d-2}$ and the integral $\int^\infty r\phi(r)^{d-2}\,dr$ are comparable. The implication that the balls are avoidable if $\int^\infty r\phi(r)^{d-2}\,dr$ is finite is now an immediate consequence of Proposition 1.

The reverse implication will follow from Theorem 1 once we check that the condition (1.2) is automatically satisfied under the regularity assumption if the balls are avoidable. We establish this in the next lemma, whose proof bears a certain resemblance to that of Lemma 1.

Lemma 2. *Suppose that the balls $\{\overline{B}(c_n, r_n))\}_{n\geq 1}$ are regularly located and that $r^2\phi(r)^{d-2}$ is an unbounded function of r. Then the collection of balls is unavoidable.*

Proof. There is a sequence of radii $\{R_j\}_{j=1}^\infty$ for which $R_j^2\phi(2R_j)^{d-2} \to \infty$ as $j \to \infty$. We put $C = A_2/2A_3$ where the particular numbers A_2 and A_3 that we need depend on the dimension, on the separation number ε and on the density number R but on nothing else, and may be worked out in principle from the proof that follows. We assume that $R_{j+1} > 4R_j$ and that $(R_j/R_{j+1})^{d-2} \leq C$ for each j. For a technical reason, we change the definition of ϕ in the following way: we set $\widetilde{\phi}(x) = \phi(2R_j)$ if $x \in [R_j, 2R_j]$ for some j and $\widetilde{\phi}(x) = \phi(x)$ elsewhere. We take new balls $B(c_n, \widetilde{\phi}(|c_n|))$. The size of the balls is thereby decreased: thus if the new balls are unavoidable then the original ones are also unavoidable. For the sake of simplicity, we will still denote by ϕ the regularized $\widetilde{\phi}$ and the new smaller balls will still be called $B(c_n, r_n)$. We write S_j for the sphere $S(0, R_j)$ and $\phi_j = \phi(R_j) = \phi(2R_j)$.

Arguing as in the proof of Theorem 1, we let Q_j be the supremum of the probabilities that Brownian motion in Ω with starting point on $S_j \cap \Omega$ hits $S_{j+1} \cap \Omega$, and wish to show that $\prod_{j=1}^\infty Q_j = 0$, that is that

$$\sum_{j=1}^\infty (1 - Q_j) = \infty.$$

We write I_j for $\{n : R_j \leq |c_n| \leq 2R_j\}$, and write

$$\Omega_j \;=\; B(0, R_{j+1}) \setminus \bigcup_{n\in I_j} \overline{B}(c_n, r_n).$$

Then Q_j is bounded above by $\hat{Q}_j$, the supremum of the probabilities that Brownian motion with starting point on $S_j \cap \Omega_j$ hits $S_{j+1} \cap \Omega_j$. We will show that, for all sufficiently large j,

$$1 - \hat{Q}_j = \inf_{x\in S_j\cap\Omega_j} \omega\Big(x, \bigcup_{n\in I_j} S(c_n, r_n); \Omega_j\Big) \geq \delta, \tag{4.1}$$

for some positive δ. Again we consider

$$u(x) = \sum_{n\in I_j} \left(\frac{r_n}{|x-c_n|}\right)^{d-2}, \quad x \in \Omega_j,$$

so that u is harmonic in Ω_j. Since ϕ is constant on $[R_j, 2R_j]$, we have $r_n = \phi(R_j) = \phi_j$ for $n \in I_j$ and

$$u(x) = \phi_j^{d-2} \sum_{n\in I_j} \frac{1}{|x-c_n|^{d-2}}.$$

Suppose that x lies on the boundary of a ball $S(c_m, r_m)$ with $m \in I_j$. It is a consequence of the separation condition that there can be at most $A2^{kd}$ balls with centre at a distance that is between $\varepsilon 2^{k-1}$ and $\varepsilon 2^k$ from x, with $k \geq 1$. Each such ball contributes at most $A2^{-k(d-2)}$ to the above sum, making for a combined contribution of at most $A2^{2k}$. We need only count those k with $\varepsilon 2^k \leq 6R_j$, as there are no balls under consideration that are more distant than $6R_j$ from x. The ball $B(c_m, r_m)$ itself contributes 1 to $u(x)$, which leads to the estimate

$$u(x) \leq 1 + \phi_j^{d-2} \sum_{k:\, \varepsilon 2^k \leq 6R_j} A2^{2k} \leq 1 + AR_j^2 \phi_j^{d-2}.$$

As $R_j^2 \phi_j^{d-2} \geq 1$ for sufficiently large j,

$$u(x) \leq A_1 R_j^2 \phi_j^{d-2} \quad \text{for} \quad x \in \bigcup_{n\in I_j} S(c_n, r_n). \tag{4.2}$$

Here A_1 is some appropriate number that depends only on the dimension and on the separation number ε.

For $x \in S_j$, we have $|x - c_n| \leq 4R_j$. At this point we use the assumption that the balls are uniformly dense to deduce that

$$u(x) = \phi_j^{d-2} \sum_{n\in I_j} \frac{1}{|x-c_n|^{d-2}} \geq \phi_j^{d-2} \frac{1}{(4R_j)^{d-2}} \sum_{n\in I_j} 1 \geq \phi_j^{d-2} \frac{A_2}{R_j^{d-2}} R_j^d$$

where the number A_2 depends only on the dimension and on the number R that appears in the definition of 'regularly located'. Thus,

$$u(x) \geq A_2 R_j^2\, \phi_j^{d-2} \quad \text{for} \quad x \in S_j. \tag{4.3}$$

Finally, for x on the sphere S_{j+1} and $n \in I_j$, we have $|x - c_n| \geq R_{j+1} - 2R_j \geq R_{j+1}/2$. Hence on S_{j+1} the function u satisfies

$$u(x) \leq \phi_j^{d-2} \frac{2^{d-2}}{R_{j+1}^{d-2}} \sum_{n\in I_j} 1 \leq A_3 \phi_j^{d-2} \frac{R_j^d}{R_{j+1}^{d-2}}.$$

Since $(R_j/R_{j+1})^{d-2} \leq C$, we obtain that

$$u(x) \leq \frac{1}{2} A_2\, R_j^2 \phi_j^{d-2} \quad \text{for} \quad x \in S_{j+1}. \tag{4.4}$$

It follows from (4.2), (4.4) and the maximum principle that, for $x \in \Omega_j$,

$$A_1 R_j^2 \phi_j^{d-2}\, \omega\Big(x, \bigcup_{n\in I_j} S(c_n, r_n); \Omega_j\Big) \geq u(x) - \frac{1}{2} A_2\, R_j^2 \phi_j^{d-2}.$$

Making use of (4.3), we deduce from this last estimate that, for $x \in S_j$,

$$A_1\, \omega\Big(x, \bigcup_{n\in I_j} S(c_n, r_n); \Omega_j\Big) \geq \frac{1}{2} A_2.$$

Thus (4.1) has been proven. □

Remark. With the same proof, one may consider a slightly more general situation where one changes the metric. Assume that a function $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ satisfies the smoothness condition $\psi(y) \simeq \psi(x)$ whenever $y < x < 2y$. We say that a sequence $\{c_n\}$ is ψ-regularly located if there is a $\delta > 0$ such that the balls $B(c_n, \delta\psi(|c_n|))$ are pairwise disjoint and there is an $R > 0$ such that any ball $B(x, R\psi(x))$ contains at least one centre c_n. Assume finally that we have a sequence of disjoint balls with ψ-regularly located centres and the radii of the balls depend on the centre, $r_n = \phi(|c_n|)$, where ϕ is a decreasing positive function. Then the balls are avoidable if and only if

$$\int^{\infty} \frac{x\phi(x)^{d-2}}{\psi(x)^d}\, dx < \infty.$$

The case $\psi = 1$ is the case previously considered.

5. Construction of the examples: Proof of Theorem 3

We wish to show by examples that the assumption (1.2) in Theorem 1 is necessary. The examples are of avoidable and separated configurations of balls for which the series $\sum \big(r_n/|c_n|\big)^{d-2}$ is divergent, in which case $r_n^{d-2}|c_n|^2$ must be unbounded by Theorem 1. In Theorem 3 it is asserted that such configurations of balls are possible even with a growth restriction on $r_n^{d-2}|c_n|^2$. Leaving the growth restriction to one side for the moment, we first give the details of a plain vanilla example that illustrates the idea behind the general construction.

Proposition 2. *There is an avoidable, separated configuration of balls, $\overline{B}(c_n, r_n)$, $n \geq 1$, in $\mathbf{R}^3$ for which*

$$\sum_{n=1}^{\infty} \frac{r_n}{|c_n|} = \infty.$$

Proof. Consider a string of closed balls $\overline{B}_1$, $\overline{B}_2$, … $\overline{B}_{2k}$, each of radius 1/4 and with centres c_i on the x_1-axis at $m + i$, $i = 1, 2, \ldots 2k$. We write $H_{m,k} = \bigcup_{i=1}^{2k} \overline{B}_i$ and wish to estimate $\omega\big(0, \partial H_{m,k}; \mathbf{R}^3 \setminus H_{m,k}\big)$. We consider, as ever,

$$u(x) = \sum_{i=1}^{2k} \frac{1}{|x - c_i|}$$

Suppose that x lies on the boundary of one of the balls $\overline{B}_j$. Then $|x - c_i| \le |i-j| + 1/4 \le 2|i-j|$ for $i \ne j$, and there are at least k balls to one side or other of any one ball. It follows that

$$u(x) \ge \frac{1}{2}\sum_{i=1}^{k}\frac{1}{i} \ge \frac{1}{2}\log k, \quad x \in \partial H_{m,k}.$$

By the maximum principle,

$$\omega\big(0, \partial H_{m,k}; \mathbf{R}^3 \setminus H_{m,k}\big) \le \frac{2}{\log k}\, u(0) = \frac{2}{\log k}\sum_{i=1}^{2k}\frac{1}{m+i} \le \frac{4k}{m\log k}.$$

We construct our counterexample as follows. Let $H_n = H_{n^2, \lfloor n/\log n\rfloor}$, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x, and

$$\Omega = \mathbf{R}^3 \setminus \bigcup_{n=n_0}^{\infty} H_n.$$

Then

$$\begin{aligned}\omega(0, \partial\Omega; \Omega) &= \sum_{n=n_0}^{\infty}\omega\big(0, \partial H_n; \Omega\big) \le \sum_{n=n_0}^{\infty}\omega\big(0, \partial H_n; \mathbf{R}^3 \setminus H_n\big) \\ &\le 4\sum_{n=n_0}^{\infty}\frac{n/\log n}{n^2\log\lfloor n/\log n\rfloor} \le 8\sum_{n=n_0}^{\infty}\frac{1}{n\log^2 n}.\end{aligned}$$

Thus n_0 may be chosen to be sufficiently large so that the balls are separated and so that $\omega(0, \partial\Omega; \Omega) < 1$, in which case the balls are avoidable.

On the other hand, the contribution of each string of balls H_n to the series $\sum r_n/|c_n|$ is comparable to $1/(n\log n)$, and this sum is divergent. □

In the examples that follow the balls are arranged in clusters rather than in higher dimensional strings, though the reason the examples work is the same: the cluster of balls as a whole contributes somewhat less to the harmonic measure than if each ball was treated individually.

Proof of Theorem 3. We consider a cluster of k^d balls, each of radius r less than $1/4$, whose centres have integer coordinates and are evenly distributed in a large ball that has radius approximately k and is centred at a distance $m/2$ from the origin. We assume that $k \le m/4$ and refer to this cluster of balls as $C_{m,k,r}$. The centre of any ball in $C_{m,k,r}$ is within a distance m of the origin. We again use the function

$$u(x) = \sum_i \frac{1}{|x - c_i|^{d-2}}, \tag{5.1}$$

the c_i being the centres of the balls. If x is a point on the boundary of one of these balls and $1 \le i \le k$, there are at least $a\, i^{d-1}$ balls whose centres lie at a distance at most $2i$ from x. Here a represents a number that depends only on the dimension.

Moreover, no ball needs to be chosen twice, that is for two different values of i. We find that, for a point x on the boundary of any ball in the cluster,

$$u(x) \geq \sum_{i=1}^{k} \frac{1}{(2i)^{d-2}}\, a i^{d-1} \geq a k^2.$$

By the maximum principle,

$$\omega\big(0, \partial C_{m,k,r}; \mathbf{R}^d \setminus C_{m,k,r}\big) \leq \frac{A}{k^2}\, u(0) \leq A\, \frac{k^d}{m^{d-2}}\, \frac{1}{k^2} = A \left(\frac{k}{m}\right)^{d-2}.$$

We suppose that an increasing unbounded function f on $[0,\infty)$ is given. To each positive integer n there corresponds a choice of variable m_n for which $f(m_n) \geq n^{2d}$ and $m_n > 2m_{n-1}$. We then choose k_n to be m_n/n^2 and choose the radius r_n so that $r_n^{d-2} m_n^2 = f(m_n)$. [We assume that the function f satisfies $f(x) \leq 4^{2-d}x^2$, so that $r_n < 1/4$.] We set

$$\Omega = \mathbf{R}^d \setminus \bigcup_{n=n_0}^{\infty} C_{m_n,k_n,r_n}$$

and write $\omega_n(x)$ for the harmonic measure at x of the finite boundary of C_{m_n,k_n,r_n} with respect to the domain $\mathbf{R}^d \setminus C_{m_n,k_n,r_n}$. Then

$$\omega(0, \partial\Omega; \Omega) \leq \sum_{n=n_0}^{\infty} w_n(0) \leq \sum_{n=n_0}^{\infty} A \left(\frac{k_n}{m_n}\right)^{d-2} = A \sum_{n=n_0}^{\infty} \frac{1}{n^{2(d-2)}},$$

which we can arrange to be strictly less than 1 by taking n_0 to be sufficiently large.

The sum in (1.1) for this collection of balls is comparable to

$$\sum_{n=n_0}^{\infty} k_n^d \left(\frac{r_n}{m_n}\right)^{d-2}$$

Since $r_n^{d-2} m_n^2 = f(m_n) \geq n^{2d}$, the general term in this last sum exceeds $n^{2d}(k_n/m_n)^d$, which in turn exceeds $n^{2d}(1/n^2)^d = 1$. The sum in (1.1) is therefore divergent. □

6. Addendum: the union of two avoidable sets is avoidable

At a certain point in our research, it seemed that it might be helpful to know if the union of two avoidable collections of balls would again be avoidable. Put another way, is it possible to split an unavoidable collection of balls into two disjoint avoidable collections? Though the solution to this problem is no longer an essential ingredient in the proofs we have presented here, we cannot resist including the elegant solution to this problem found by Professor Rosay. We are grateful to him for granting us permission to include his proof in this article.

A set A is called *avoidable from* p if Brownian motion in $\mathbf{R}^d$ starting at p has a probability smaller than one of hitting A. We assume that $\mathbf{R}^d \setminus A$ is connected: then, by the maximum principle, if A is avoidable from one point it is avoidable from any other point. In this case we just say that the set A is *avoidable*. Equivalently,

A is avoidable whenever there is a positive harmonic function u in $\mathbf{R}^d \setminus A$ such that $u \equiv 1$ q.e. (that is, apart from a polar set) on the boundary of A but $\inf u = 0$.

Proposition 3. *If two avoidable sets A and B satisfy $\mathbf{R}^d \setminus (A \cup B)$ is connected then $A \cup B$ is avoidable.*

The basic lemma required to prove this proposition is the following:

Lemma 3. *If E is avoidable and u_E is the associated positive harmonic function in $\mathbf{R}^d \setminus E$, with $u_E \equiv 1$ q.e. on the boundary of E and $\inf u_E = 0$, then there is an R_0 such that for all $R \geq R_0$ the set of points*

$$S_E^R = \{x \in S(0,R) \setminus E : \ u_E(x) \leq 1/4\}$$

satisfies $|S_E^R| > \frac{3}{4}|S(0,R)|$. Here the measure indicated by $|\cdot|$ is Lebesgue area measure on $S(0,R)$.

Proof. We take a point q where $u_E(q) < 1/32$. For any R with $R > |q|$, we denote by μ_R the harmonic measure on the boundary of $B(0,R) \setminus E$ with respect to q. Then, since $u_E = 1$ q.e. on $\partial E \cap B(0,R)$, we have

$$\frac{1}{32} > \mu_R\bigl(\partial E \cap B(0,R)\bigr) + \frac{1}{4}\bigl[1 - \mu_R\bigl(\partial E \cap B(0,R)\bigr) - \mu_R(S_E^R)\bigr]$$

from which it follows that $\mu_R(S_E^R) > 7/8$. We denote by σ_R harmonic measure with base point q with respect to the ball $B(0,R)$, so that $\sigma_R \geq \mu_R$ on $S(0,R)$. Thus, $\sigma_R(S_E^R) > 7/8$ for all $R > |q|$. The harmonic measure σ_R can be given explicitly, but the key property is that as $R \to \infty$ it is more and more similar to the normalized area measure on $S(0,R)$. Thus $|S_E^R| > \frac{3}{4}|S(0,R)|$ for all large R. □

Proof of Proposition 3. For the sets A and B we take the corresponding functions u_A and u_B. We take R so that $|S_A^R| > \frac{3}{4}|S(0,R)|$ and $|S_B^R| > \frac{3}{4}|S(0,R)|$. This means that there is point p that lies in the intersection $S_A^R \cap S_B^R$. We define $u = u_A + u_B$: it is a positive and bounded harmonic function defined outside $A \cup B$. On the boundary of $A \cup B$ it satisfies $u \geq 1$ and on the other hand $u(p) \leq 1/2$. Thus $A \cup B$ is avoidable from p. Since the complement $\mathbf{R}^d \setminus (A \cup B)$ is connected, then it is avoidable from any point. □

References

[1] Akeroyd, J. R.: Champagne subregions of the unit disk whose bubbles carry harmonic measure. - Math. Ann. 323, 2002, 267–279.

[2] Ortega-Cerdà, J., and K. Seip: Harmonic measure and uniform densities. - Indiana Univ. Math. J. 53:3, 2004, 905–923.

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