# Elastodynamic Scattering From Inclusions Surrounded by Thin Interface Layers

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*The scattering of elastic waves by elastic inclusions surrounded by interface layers is a problem of interest for nondestructive evaluation of interfaces in composites. In the present paper the scattering by a single elastic inclusion is studied. The scattering problem is solved by means of the null field approach and the properties of the interface layer enters through the boundary conditions on the inclusion. Various ways of doing this have been tried, from the simpler approach of just keeping the inertia of the layer, to using a membrane type of approximation or a more sophisticated method that includes all effects to first order in the layer thickness. The results obtained by using these different methods are compared numerically and with the exact solution for a layered sphere and with some recent results for a spheroid obtained using a hybrid finite element and wave function expansion technique. The numerical results show significant dependence on parameters containing the thickness and stiffness of the interface layer.*

## 1 Introduction

Three-dimensional scattering of elastic waves by inclusions has been the subject of investigation since the turn of the last century. Early studies were concerned only with spherical inclusions. A comprehensive review of these was published by Pao and Mow (1973). Elastodynamic scattering by inclusions of arbitrary shape is not amenable to exact solutions and only in recent years it has been possible to obtain numerical and approximate asymptotic solutions for scattering by spheroidal inclusions. The approximate solutions presented by Datta and Sangster (1974), Datta (1977), Gubernatis (1979), and Willis (1980) are valid at low frequenies. On the other hand, numerical solutions valid at arbitrary frequencies have been reported by Varadan and Varadan (1979), Boström (1980), Opsal and Visscher (1985), Olsson (1985), and others. In all these reported studies it has been assumed that the inclusion is perfectly bonded with the surrounding elastic matrix.

In composites, particularly metal-matrix composites reinforced by fibers or particles, it is often the case that there is an interface layer surrounding the particles (or fibers) induced by processing conditions. The strength and fracture behavior of the composite is significantly influenced by this interface layer. It is, however, difficult to characterize this layer nondestructively. The purpose of this paper is to analyze the effect of an interface layer on the scattering cross-section of a spheroidal inclusion. It is hoped that this study will help in assessing the feasibility of determining interface characteristics by ultrasonic means. In an earlier study Mal and Bose (1974) considered the effect of thin viscous layers on long wavelength scattering by spherical inclusions. More recently, Datta et al. (1988) have studied scattering by spherical inclusions surrounded by thin elastic interface layers. In both these studies it is assumed that the tractions are continuous across the layer, whereas the displacements satisfy jump conditions that are linear in the thickness of the layer. It may be noted that the approximate boundary conditions used in these studies are based on the assumption that inertial and curvature effects are negligible.

In the present study the interface layer is modeled as a thin shell, the equations of motion of which enter the boundary conditions on the surface of the inclusion. The scattering problem is solved by the null field approach and results for some simple shell approximations are compared with one another and with the results obtained by Paskaramoorthy et al. (1988) using a finite element and eigenfunction expansion method. It is found that if all the terms of $0(h)$ are kept in the boundary conditions, then the results agree very well with the calculations based on modeling the interface layer exactly as a third phase. It is also found that the shell approximations give meaningful results at low frequencies, but tend to underestimate the scattering cross-section at moderate frequencies.

## 2 The Null Field Approach

Consider an elastic inclusion surrounded by a thin interface layer in an otherwise homogeneous elastic matrix. The matrix

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has density $\rho$, Lamé parameters, $\lambda$ and $\mu$, and wave numbers, $k_p$ and $k_s$ (the time factor $\exp(-i\omega t)$ is assumed throughout). The corresponding quantities in the inclusion and layer are $\rho_1$, $\lambda_1$, $\mu_1$, $k_{p1}$, $k_s$, and $\rho_0$, $\lambda_0$, $\mu_0$, $k_{p0}$, $k_{so}$, respectively. The thickness $h$ of the layer is assumed to be small. The surface of the inclusion is denoted $S_0$ and its outward-pointing unit normal $\hat{n}$. The outer surface of the layer is denoted $S_h$.

In the null field approach all pertinent fields are expanded in spherical vector waves (Boström, 1980; Olsson, 1985). In the matrix the regular waves are denoted $\mathrm{Re}\psi_n$ and the outgoing ones are denoted $\psi_n$, where $n$ is a quadruple index. The corresponding regular waves in the inclusion are $\mathrm{Re}\psi_n^1$. The incoming and scattered displacement fields are then expanded as

$$\mathbf{u}^{in} = \sum_{n} a_n \mathrm{Re}\psi_n \tag{1}$$

$$\mathbf{u}^{s} = \sum_{n} f_n \psi_n. \tag{2}$$

Here, $a_n$ is assumed known and $f_n$ is to be determined, i.e., the problem is to determine the transition matrix, $T_{nn'}$ that gives the linear relationship between $a_n$ and $f_n$:

$$f_n = \sum_{n'} T_{nn'} a_{n'}. \tag{3}$$

Following the usual procedure in the null field approach, the outer integral representation gives relation between $a_n, f_n$, and the surface displacement $\mathbf{u}_+$ and traction $\mathbf{t}_+$:

$$a_n = -ik_s/\mu \int_{S_h} [\mathbf{u}_+ \cdot \mathbf{t}(\psi_n) - \mathbf{t}_+ \cdot \psi_n]\, dS \tag{4}$$

$$f_n = ik_s/\mu \int_{S_h} [\mathbf{u}_+ \cdot \mathbf{t}(\mathrm{Re}\psi_n) - \mathbf{t}_+ \cdot \mathrm{Re}\psi_n]\, dS. \tag{5}$$

The index + denotes the limit from the outside and $\mathbf{t}(\cdot)$ is the traction operator. The surface field on the inclusion is expanded in the regular spherical waves within the inclusion

$$\mathbf{u}_- = \sum_{n} \alpha_n \mathrm{Re}\psi_n^1 \tag{6}$$

and the inner integral representation then gives an expansion of the surface traction on the inclusion as

$$\mathbf{t}_- = \sum_{n} \alpha_n \mathbf{t}^1(\mathrm{Re}\psi_n^1) \tag{7}$$

where $\mathbf{t}^1(\cdot)$ is the traction operator for the inclusion.

The question is now how to relate $\mathbf{u}_+$ and $\mathbf{t}_+$ on $S_h$ appearing in equations (4) and (5) to $\mathbf{u}_-$ and $\mathbf{t}_-$ on $S_o$ appearing in equations (6) and (7). One way to proceed would be to use the null field approach for a layered medium (Boström, 1980), but in the present context with a thin layer one would then violate the requirement that the circumscribing sphere of the inner surface should not intersect the outer surface (Boström, 1984). Instead the assumed smallness of $h$ is used to expand the right-hand sides of equation (4) to (5) to order $h$ and it must then be remembered that the surface, the surface fields, and the partial waves all depend on the location of the outer surface. Thus, one gets

$$a_n = ik_s/\mu \int_{S_o} [\mathbf{u}_- \cdot \mathbf{t}(\psi_n) - \mathbf{t}_- \cdot \psi_n]\, dS$$

$$-ik_s/\mu \int_{S_o} h\left[\frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{t}(\psi_n) - \frac{\partial \mathbf{t}}{\partial n} \cdot \psi_n\right] dS$$

$$-ik_s/\mu \int_{S_o} h\left[\mathbf{u}_- \cdot \frac{\partial}{\partial n}\mathbf{t}(\psi_n) - \mathbf{t}_- \cdot \frac{\partial}{\partial n}\psi_n\right] dS$$

$$-ik_s/\mu \int_{S_o} [\mathbf{u}_- \cdot \mathbf{t}(\psi_n) - \mathbf{t}_- \cdot \psi_n]\, h\frac{\partial}{\partial n} dS \tag{8}$$

and similarly for $f_n$. The first integral is what one gets if there were no interface layer, the second will reflect the properties of the layer as will be presented, and the third and fourth are geometrical terms which arise because $S_o$ and $S_h$ are different surfaces.

The normal derivatives of the displacement and traction in the second integral in equation (8) must now be expressed in terms of $\mathbf{u}_-$ and $\mathbf{t}_-$. This is done by using the properties of the interface layer and a consistent approach is to start with the three-dimensional constitutive equation and equations of motion and extract the desired quantities with the normal derivatives present. For a rotationally symmetric inclusion when the azimuthal unit vector $\hat{\Phi}$ and the unit vector $\hat{\tau} = \hat{\Phi} x \hat{n}$ are tangential to $S_o$ this leads to

$$\left(\frac{\partial \mathbf{u}}{\partial n}\right)_{\tan} = [(\mathbf{t}_-)_{\tan}/\mu_o - \hat{n} x(\nabla_s x \mathbf{u}_-)] \tag{9}$$

$$\hat{n} \cdot \frac{\partial \mathbf{u}}{\partial n} = \frac{1}{(\lambda_o + 2\mu_o)}[\hat{n} \cdot \mathbf{t}_- - \lambda_o \nabla_s \cdot \mathbf{u}_-] \tag{10}$$

and

$$\frac{\partial \mathbf{t}}{\partial n} = \mathbf{B}(\mathbf{u}_-, \mathbf{t}_-) \tag{11}$$

where

$$\mathbf{B}(\mathbf{u}, \mathbf{t}) = -[\rho_o \omega^2 \mathbf{u} + \nabla_s \cdot \sigma^s(\mathbf{u}) + \hat{n} \nabla_s \cdot \mathbf{t} + \mathbf{t}_{\tan} \nabla_s \cdot \hat{n} + \tau \cdot \mathbf{t} \frac{\partial \hat{n}}{\partial \tau} + \hat{\phi} \cdot \mathbf{t}(\hat{\phi} \cdot \nabla)\hat{n} + \nabla_s \cdot \left\{(\hat{\tau}\hat{\tau} + \hat{\phi}\hat{\phi})\frac{\lambda_o}{\lambda_o + 2\mu_o}(\hat{n} \cdot \hat{t} - \lambda_o \nabla_s \cdot \mathbf{u})\right\}. \tag{12}$$

The tangential stress tensor in the layer is

$$\sigma^s_{\tau\tau}(\mathbf{u}) = \lambda_o \nabla_s \cdot \mathbf{u} + 2\mu_o \hat{\tau} \cdot \frac{\partial \mathbf{u}}{\partial \tau} \tag{13}$$

$$\sigma^s_{\phi\phi}(\mathbf{u}) = \lambda_o \nabla_s \cdot \mathbf{u} + 2\mu_o \hat{\phi} \cdot (\hat{\phi} \cdot \nabla)\mathbf{u} \tag{14}$$

$$\sigma^s_{\tau\phi}(\mathbf{u}) = \mu_o \left[\hat{\phi} \cdot \frac{\partial \mathbf{u}}{\partial \tau} + \hat{\tau} \cdot (\hat{\phi} \cdot \nabla)\mathbf{u}\right] \tag{15}$$

In the foregoing formulas, $\nabla_s \cdot$ and $\nabla_s x$ are the surface divergence and curl, respectively. It should be noted that earlier investigations of problems of the present type (see Datta et al. (1988) for a recent example) have usually assumed the traction to be continuous across the layer; i.e., it has been assumed that $\mathbf{B} = 0$. Moreover, curvature effects have usually been neglected so that the last terms in equations (9) and (10) are dropped.

A straightforward collection of all the previous formulas now determines the transition matrix as

$$T_{nn'} = -\sum_{n''} \mathrm{Re}Q_{nn''}(Q^{-1})_{n''n'} \tag{16}$$

with

$$Q_{nn'} = k_s/\mu \int_{S_o} [\mathbf{t}(\psi_n) \cdot \mathrm{Re}\psi_{n'} - \psi_n \cdot \mathbf{t}^1(\mathrm{Re}\psi^1_{n'})]\, dS$$

$$+ k_s/\mu \int_{S_o} h[\mathbf{t}(\psi_n) \cdot \mathbf{B}(\mathrm{Re}\psi^1_n, \mathbf{t}^1(\mathrm{Re}\psi^1_{n'}))$$

$$-\psi_n \bullet \mathbf{C}(\mathrm{Re}\psi^1_{n'}, \mathbf{t}^1(\mathrm{Re}\psi^1_{n'}))]\,dS$$

$$+k_s/\mu \int_{S_o} h[\mathrm{Re}\psi^1_{n'} \bullet \frac{\partial}{\partial n}\mathbf{t}(\psi_n)$$

$$-\mathbf{t}^1(\mathrm{Re}\psi^1_{n'}) \bullet \frac{\partial}{\partial n}\psi_n]\,dS + k_s/\mu \int_{S_o} [\mathbf{t}(\psi_n) \bullet \mathrm{Re}\psi^1_{n'}$$

$$-\psi_n \bullet \mathbf{t}^1(\mathrm{Re}\psi^1_{n'})]\,h\frac{\partial}{\partial n}dS \qquad (17)$$

where

$$\mathbf{C}(\mathbf{u},\mathbf{t}) = \frac{\hat{n}}{\lambda_o + 2\mu_o}(\hat{n}\bullet\mathbf{t} - \lambda_o \nabla_s \bullet \mathbf{u}) - \mathbf{t}_{\tan}/\mu_o - \hat{n}x(\nabla_s x\mathbf{u}) \qquad (18)$$

and $\mathrm{Re}Q_{nn'}$ contains $\mathrm{Re}\psi_n$ instead of $\psi_n$ in all places. For some details concerning the computation of $\partial/\partial n\, dS$ and $\partial/\partial n\, \mathbf{t}(\psi_1)$, see the Appendix.

The exact order $h$ expressions given previously are hideously complicated to use in actual computations and there are, therefore, good reasons to look for simplified theories that might contain the most important effects. One way is to use a *membrane shell type of approximation*, i.e., to skip the last two integrals, the operator **C** and all of **B** except the first term in equation (17). This has been shown earlier to lead to meaningful results (Olsson et al., 1988). In fact, one could also skip the second term in **B** and only keep the first inertial term, and this actually captures the main effects of the interface layer at low frequencies. As mentioned previously, another approach is to put $\mathbf{B}=0$ and to skip the second and fourth terms in **C** in equation (18).

## 3 Numerical Results and Discussion

The present section contains some examples of computations performed by means of the approach developed in the previous section. The total scattering cross-sections for S and P wave scattering from spherical and spheroidal inclusions have been computed. The normalization chosen is with respect to the square of the radius of the sphere circumscribing the inclusion inside the layer. The main emphasis is on checking, by comparing the results from other methods, that the approximate boundary conditions derived in the previous section indeed yield meaningful results.

**Table 1 Material parameters for the SiC inclusion in *A*1**

| | |
|---|---|
| Matrix | $\rho = 2.706$ kg m$^{-3}$, $(\lambda + 2\mu, \mu) = (1.105, 0.267) \times 10^{11}$ Pa |
| Inclusion | $\rho_1 = 3.181$ kg m$^{-3}$, $(\lambda_1 + 2\mu_1, \mu_1) = (4.742, 1.881) \times 10^{11}$ Pa |
| Layer | $\rho_0 = (\rho + \rho_1)/2$, $\lambda_0 = (\lambda + \lambda_1)/2$, $\mu_0 = (\mu + \mu_1)/2$ |

The inclusion is taken to be a SiC inclusion used by Paskaramoorthy, Datta, and Shah (1988). The properties are given in Table 1.

To check the null field computer program, various tests have been performed. One is that of taking the center of a spherical inclusion to be displaced from the origin of the spherical coordinate system employed. For an incident P wave at $k_s a=2$, where $a$ is the radius of the sphere, and thickness $h=0.1a$, the results from calculations for a sphere displaced by $0.5a$ differ by less than $5\times10^{-6}$ from those for a sphere centered at the origin. This should be compared to the value of the scattering cross-section, which is roughly 1.5.

In Table 2 a comparison is made between the results of the various approximations for a thin layer on a spherical inclusion, and the exact analytical results. A conclusion from the table is that in the frequency interval considered the results of including only the jump in the displacement to order $h$ (and no geometrical effects, etc.) are inferior to the full-order $h$ results, as well as to the membrane and inertial approximations described in Section 2. The importance of the geometrical terms is also borne out in the table. Only the full-order $h$ column contains the effects of including the last two terms, the geometrical ones, in equation (17). Note that the last two columns contain values which are actually below the values for a sphere without a surrounding layer. The corresponding boundary conditions thus represent a kind of smoothing of the contrast between the inclusion and the matrix.

Turning now to the plots, in Figs. 1 and 2 are found results for scattering from spherical inclusions. The full-order $h$ results are here compared to the results from exact separation of variables calculations. A good agreement is found between the exact results and those of the order $h$ theory in the frequency range considered, but it can be noted that the P wave results are in close agreement over a larger interval than are the S wave results. This is only to be expected from the difference in wavelength between P and S waves, $k_s h$ being roughly twice as large as $k_p h$. Note also that the S wave curve peaks at $k_s a \approx 4.5$, while the P wave curve does not peak in the interval considered. For larger frequencies the curves are of course expected to approach a constant value. The great effect of the interface layer on the cross-section can be clearly seen in these and the following plots, since results for inclusions without a layer are also included.

Figure 3 contains results for the total scattering cross-section for an SV wave incident at 45 deg angle to the symmetry axis on an 1:2 oblate spheriod. For comparison, data points from the FEM calculations of Paskaramoorthy, Datta, and Shah (1988) are included for scatterers with and without interface layers. The agreement between the results is in this case very good. (It should be noted that the frequency variable and the normalization used here differs from that of the previous reference. The difference only matters for

**Table 2 Comparison of analytical and numerical results for the total scattering cross-section for a plane P wave incident on a coated SiC sphere in *A*1.*h* = *a*/10**

| $k_s a$ | Analytical | Full order h | Membrane | Inertial | Only jump in u | Order h without geometrical effects |
|---|---|---|---|---|---|---|
| 0.1 | $4.232 \times 10^{-5}$ | $4.211 \times 10^{-5}$ | $3.880 \times 10^{-5}$ | $3.741 \times 10^{-5}$ | $2.346 \times 10^{-5}$ | $2.414 \times 10^{-5}$ |
| 0.5 | $2.212 \times 10^{-2}$ | $2.201 \times 10^{-2}$ | $2.142 \times 10^{-2}$ | $2.067 \times 10^{-2}$ | $1.264 \times 10^{-2}$ | $1.313 \times 10^{-2}$ |
| 1.0 | 0.2339 | 0.2328 | 0.2443 | 0.2355 | 0.1408 | 0.1482 |
| 1.5 | 0.7504 | 0.7469 | 0.7559 | 0.7234 | 0.4675 | 0.4874 |
| 2.0 | 1.5574 | 1.5474 | 1.4113 | 1.3379 | 0.9831 | 1.0029 |

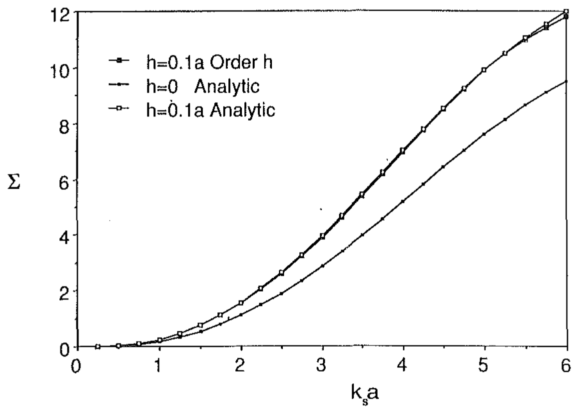


**Fig. 1 The total scattering cross-section normalized by $a^2$ for P wave scattering from a sphere of radius $a$**

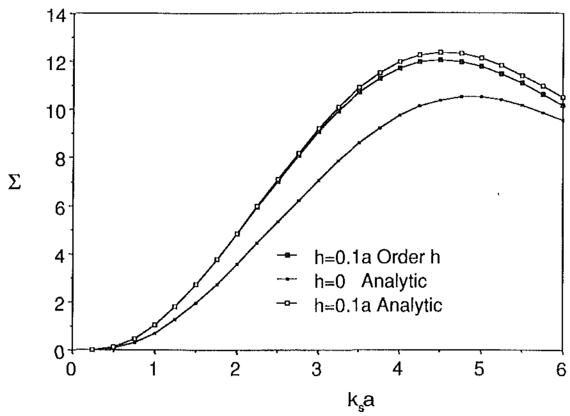


**Fig. 2 Same as Fig. 1 but for S wave scattering**

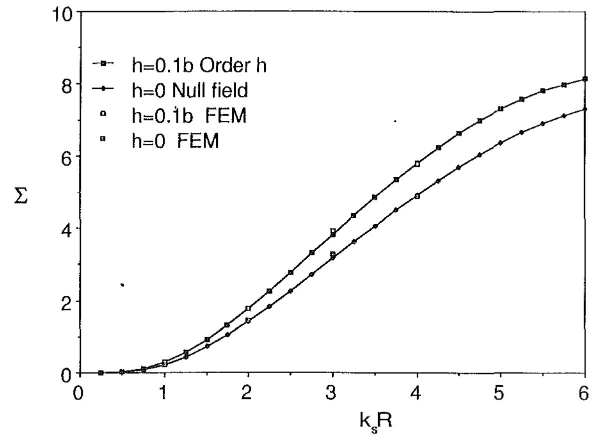


**Fig. 3 The total scattering cross-section normalized by $R^2$ for SV wave scattering from an oblate spheroid; major axis $a = R$, minor axis $b = a/2$; angle of incidence 45 deg from axis of symmetry**

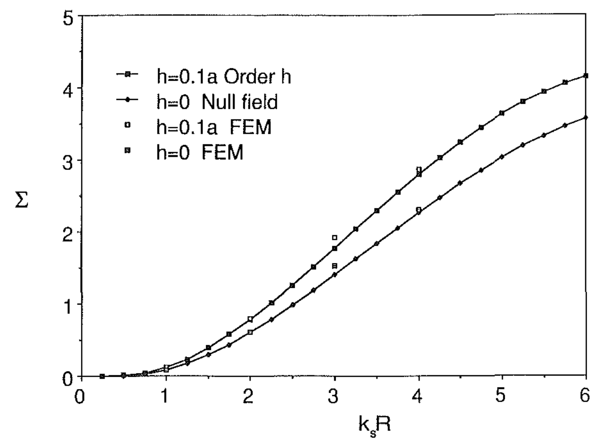


**Fig. 4 The total scattering cross-section normalized by $R^2$ for SV wave scattering from a prolate spheroid; major axis $b = R$, minor axis $a = b/2$; angle of incidence 45 deg from axis of symmetry**

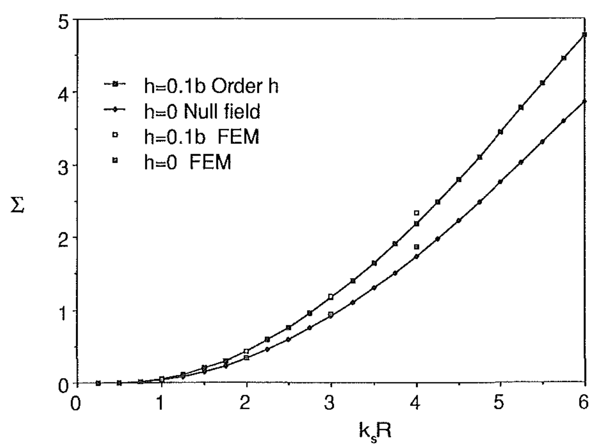


**Fig. 5 Same as Fig. 3 but for P wave scattering**

nonspherical bodies.) The discrepancy is not appreciably larger for the inclusion with a layer than for the one without a layer.

A less good agreement with FEM is found in the next plot, Fig. 4, which shows the scattering cross-section for SV incidence at 45 deg on a prolate 2:1 spheriod. However, we again find that the disagreement is roughly the same for the cases $h = 0.1a$ and $h = 0$. In the latter case, the case of an inclusion without a layer, the present method reduces to the ordinary (well-tested) null field approach. The discrepancy is possibly due to discretization errors in the FEM calculation. Curiously, the largest differences occur at an intermediate frequency.

Similar remarks can be made concerning Fig. 5, which contains results for P wave scattering from a coated 1:2 oblate spheroid, except that here the largest differences occur at the highest frequency compared. Note also that while the curves in Fig. 3 seem to be approaching a peak just outside the frequency interval plotted, the curves in Fig. 5 have just reached at most an inflection point. As for the sphere, this is a result of the different wavelengths of P and S waves.

## 4 Conclusion

The main conclusions of the present study are the following. First, the results of using approximate boundary conditions coupled with the null field method are in agreement with those obtained by Paskaramoorthy et al. (1988), who used a hybrid finite element (FE) and eigenfunction expansions technique to take into account the interface layer properties in an exact manner. These results indicate that the effect of an interface layer is to significantly increase the scattering cross-section. Second, the results of the simpler membrane shell and "phenomenological" approaches presented here and by Olsson et al. (1988) are in close agreement at low frequencies ($k_s R = 2$) with the exact calculations for the sphere and with the hybrid FE results for spheroids. As seen from Table 2, the usual approximation of considering the jump in $u$ without the geometrical effects underestimates the scattering cross-sections considerably. Third, the full $O(h)$ calculations including the geometrical effects presented in this study are found to agree quite well with the exact results for the sphere even at moderately high frequencies ($k_s R = 6$). Thus, it may be concluded that the "phenomemological" approach is meaningful at low frequencies in describing the effect of interface thin layers in composite materials with arbitrarily shaped inclusions. At higher frequencies, however, one needs to take into account all terms of $O(h)$ in the boundary conditions.

Although only a single scattering problem is considered here, the results obtained in this case can be used to study dispersion and attenuation of plane waves in the presence of a distribution of inclusions. We plan to present these results in a future communication.

In the case of the "phenomenological" approach of taking $\mathbf{t}_+ - \mathbf{t}_-$ proportional to $\mathbf{u}_-$, we have here simply taken the constant of proportionality to be the same as that of the inertial term of (11). To achieve better results, one could possible vary this constant, and even have different constants for the normal and tangential components, respectively. We hope to pursue these possibilities further in the future.


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## APPENDIX

Consider a family of surfaces $S_\epsilon$, $0 \le \epsilon \le h$, given by the parametrization

$$\gamma = \gamma_\epsilon(\theta, \phi) \equiv \gamma_o(\theta, \phi) + \epsilon \hat{n}_o(\theta, \phi)$$

where $\gamma = \gamma_o(\theta, \phi)$ is a parametrization of $S_o$, and $\hat{n}_o$ is the outward pointing unit normal of $S_o$.
The normal derivative of the surface element at $S_o$ can then be computed as

$$\frac{\partial}{\partial n} dS = \left\{ \frac{\partial}{\partial \epsilon} \left| \frac{\partial \gamma_\epsilon}{\partial \theta} \times \frac{\partial \gamma_\epsilon}{\partial \phi} \right| \right\} \bigg|_{\epsilon=0} d\theta d\phi \, .$$

Quantities like the normal derivative of the unit normal at $S_o$ can similarly be computed as

$$\frac{\partial \hat{n}}{\partial n} = \frac{\partial}{\partial \epsilon} \hat{n}_\epsilon \bigg|_{\epsilon=0}$$

where

$$\hat{n}_\epsilon = \frac{\dfrac{\partial \gamma_\epsilon}{\partial \theta} \times \dfrac{\partial \gamma_\epsilon}{\partial \phi}}{\left| \dfrac{\partial \gamma_\epsilon}{\partial \theta} \times \dfrac{\partial \gamma_\epsilon}{\partial \phi} \right|}$$

defines the unit normal of $S_\epsilon$. Incidentally, $\partial \hat{n}/\partial n$ turns out to be zero.

Thus,

$$\frac{\partial}{\partial n} \mathbf{t}(\mathbf{u}) = \hat{n} \cdot \frac{\partial}{\partial n} \sigma(\mathbf{u})$$

where $\sigma$ is the stress tensor.