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# Positions of convex bodies associated to extremal problems and isotropic measures

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**Abstract**

We show that there are close relations between extremal problems in dual Brunn–Minkowski theory and isotropic-type properties for some Borel measures on the sphere. The methods we use allow us to obtain similar results in the context of Firey–Brunn–Minkowski theory. We also study reverse inequalities for dual mixed volumes which are related with classical positions, such as $\ell$-position or isotropic position.


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## 1. Introduction and notation

An *isotropic measure* $\mu$ in $\mathbb{R}^n$ is a positive, finite Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} x_i x_j \, d\mu(x) = L^2 \delta_{ij}$$

for all $1 \leqslant i, j \leqslant n$, where $L$ is a constant.

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If $K$ is a convex body in $\mathbb{R}^n$ we shall say that $K$ is in *isotropic position* if its centroid is the origin and the measure $\mu = \chi_K(x)\,dx$ is isotropic, i.e. for some constant $L$

$$\int_K \langle x, \theta \rangle^2 \, dx = L^2$$

for all $\theta \in S^{n-1}$. Given a convex body $K$ in $\mathbb{R}^n$ we consider the family of its *positions* $\{t + SK; t \in \mathbb{R}^n, S \in SL(n)\}$, where $SL(n)$ denotes the family of $n \times n$ real matrices with determinant equal to $\pm 1$. It is well known that any convex body has a unique (up to orthogonal transformations) isotropic position.

Giannopoulos and Milman associated isotropic measures to extremal positions of convex bodies in $\mathbb{R}^n$ as a tool to discuss an isometric approach to the study of the different positions for convex bodies, which have been introduced in the local theory of Banach spaces (see [6,7]).

For instance, the isotropic position of a convex body, defined above, is the solution of the extremal problem $\min\{\int_K |t + Sx|^2 \, dx; S \in SL(n), t \in \mathbb{R}^n\}$ (see [19] for the symmetric case and [4] for the non-symmetric one).

In the same way, the euclidean ball $D_n$ is the ellipsoid of maximal volume among all ellipsoids contained in a symmetric convex body $K$ (*John's ellipsoid*) if and only if the identity $I_n$ is the solution of the extremal problem $\min\{\|S : \ell_2^n \to X_K\|\}$ and this situation is characterized by the existence of an isotropic measure $\mu$ supported on the contact points of $K$ and $D_n$ ($X_K$ represents the normed space $\mathbb{R}^n$ endowed with the gauge function of $K$). In [9] the authors characterize the extremal volume position between two convex bodies in terms of decompositions of the identity (see [3] for a non-convex case). These decompositions of the identity can also be understood as the existence of some *generalized isotropic measures* supported on the contact pairs, which emphasizes the close relation between extremal problems of convex bodies and measures with isotropic-type properties.

The minimal surface position is another example of this phenomenon. If $K$ is a convex body, we denote by $|\partial K|$ its surface area. Then $K$ is in minimal surface area position (i.e. $|\partial K| \leqslant |\partial(SK)|$, for all $S \in SL(n)$) if and only if the area measure $\sigma_K$ of $K$ is isotropic. Recall that $\sigma_K$ is the measure supported on $S^{n-1}$ defined on each Borel subset $A$ in $S^{n-1}$ as the measure of the set of points in the boundary of $K$ whose outer normal is in $A$ (see [8,22]).

The mean width of a convex body $\omega(K)$ is defined by

$$\omega(K) = 2 \int_{S^{n-1}} h_K(u) \, d\sigma(u),$$

where $h_K$ is the support function and $d\sigma$ is Lebesgue measure on $S^{n-1}$. In [6] the authors show that a "*smooth enough*" convex body $K$ (that is, $h_K$ is twice continuously differentiable) is in minimal mean width position if and only if the measure $h_K(u)\,d\sigma(u)$, supported on the unit sphere $S^{n-1}$, is isotropic.

These last two cases are also the extreme cases which, respectively, minimize the quermassintegrals of $K$, $W_i(K)$ for the values $i = n-1$ and 1. In the same paper [6], the authors also consider the remaining cases and they obtain necessary conditions for minimizing the corresponding $W_i(K)$.

The main goal of this paper is to extend these ideas of Giannopoulos and Milman and to show that a similar situation occurs when we consider the dual mixed volumes, $\tilde{V}_i(K,L)$, and dual quermassintegrals, $\tilde{W}_i(K)$. If $K \subseteq \mathbb{R}^n$ is a star-shaped body and $i \in \mathbb{R}$ (not necessarily an integer), we consider the dual quermassintegral $\tilde{W}_i(K)$ defined by

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u)\, d\sigma(u)$$

(see [13]). We will consider the following extremal problems:

$$\max\{\tilde{W}_i(SK), S \in SL(n)\} \quad \text{if } i \in (0,n),$$

$$\min\{\tilde{W}_i(SK), S \in SL(n)\} \quad \text{if } i \notin [0,n].$$

In Section 2 we study the positions of a convex body which are solutions of these extremal problems and we show that there is a close relation between these *extremal* positions and properties of isotropic type of some measures. In fact, we prove that if $i \in (-\infty, 0) \cap [n+1, +\infty)$, the isotropy of some Borel measures on $S^{n-1}$ is necessary and sufficient for a convex body $K$ (symmetric when $i \geqslant n+1$) to be in the position that minimizes $\tilde{W}_i(TK)$. If $i \in (0,n)$ the phenomenon is not so clear and, in general, we can only ensure that the isotropy of some measure is a necessary condition for a convex body $K$ to optimize $\tilde{W}_i(\cdot)$, but we do not know if this condition is also sufficient. The methods we use include general results about isotropic measures on $S^{n-1}$ (see Lemma 2.7) which allow us to characterize the solutions of some extremal problems in the context of the Brunn–Minkowski–Firey theory ($L^p$-mixed volumes) introduced by Lutwak (see [13–15] and the references therein).

It is well known that inequalities like Brunn–Minkowski, or even its most important consequence the isoperimetric inequality, cannot be reversed, as simple examples show. However Milman in the very remarkable paper [18] proved that we can reverse the Brunn–Minkowski inequality, up to an absolute constant, if we consider different positions for the convex bodies, i.e., there exist positions which are now called $M$-positions which allow one to reverse the inequality of Brunn–Minkowski (see [23] for another approach to the problem using interpolation of operators and [2] for its extension to the non-convex case).

In the same spirit, Ball [1] proved that among all the positions of a convex body, John's position leads to the reverse isoperimetric inequality; i.e. this is the one position of a convex body for which the surface area is less or equal to the one of a cube (in the symmetric case) or a simplex (in the non-symmetric one) with the same volume.

In Section 3 we consider the same problem for dual quermassintegrals, $\tilde{W}_i(K)$, and we study the corresponding *reverse dual* Minkowski inequalities. Apart from the interest of these reverse inequalities as a natural complement of the dual mixed volumes theory, it is also interesting that in these reverse inequalities we come across the classical hyperplane conjecture. Moreover, Theorem 3.1 will allow us to reformulate the hyperplane conjecture in terms of reverse Minkowski inequalities for $-\infty < i < 1$. In this section we also study reverse inequalities for other indices $i \in (n-1, +\infty)$ and we find out that they are related to different classical positions of convex bodies such as $\ell$-positions. In the range $1 \leqslant i < n-1$, we can say something more for the balls $B_p^n$, $1 \leqslant p \leqslant \infty$, by using sharp estimates given in [24].

Next we introduce some notation. As usual we let $||x||_p = (\sum_1^n |x_i|^p)^{1/p}$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $0 < p < \infty$. $D_n$ denotes the euclidean unit ball, i.e. $\{x \in \mathbb{R}^n; ||x||_2 \leqslant 1\}$. $B_p^n$ is the unit ball of the norm $||\cdot||_p$. If $A \subseteq \mathbb{R}^n$, $|A|$ will represent the $n$-dimensional Lebesgue measure of $A$. Notice that $|\cdot|$ may also represent the absolute value of a real number and the euclidean norm of a vector, i.e. $||\cdot||_2$, since the context will avoid any confusion.

A set $K \subseteq \mathbb{R}^n$ is star shaped at 0 if $\lambda x \in K$, whenever $0 \leqslant \lambda \leqslant 1$ and $x \in K$. If $K$ is non-empty, compact and star shaped at 0, its *radial function* $\rho_K$ is defined by

$$\rho_K(x) = \max\{\lambda \geqslant 0 \colon \lambda x \in K\}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. This function is homogeneous of degree $-1$. We say that a body (compact with non-empty interior) $K$ is a *star body at* 0 if it is star shaped at 0, 0 belongs to the interior of $K$ and the restriction of its radial function $\rho_K$ is continuous on the sphere $S^{n-1}$. Every convex body with 0 in its interior is a star body at 0. In this case

$$\rho_K(x)^{-1} = ||x||_K = h_{K^\circ(x)},$$

where $||\cdot||_K$ is the gauge of $K$, $K^\circ$ denotes the polar set of $K$ and $h_{K^\circ}$ its support function (all these notions are fully explained in [25]).

We recall the definition of the *isotropy constant* $L_K$ of a convex body $K$

$$nL_K^2|K|^{2/n} = \inf_{\substack{S \in SL(n) \\ t \in \mathbb{R}^n}} \frac{1}{|K|} \int_K |t + Sx|^2 \, dx.$$

Throughout this paper, unless otherwise stated, we will use $C$ to denote a positive absolute constant, which can assume different values in different occurrences.

## 2. Extremal positions for dual and $L^p$ mixed volumes

The Brunn–Minkowski theory is the natural framework to work with shadows (projections) of convex bodies but when the data concern sections through a fixed point the dual Brunn–Minkowski theory provides a natural setting. In 1975, Lutwak (see [13,14] and the references therein) introduced the concept of *dual mixed volumes*. If $K, L \subseteq \mathbb{R}^n$ are star bodies at 0, the dual mixed volumes $\tilde{V}_i(K, L)$ are defined for all $i \in \mathbb{R}$ by

$$\tilde{V}_i(K,L) = \tilde{V}(K,n-i;L,i) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\,d\sigma(u), \tag{1}$$

where $\rho_K(\cdot)$, $\rho_L(\cdot)$ are the radial functions of $K, L$ and $\sigma$ is the Lebesgue measure on $S^{n-1}$. General properties of dual mixed volumes can be also found in [5]. By changing variables, it is clear that

$$\tilde{V}_i(K,L) = \tilde{V}_{n-i}(L,K) = |\det T|^{-1}\tilde{V}_i(TK,TL) \tag{2}$$

for all $T \in GL(n)$.

It seems that many of the results found in Brunn–Minkowski theory have analogues in this dual Brunn–Minkowski theory. A clear example of this is the Minkowski inequality. A simple use of the Hölder inequality implies that

$$\tilde{V}_i(K,L) \leqslant |K|^{(n-i)/n}|L|^{i/n} \quad \text{if } i \in [0,n], \tag{3}$$

$$\tilde{V}_i(K,L) \geqslant |K|^{(n-i)/n}|L|^{i/n} \quad \text{if } i \notin (0,n), \tag{4}$$

which can be understood as a *dual* of the well-known result of Minkowski. These inequalities make us wonder when

$$\tilde{V}_i(K,L) = \max\{\tilde{V}_i(SK,L); S \in SL(n)\} \quad \text{if } i \in [0,n] \tag{5}$$

or

$$\tilde{V}_i(K,L) = \min\{\tilde{V}_i(SK,L); S \in SL(n)\} \quad \text{if } i \notin (0,n). \tag{6}$$

We should note that the origin now plays an important role. This theory is not translation invariant, so we should only consider linear positions of convex bodies, i.e. $\{SK; S \in SL(n)\}$.

In this section we study necessary and sufficient conditions for $K$ and $L$ to solve the extremal problems stated in (5) and (6) and we show that the necessary and sufficient conditions for $K$ and $L$ to be solutions of these extremal problems are related to the existence of measures with "*isotropic*"-*type properties*, extending the ideas of Giannopoulos and Milman [6].

**Proposition 2.1.** *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies having 0 in their interior such that $K^\circ$ and $L^\circ$ are "smooth enough" (that is, $h_{K^\circ}$ and $h_{L^\circ}$ are twice continuously differentiable). Then each of the following conditions*

(i) *for some $i \in (0,n)$, $\tilde{V}_i(K,L) = \max\{\tilde{V}_i(SK,L); S \in SL(n)\}$,*
(ii) *for some $i \notin [0,n]$, $\tilde{V}_i(K,L) = \min\{\tilde{V}_i(SK,L); S \in SL(n)\}$,*

*implies that*

$$\frac{\operatorname{tr} T}{n}\,\tilde{V}_i(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle\, d\sigma(u) \tag{7}$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) \tag{8}$$

*for all* $T \in GL(n)$.

**Proof.** We only prove case (i) since (ii) is similar. It is also clear (see (2)) that $\tilde{V}_i(SK,L) = \tilde{V}_{n-i}(T^{-1}L,K)$, for all $S \in SL(n)$, so we only need establish (7).

If we take $T \in GL(n)$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we can define

$$S_\varepsilon = \frac{I + \varepsilon T}{|\det(I+\varepsilon T)|^{1/n}}.$$

By hypothesis, $\tilde{V}_i(K,L) \geqslant \tilde{V}_i(S_\varepsilon^{-1}K, L) = \tilde{V}_i(K, S_\varepsilon L)$, that is

$$\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\, d\sigma(u) \geqslant \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_{S_\varepsilon L}^i(u)\, d\sigma(u),$$

but since $\rho_{\phi(L)}(u) = \rho_L(\phi^{-1}u)$,

$$|\det(I+\varepsilon T)|^{i/n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\, d\sigma(u) \geqslant \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i((I+\varepsilon T)^{-1}u)\, d\sigma(u).$$

It is easy to prove that if $||\varepsilon T|| < 1$

$$|\det(I+\varepsilon T)|^{i/n} = 1 + \frac{i\varepsilon(\operatorname{tr} T)}{n} + O(\varepsilon^2),$$
$$(I+\varepsilon T)^{-1}(u) = u - \varepsilon Tu + O(\varepsilon^2)$$

and

$$\rho_L^i(u - \varepsilon Tu + O(\varepsilon^2)) = \left(\frac{1}{h_{L^\circ}(u) - \varepsilon\langle \nabla h_{L^\circ}(u), Tu\rangle + O(\varepsilon^2)}\right)^i$$
$$= \frac{1}{h_{L^\circ}^{i+1}(u)}(h_{L^\circ}(u) + i\varepsilon\langle \nabla h_{L^\circ}(u), Tu\rangle + O(\varepsilon^2))$$

when $\varepsilon \to 0$. Hence,

$$\left(1+\frac{i\varepsilon(\operatorname{tr} T)}{n}+O(\varepsilon^2)\right)\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\,d\sigma(u)$$

$$\geqslant \int_{S^{n-1}} \frac{\rho_K^{n-i}(u)\,d\sigma(u)}{(h_{L^\circ}(u)-\varepsilon\langle \nabla h_{L^\circ}(u), Tu\rangle + O(\varepsilon^2))^i}$$

$$= n\tilde{V}_i(K,L) + i\varepsilon \int_{S^{n-1}} \rho_K^{n-i}(u)\frac{\langle \nabla h_{L^\circ}(u), Tu\rangle}{h_{L^\circ}^{i+1}(u)}\,d\sigma(u) + O(\varepsilon^2).$$

Then, if $\varepsilon \to 0^+$

$$\frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L) \geqslant \int_{S^{n-1}} \rho_K^{n-i}(u)\frac{1}{h_{L^\circ}^{i+1}(u)}\langle \nabla h_{L^\circ}(u), Tu\rangle\,d\sigma(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle\,d\sigma(u).$$

But if we replace $T$ by $-T$ we conclude that

$$\frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle\,d\sigma(u). \qquad \square$$

Note that conditions (7) and (8) can be understood as *non-commutative isotropic conditions* for the measures $\rho_K^{n-i}(\cdot)\rho_L^{i+1}(\cdot)\,d\sigma(\cdot)$ and $\rho_K^{n-i+1}(\cdot)\rho_L^{i}(\cdot)\,d\sigma(\cdot)$, respectively.

Next, we show that these necessary conditions appearing in Proposition 2.1 are also sufficient in some cases, but first of all we shall study relations between these two conditions (7) and (8). This is stated in the following result.

**Proposition 2.2.** *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies having 0 in their interior and such that $K^\circ$ and $L^\circ$ are "smooth enough". The following assertions are equivalent*:

(i) *For every $T \in GL(n)$ symmetric*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle\,d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L).$$

(ii) *For every $T \in GL(n)$ symmetric*,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\,d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L).$$

**Proof.** Since for every $T \in GL(n)$ symmetric, there exist $\theta_i \in S^{n-1}$ and $\lambda_i > 0$ $(i = 1, \dots, n)$ such that

$$T = \sum_{i=1}^{n} \lambda_i \theta_i \otimes \theta_i,$$

it is enough to prove that the following assertions are equivalent:

(i) For every $\theta \in S^{n-1}$,

$$\int_{S^{n-1}} \rho_K^{n-i}(u) \rho_L^{i+1}(u) \langle \nabla h_{L^\circ}(u), \theta \rangle \langle u, \theta \rangle \, d\sigma(u) = \tilde{V}_i(K, L).$$

(ii) For every $\theta \in S^{n-1}$,

$$\int_{S^{n-1}} \rho_K^{n-i+1}(u) \rho_L^{i}(u) \langle \nabla h_{K^\circ}(u), \theta \rangle \langle u, \theta \rangle \, d\sigma(u) = \tilde{V}_i(K, L).$$

Take $\theta \in S^{n-1}$. We shall use the Laplace-Beltrami operator. If we define $F : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ by

$$F(x) = \frac{\langle x, \theta \rangle^2}{2|x|^2}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

it is easy to check that for every $u \in S^{n-1}$, $\nabla F(u) = \langle u, \theta \rangle \theta - \langle u, \theta \rangle^2 u$ and $\Delta F(u) = 1 - n \langle u, \theta \rangle^2$.

On the other hand, we define $H : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ by

$$H(x) = h_{K^\circ}\left(\frac{x}{|x|}\right)^{i-n} h_{L^\circ}\left(\frac{x}{|x|}\right)^{-i} \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Since support functions are 1-homogeneous, it can be proved that for every $u \in S^{n-1}$,

$$\begin{aligned} \nabla H(u) = {} & (i-n) h_{L^\circ}(u)^{-i} h_{K^\circ}(u)^{i-n-1} (\nabla h_{K^\circ}(u) - h_{K^\circ}(u) u) \\ & - i h_{L^\circ}(u)^{-i-1} h_{K^\circ}(u)^{i-n} (\nabla h_{L^\circ}(u) - h_{L^\circ}(u) u). \end{aligned}$$

Now, if we integrate on the sphere and use Green's formula for the Beltrami operator (see for instance [10, p. 7]), we get that

$$\int_{S^{n-1}} H(u) \Delta F(u) \, d\sigma(u) = -\int_{S^{n-1}} \langle \nabla F(u), \nabla H(u) \rangle \, d\sigma(u).$$

Hence we deduce

$$(n-i)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^i(u)\langle \nabla h_{K^\circ}(u), \theta\rangle \langle u, \theta\rangle \, d\sigma(u)$$
$$= n\tilde{V}_i(K,L) - i\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), \theta\rangle \langle u, \theta\rangle \, d\sigma(u),$$

for all $\theta \in S^{n-1}$ which completes the proof. $\square$

We do not know if this result is true for general transformations. We can achieve a complete characterization only in special cases, for example when one of the bodies is the euclidean ball. We also remark that, if $L = D_n$, condition (i) in the last proposition means that the measure $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ is isotropic.

We now study whether assertion (7) or (8) is sufficient to ensure that $K$ solves the extremal problem (5) or (6).

**Proposition 2.3.** *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies having* 0 *in their interior and such that $K^\circ$ and $L^\circ$ are "smooth enough". If $i \leqslant -1$ and $L$ is* 0*-symmetric, then the following assertions are equivalent*:

(i) $\tilde{V}_i(K,L) = \min\{\tilde{V}_i(SK,L)\}$, *when the minimum runs over all symmetric, positive definite matrices $S \in SL(n)$.*
(ii) *For every $T$ symmetric, positive definite matrix in $GL(n)$,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L).$$

*Moreover $K$ is the unique symmetric positive definite position satisfying* (i) *or* (ii). *Furthermore, if $i = -1$ the result holds without any symmetry assumptions on $L$.*

**Proof.** Implication (i) $\Rightarrow$ (ii) can be proved by using the same ideas as in Proposition 2.1.

(ii) $\Rightarrow$ (i) We shall assume $i < -1$. If we take $S \in SL(n)$, by using (2)

$$\tilde{V}_i(SK,L) = \tilde{V}_i(K,S^{-1}L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_{S^{-1}L}^i(u)\, d\sigma(u)$$
$$= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u) h_{S^\star(L^\circ)}^{-i}(u)\, d\sigma(u).$$

By using Hölder's inequality with respect to the measure $\frac{1}{n}\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ we get

$$\tilde{V}_i(SK,L) \geqslant \left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\, d\sigma(u)\right)^{i+1}$$
$$\times \left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u) h_{S^\star(L^\circ)}(u)\, d\sigma(u)\right)^{-i}.$$

Since $\langle \nabla h_{L^\circ}(u), Su\rangle \leqslant h_{S^\star(L^\circ)}(u)$ for all $u \in S^{n-1}$ (see [25, p. 40]) and the symmetry of $L$ implies that also

$$|\langle \nabla h_{L^\circ}(u), Su\rangle| \leqslant h_{S^\star(L^\circ)}(u),$$

if $S \in SL(n)$ is positive definite, we get that

$$\begin{aligned}\tilde{V}_i(SK, L) &\geqslant (\tilde{V}_i(K, L))^{i+1}\left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Su\rangle \, d\sigma(u)\right)^{-i}\\ &= (\tilde{V}_i(K, L))^{i+1}\left(\frac{\operatorname{tr} S}{n}\tilde{V}_i(K, L)\right)^{-i}\\ &\geqslant (\det S)^{-i/n}\tilde{V}_i(K, L) = \tilde{V}_i(K, L),\end{aligned}$$

so we obtain the result for $i < -1$.

The uniqueness is a consequence of the fact that for symmetric definite matrices $(\operatorname{tr} S)/n = (\det S)^{1/n}$ if and only if $S$ is the identity.

The case $i = -1$ can be proved by analogous methods and we do not need any symmetry property on $L$. □

**Remark 2.4.** Since $\tilde{V}_i(K, L) = \tilde{V}_{n-i}(L, K)$, by using the last result we can state a similar proposition for $\tilde{V}_i(K, L)$, with $K$ 0-symmetric and $i \geqslant n+1$.

As we said before we can improve our results if one of the bodies is the euclidean ball $D_n$. In the sequel

$$\tilde{W}_i(SK) = \tilde{V}_i(SK, D_n) = \frac{1}{n}\int_{S^{n-1}} \frac{\rho_K^{n-i}(u)}{|Su|^i}\, d\sigma(u),$$

where $S \in SL(n)$ and $K \subseteq \mathbb{R}^n$ is a convex body having 0 in its interior. By using the symmetry properties of $D_n$, it is easy to check that we only have to consider $S \in SL(n)$ which are symmetric and positive definite in order to optimize the dual quermassintegrals.

As an application of dual Minkowski inequalities (3), (4) and the next lemma we can ensure the existence of extremal positions for the dual mixed volumes.

**Lemma 2.5.** *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies with 0 in their interior, then*

$$\lim_{\substack{S\in SL(n)\\ \|S\|\to\infty}} \tilde{V}_i(SK, L) = \begin{cases} 0 & \text{if } i \in (0, n),\\ +\infty & \text{if } i \in (-\infty, 0)\cup(n, \infty).\end{cases} \tag{9}$$

**Proof.** Since $C_1(L)\tilde{W}_i(SK) \leqslant \tilde{V}_i(SK, L) \leqslant C_2(L)\tilde{W}_i(SK)$, where $C_1(L), C_2(L) > 0$ are constants only depending on $L$, it is enough to prove the result for $\tilde{W}_i(SK)$.

First of all we suppose $S \in SL(n)$ is diagonal, with diagonal elements $d_1, \dots, d_n > 0$ such that $\prod_{j=1}^n d_j = 1$.

If $0 < i < n$, by using polar coordinates it is clear that

$$\begin{aligned}\tilde{W}_i(SK) &= \frac{1}{n}\int_{S^{n-1}} \rho_{SK}^{n-i}(u)\, d\sigma(u) \\ &= \frac{n-i}{n}\int_K \frac{dx}{|Sx|^i} = \frac{n-i}{n}\int_K \frac{dx}{\left(\sum_{j=1}^n d_j^2 x_j^2\right)^{i/2}} \\ &\leqslant C(n,i)\int_K \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i},\end{aligned}$$

where $C(n,i)$ denotes a constant depending on $n$ and $i$, which could vary from line to line. If we let $B_\infty^n = Q_n = \{x \in \mathbb{R}^n; |x_i| \leqslant 1\}$, there exist $r$, $R > 0$ such that $rQ_n \subseteq K \subseteq RQ_n$. Therefore, if $d_1 = \max\{d_j\colon\ 1 \leqslant j \leqslant n\}$, by using Fubini's theorem

$$\begin{aligned}\tilde{W}_i(SK) &\leqslant C(n,i)\int_{RQ_n} \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i} \\ &\leqslant C(n,i) R^{n-i}\int_{Q_n} \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i} \\ &= C(n,i) R^{n-i}\left(\int_0^1 \int_0^{d_2} \cdots \int_0^{d_n} \frac{dy_1 \dots dy_n}{\sum_{j=1}^n y_j^i}\right. \\ &\qquad \left. + \int_1^{d_1}\int_0^{d_2} \cdots \int_0^{d_n} \frac{dy_1 \dots dy_n}{\sum_{j=1}^n y_j^i}\right).\end{aligned}$$

Notice that

$$\begin{aligned}\int_0^1 \int_0^{d_2} \cdots \int_0^{d_n} \frac{dy_1 \dots dy_n}{\sum_{j=1}^n |y_j|^i} &\leqslant C(n)\int_0^1 \frac{dy_1}{|y_1|^{i/n}}\left(\prod_{j=2}^n \int_0^{d_j} \frac{dy_j}{|y_j|^{i/n}}\right) \\ &\leqslant C(n)\left(\prod_{j=2}^n d_j\right)^{(n-i)/n} = C(n) d_1^{(i-n)/n} \to 0,\end{aligned}$$

when $||S|| \to +\infty$. On the other hand, if $i \neq 1$

$$0 \leqslant \int_1^{d_1}\int_0^{d_2} \cdots \int_0^{d_n} \frac{dy_1 \dots dy_n}{\sum_{j=1}^n |y_j|^i} \leqslant \left(\prod_{j=2}^n d_j\right)\int_1^{d_1} \frac{dy_1}{y_1^i} = C(i)\left(\frac{1}{d_1^i} - \frac{1}{d_1}\right) \to 0,$$

when $||S||\to+\infty$ and if $i=1$

$$0\leqslant\int_1^{d_1}\int_0^{d_2}\cdots\int_0^{d_n}\frac{dy_1\dots dy_n}{\sum_{j=1}^n|y_j|}\leqslant\frac{1}{d_1}\int_1^{d_1}\frac{dy_1}{y_1}=C(i)\frac{\log d_1}{d_1}\to 0,$$

when $||S||\to+\infty$, therefore $\tilde{W}_i(SK)\to 0$, when $||S||\to+\infty$.

If $-\infty<i<0$ the proof is almost the same, but the case $i>n$ is different. Following the same ideas as before we get that

$$\tilde{W}_i(SK)\geqslant\frac{C(n,i)}{R^{n-i}}\int_{Q_n^c}\frac{dx}{\sum_{j=1}^n d_j^i|x_j|^i},$$

where $Q_n^c$ is the complementary set of $Q_n$. If $d_1=\min\{d_j\colon 1\leqslant j\leqslant n\}$, we have

$$\begin{aligned}\tilde{W}_i(SK)&\geqslant C(n,i)\int_{Q_n^c}\frac{dx}{\sum_{j=1}^n d_j^i|x_j|^i}\\&\geqslant C(n,i)\int_1^2 dx_1\int_0^{d_1/d_2}\cdots\int_0^{d_1/d_n}\frac{dx_2\dots dx_n}{d_1^i2^i+\sum_{j=2}^n d_1^i}\\&=C(n,i)\frac{1}{d_1^i(2^i+n-1)}\prod_{j=2}^n\frac{d_1}{d_j}\\&=C(n,i)d_1^{n-i}\to\infty\end{aligned}$$

when $||S||\to+\infty$.

If $S\in SL(n)$ is a symmetric, positive definite matrix, there exist an orthogonal matrix $V\in O(n)$ and a diagonal matrix $D$ with diagonal elements $d_1,\dots,d_n>0$ such that $\prod_{j=1}^n d_j=1$ and $S=V^*DV$. Henceforth if we assume $rD_n\subseteq K\subseteq RD_n$,

$$\begin{aligned}\tilde{W}_i(SK)&=\frac{1}{n}\int_{S^{n-1}}\frac{\rho_K^{n-i}(u)}{|Su|^i}\,d\sigma(u)\\&\simeq\int_{S^{n-1}}\frac{1}{|DVu|^i}\,d\sigma(u)\\&\simeq\int_{S^{n-1}}\frac{1}{|Dv|^i}\,d\sigma(v)\end{aligned}$$

where $A\simeq B$ means here that the quotient $A/B$ is bounded from above and from below for constants depending only on $n$, $i$, $R$ and $r$. Hence

$$\lim_{\substack{S\in SL(n)\\||S||\to\infty}}\tilde{W}_i(SK)=\lim_{\substack{D\in SL(n)\\D\ \text{diagonal}\\||D||\to\infty}}\tilde{W}_i(DK)=\begin{cases}0 & \text{if } i\in(0,n),\\+\infty & \text{if } i\in(-\infty,0)\cup(n,\infty).\end{cases}\qquad\square$$

The isotropy of some measure characterizes exactly when $K$ optimizes the dual quermassintegrals in the range $i\in(-\infty,0)$, as is shown in the following result.

**Theorem 2.6.** *Let $K \subseteq \mathbb{R}^n$ be a convex body having 0 in its interior. Suppose that $K^\circ$ is "smooth enough". Let $i \in (-\infty, 0)$. Then the following assertions are equivalent*:

(i) $\tilde{W}_i(K) = \min\{\tilde{W}_i(SK); S \in SL(n)\}$.

(ii) *For every $T \in GL(n)$,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n} \tilde{W}_i(K).$$

(iii) *For every $T \in GL(n)$ symmetric,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle d\sigma(u) = \frac{\operatorname{tr} T}{n} \tilde{W}_i(K).$$

(iv) *The measure given by $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ is isotropic in $S^{n-1}$.*

(v) *For every $\alpha > 0$ and for every $S \in SL(n)$*

$$\int_K \left(\frac{|Sx|}{|x|}\right)^\alpha |x|^{-i}\, dx \geqslant \int_K |x|^{-i}\, dx.$$

(vi) *For some $\alpha_0 > 0$ and for every $S \in SL(n)$*

$$\int_K \left(\frac{|Sx|}{|x|}\right)^{\alpha_0} |x|^{-i}\, dx \geqslant \int_K |x|^{-i}\, dx.$$

*Moreover, any one of these six assertions implies that K is the unique position, up to orthogonal transformation, that minimizes $\tilde{W}_i(SK)$.*

**Proof.** Implication (i) $\Rightarrow$ (ii) is a consequence of Proposition 2.1.

Implication (ii) $\Rightarrow$ (iii) is trivial.

Implication (iii) $\Rightarrow$ (iv) is deduced from Proposition 2.2, since for $L = D_n$ condition (i) in that theorem is just the isotropy of the measure $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$.

Implication (iv) $\Rightarrow$ (v) is a consequence of the following Lemma 2.7 applied to the measure $d\mu(\cdot) = \rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ and the use of polar coordinates.

Implication (v) $\Rightarrow$ (vi) is trivial.

Implication (vi) $\Rightarrow$ (i) is also a consequence of the following Lemma 2.7 applied to $d\mu(\cdot) = \rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ for $\alpha_0 = -i$ and the use of polar coordinates.

The uniqueness can be proved as in Proposition 2.3. □

We want to point out that simply by using Remark 2.4 we could obtain Theorem 2.6, but only in the range $i \in (-\infty, -1]$.

**Lemma 2.7.** *Let $\mu$ be a positive and finite Borel measure on $S^{n-1}$. The following assertions are equivalent*:

(i) *$\mu$ is isotropic on $S^{n-1}$.*
(ii) *For every $S \in SL(n)$ and for every $\alpha > 0$*

$$\int_{S^{n-1}} |Su|^{\alpha} \, d\mu(u) \geqslant \int_{S^{n-1}} d\mu(u). \tag{10}$$

(iii) *There exists $\alpha_0 > 0$ such that for every $S \in SL(n)$*

$$\int_{S^{n-1}} |Su|^{\alpha_0} \, d\mu(u) \geqslant \int_{S^{n-1}} d\mu(u). \tag{11}$$

**Proof.** (i) $\Rightarrow$ (ii) We first prove that (10) holds for every $S \in SL(n)$ diagonal, with diagonal elements $d_1, \ldots, d_n > 0$ such that $\prod d_i = 1$.

If $\alpha \in (0, 2]$, then $f(x) = x^{\alpha/2}$ is concave in $[0, +\infty)$ and since $\sum u_j^2 = 1$

$$\begin{aligned}
\int_{S^{n-1}} |S(u)|^{\alpha} \, d\mu(u) &= \int_{S^{n-1}} \left( \sum_{j=1}^{n} d_j^2 u_j^2 \right)^{\alpha/2} d\mu(u) \geqslant \int_{S^{n-1}} \sum_{j=1}^{n} d_j^{\alpha} u_j^2 \, d\mu(u) \\
&= \sum_{j=1}^{n} d_j^{\alpha} \int_{S^{n-1}} u_j^2 \, d\mu(u) = \sum_{j=1}^{n} d_j^{\alpha} \frac{1}{n} \int_{S^{n-1}} |u|^2 \, d\mu(u) \\
&\geqslant \left( \prod_{j=1}^{n} d_j^{\alpha} \right)^{1/n} \int_{S^{n-1}} |u|^2 \, d\mu(u) = \int_{S^{n-1}} d\mu(u).
\end{aligned}$$

If $\alpha \in (2, +\infty)$, let us consider $p = \alpha/2 \in (1, +\infty)$ and if $p^{-1} + q^{-1} = 1$, by using Hölder's inequality we get that

$$\begin{aligned}
\int_{S^{n-1}} |S(u)|^2 \, d\mu(u) &= \int_{S^{n-1}} \sum_{j=1}^{n} d_j^2 u_j^2 \, d\mu(u) \\
&\leqslant \left( \int_{S^{n-1}} \left( \sum_{j=1}^{n} d_j^2 u_j^2 \right)^{\alpha/2} d\mu(u) \right)^{2/\alpha} \left( \int_{S^{n-1}} d\mu(u) \right)^{1/q}.
\end{aligned}$$

Therefore,

$$\int_{S^{n-1}} \left( \sum_{j=1}^{n} d_j^2 u_j^2 \right)^{\alpha/2} d\mu(u) \geqslant \left( \int_{S^{n-1}} \sum_{j=1}^{n} d_j^2 u_j^2 \, d\mu(u) \right)^{p} \left( \int_{S^{n-1}} d\mu(u) \right)^{-p/q}.$$

But, notice that

$$\begin{aligned}\int_{S^{n-1}} \sum_{j=1}^{n} d_j^2 u_j^2 \, d\mu(u) &= \sum_{j=1}^{n} d_j^2 \left( \int_{S^{n-1}} u_j^2 \, d\mu(u) \right) = \sum_{j=1}^{n} d_j^2 \left( \frac{1}{n} \int_{S^{n-1}} d\mu(u) \right) \\ &\geqslant \left( \prod_{j=1}^{n} d_j^2 \right)^{1/n} \int_{S^{n-1}} d\mu(u) = \int_{S^{n-1}} d\mu(u).\end{aligned}$$

So

$$\int_{S^{n-1}} |S(u)|^{\alpha} \, d\mu(u) \geqslant \left( \int_{S^{n-1}} d\mu(u) \right)^{p-p/q} = \int_{S^{n-1}} d\mu(u).$$

Now, if $S \in SL(n)$, there exist orthogonal matrices $V, W \in O(n)$ and diagonal matrix $D$ with diagonal elements $d_1, \dots, d_n > 0$ such that $\prod d_j = 1$ and $S = WDV$ (in this case we cannot restrict to symmetric, positive definite matrices). Then,

$$\begin{aligned}\int_{S^{n-1}} |S(u)|^{\alpha} \, d\mu(u) &= \int_{S^{n-1}} |WDV(u)|^{\alpha} \, d\mu(u) = \int_{S^{n-1}} |DV(u)|^{\alpha} \, d\mu(u) \\ &= \int_{S^{n-1}} |D(u)|^{\alpha} \, dV(\mu)(u),\end{aligned}$$

where $V(\mu)$ denotes the *image measure* of $\mu$ by $V$. It is easy to check that if $\mu$ is a Borel isotropic measure in $S^{n-1}$, then for every orthogonal transformation $V \in O(n)$, $V(\mu)$ is also a Borel isotropic measure in $S^{n-1}$ and $\mu(S^{n-1}) = V(\mu)(S^{n-1})$. Hence,

$$\begin{aligned}\int_{S^{n-1}} |S(u)|^{\alpha} \, d\mu(u) &= \int_{S^{n-1}} |D(u)|^{\alpha} \, dV(\mu)(u) \\ &\geqslant \int_{S^{n-1}} dV(\mu)(u) = \int_{S^{n-1}} d\mu(u).\end{aligned}$$

Implication (ii) $\Rightarrow$ (iii) is trivial.

In order to prove (iii) $\Rightarrow$ (i), it is enough to show that for every $T \in GL(n)$

$$\int_{S^{n-1}} \langle Tu, u \rangle \, d\mu(u) = \frac{\operatorname{tr} T}{n} \int_{S^{n-1}} d\mu(u). \qquad (12)$$

If we take $T \in GL(n)$, we consider for every $0 < \varepsilon < \varepsilon_0$

$$S_{\varepsilon} = \frac{I + \varepsilon T}{|\det(I + \varepsilon T)|^{1/n}}.$$

It can be shown that by using the same variational technique as in Proposition 2.1, we obtain (12). □

**Remark 2.8.** If $K$ has its centroid at 0 and $i=-2$, taking $\alpha=2$, the last theorem ensures that $K$ is in a position that minimizes $\tilde{W}_{-2}(SK)$ if and only if $K$ is in isotropic position.

The preceding lemma allows us to investigate the solution of extremal problems in the context of the Brunn–Minkowski–Firey theory of $L^p$-mixed volumes. If $K, L \subseteq \mathbb{R}^n$ are convex bodies with 0 in their interior, $1 \leqslant p < \infty$ and $i = 0, \dots, n-1$, Lutwak [15] defined the $L^p$-mixed volume $W_{p,i}(K,L)$ as

$$W_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(u) h_K^{1-p}(u)\, dS_i(K,u),$$

which fits the Firey addition of sets (see [14,15] and the references therein). A direct consequence of the preceding lemma shows that, by using the same ideas as in the dual Brunn–Minkowski theory, for a convex body $K \subseteq \mathbb{R}^n$ with 0 in its interior, the following assertions are equivalent:

(i) $S_i(K,\cdot)$ is isotropic on $S^{n-1}$.
(ii) The $L^p$-mixed volume $W_{p,i}(K, D_n) = \min\{W_{p,i}(SK, D_n);\;\; S \in SL(n)\}$.

In particular for $i=0$ and $p=2$ we can characterize when the Lutwak–Yang–Zhang ellipsoid $\Gamma_{-2}(K)$ is an euclidean ball (see [16,17]) in terms of extremal $L^p$-mixed volumes. This is stated explicitly as follows:

**Proposition 2.9.** *For a convex body $K \subseteq \mathbb{R}^n$ with 0 in its interior, the following assertions are equivalent*:

(i) $W_{2,0}(K, D_n) = \min\{W_{2,0}(SK, D_n);\; S \in SL(n)\}$.
(ii) $\Gamma_{-2}(K)$ *is a multiple of the euclidean unit ball.*

In the range $i \in [n+1, \infty)$, the results we gather in the dual Brunn–Minkowski theory are not so complete as the preceding ones and are a consequence of Remark 2.4.

**Corollary 2.10.** *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body with 0 in its interior. Suppose that $K^\circ$ is "smooth enough". Let $i \in [n+1, \infty)$. Then the following assertions are equivalent*:

(i) $\tilde{W}_i(K) = \min\{\tilde{W}_i(SK); S \in SL(n)\}$.
(ii) *For every $T \in GL(n)$,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u) \langle \nabla h_{K^\circ}(u), Tu \rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n} \tilde{W}_i(K).$$

(iii) *For every $T \in GL(n)$ symmetric,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u) \langle \nabla h_{K^\circ}(u), Tu \rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n} \tilde{W}_i(K).$$

(iv) *The measure given by* $\rho_K^{n-i}(\cdot)\,d\sigma(\cdot)$ *is isotropic in* $S^{n-1}$.

*Moreover* $K$ *is the unique position, up to orthogonal transformation, that minimizes* $\tilde{W}_i(TK)$.

**Proof.** Implication (i) $\Rightarrow$ (ii) is consequence of Proposition 2.1.

Implication (ii) $\Rightarrow$ (iii) is trivial.

Implication (iii) $\Rightarrow$ (iv) is deduced from Proposition 2.2 and (iii) $\Rightarrow$ (i) is deduced from Remark 2.4. $\square$

## 3. Reverse isoperimetric inequalities

In this section we study reverse inequalities for Minkowski dual inequalities associated to dual quermassintegrals. Let $K$ be a convex body with 0 in its interior. According to (3) and (4) we have

$$\tilde{W}_i(K) \leqslant |K|^{(n-i)/n} |D_n|^{i/n} \quad \text{if } i \in [0, n], \tag{13}$$

$$\tilde{W}_i(K) \geqslant |K|^{(n-i)/n} |D_n|^{i/n} \quad \text{if } i \notin (0, n). \tag{14}$$

It is well known that we cannot reverse these inequalities since this would imply that $K$ is homothetic to $D_n$. We want to reverse the inequalities by using different affine positions of $K$, as was done by Milman and Ball in other situations (see the Introduction). This problem is closely related to that of the previous section. Indeed, we can define the function

$$\psi_{i,K}(t, T) = \tilde{W}_i(t + TK),$$

where $t \in \mathbb{R}^n$ and $T$ varies on $SL(n)$ in such a way that 0 is in the interior of $t + TK$. Since $\psi_{i,K}(t, T)$ is bounded (see (13) or (14)) and it has a suitable behaviour on the boundary of $SL(n)$ (cf. Lemma 2.7) we know that for a fixed $t$ the function $\psi_{i,K}(t, T)$ attains its extreme value. In Section 2, we obtained necessary and/or sufficient conditions for a position to be extreme. What we shall do now is to estimate how close are the universal bounds given in (13) or (14) from the corresponding extreme values of the function $\psi_{i,K}(t, T)$. The results we get depend on the range of $i$'s and, as before, they are sharp for the interval $i \in (-\infty, 1)$.

**Theorem 3.1.** *Let* $K \subseteq \mathbb{R}^n$ *be a convex body and let* $i \in (-\infty, 1)$, $i \neq 0$. *Then, there exists an affine position of* $K$, $t + TK$, *with* $t \in \mathbb{R}^n$ *and* $T \in SL(n)$ *such that* 0 *belongs to the interior of* $t + TK$ *and*

$$C_1^{|i|} \leqslant \frac{\tilde{W}_i(t + TK)}{L_K^{-i} |K|^{(n-i)/n} |D_n|^{i/n}} \leqslant (C_2|i|)^{|i|+1},$$

*for* $-\infty < i < 0$; *and*

$$C_1 \leqslant \frac{\tilde{W}_i(t+TK)}{L_K^{-i}|K|^{(n-i)/n}|D_n|^{i/n}} \leqslant \frac{C_2}{(1-i)^i},$$

*for* $0<i<1$, *where* $C_1$, $C_2$ *are absolute constants and* $L_K$ *is the isotropy constant of* $K$.

**Proof.** There exists $t \in \mathbb{R}^n$ and $T \in SL(n)$ such that $t+TK$ is in isotropic position (see [4,19]). Then the origin is the centroid of $t+TK$ and

$$\tilde{W}_i(t+TK) = \frac{1}{n}\int_{S^{n-1}} \rho_{t+TK}^{n-i}(u)\,d\sigma(u) = \frac{n-i}{n}\int_{t+TK} \frac{dx}{|x|^i}.$$

Since $i \in (-\infty, 1)$, by using well-known results about equivalence of moments of order $-i \in (-1, +\infty)$ of a norm on any convex body (see for instance [11,12,19,21]) we obtain that for some absolute constant $C>0$ we have

$$\begin{aligned}\frac{1+\min\{-i,0\}}{C}\left(\frac{1}{|K|}\int_{t+TK}|x|^2\,dx\right)^{1/2} &\leqslant \left(\frac{1}{|K|}\int_{t+TK}|x|^{-i}\,dx\right)^{-1/i}\\ &\leqslant C\max\{2,-i\}\left(\frac{1}{|K|}\int_{t+TK}|x|^2\,dx\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1+\min\{-i,0\}}{C}n^{1/2}L_K|K|^{1/n} &\leqslant \left(\frac{n}{n-i}\frac{\tilde{W}_i(t+TK)}{|K|}\right)^{-1/i}\\ &\leqslant C\max\{2,-i\}n^{1/2}L_K|K|^{1/n}.\end{aligned}$$

On the one hand, if $i<0$, we get that

$$\begin{aligned}C_1^{|i|} \leqslant \frac{n-i}{n}C_1^{-i} \leqslant \frac{\tilde{W}_i(t+TK)}{n^{-i/2}L_K^{-i}|K|^{\frac{n-i}{n}}} &\leqslant \frac{n-i}{n}C_2^{-i}\max\{2,-i\}^{-i}\\ &\leqslant (C_2|i|)^{|i|+1}.\end{aligned}$$

On the other hand, if $0<i<1$ and $\geqslant 2$, then

$$\begin{aligned}C_2' \leqslant \frac{n-i}{n}C_2^{-i} \leqslant \frac{\tilde{W}_i(t+TK)}{n^{-i/2}L_K^{-i}|K|_n^{\frac{n-i}{n}}} &\leqslant \frac{n-i}{n}C_1^{-i}(1-i)^{-i}\\ &\leqslant \left(\frac{C_1'}{1-i}\right)^i \leqslant \frac{C_1''}{(1-i)^i}. \qquad \square\end{aligned}$$

The estimate we obtained is sharp in the following sense. Suppose $i = -1$. Our result says that for any convex body $K$ of volume equal to 1, we can find a position such that

$$C_1 L_K |D_n|^{-1/n} \leqslant \tilde{W}_{-1}(t + TK) \leqslant C_2 L_K |D_n|^{-1/n}.$$

Furthermore, to prove that "*for any convex body $K$ of volume equal to* 1 *there exists a position such that*

$$C_1 |D_n|^{-1/n} \leqslant \tilde{W}_{-1}(t + TK) \leqslant C_2 |D_n|^{-1/n},$$

*where $C_1, C_2 > 0$ are absolute constants*" is a reformulation of the hyperplane conjecture (see [19]). Note that the case $i = -2$ would be exactly the hyperplane conjecture. Now we know that we can reformulate the hyperplane conjecture in terms of sharp estimates for the dual quermassintegrals of the convex bodies in the range $i \in (-\infty, 0) \cup (0, 1)$.

Apart from this reformulation of the hyperplane conjecture, if we consider $i \to 0$ in the last theorem, since

$$\left( \frac{1}{|K|} \int_{t+TK} |x|^{-i} \, dx \right)^{-1/i} \underset{i \to 0}{\longrightarrow} \exp\left( \frac{1}{|K|} \int_{t+TK} \log |x| \, dx \right),$$

we get that there exist affine position $t + TK$ such that

$$C_1 L_K \left( \frac{|K|}{|D_n|} \right)^{1/n} \leqslant \exp\left( \frac{1}{|s + T'K|} \int_{t+TK} \log |x| \, dx \right) \leqslant C_2 L_K \left( \frac{|K|}{|D_n|} \right)^{1/n}.$$

Moreover, if we could prove that there exists an absolute constant $C$ such that for every dimension $n$ and every convex body $K$ with $|K| = 1$ there exists an affine position $\hat{K} = s + T'K$ such that

$$\exp\left( \frac{1}{n} \int_{S^{n-1}} \rho^n_{\hat{K}}(u) \log \rho_{\hat{K}}(u) \sigma(u) \right) \leqslant C\sqrt{n}, \tag{15}$$

then we would have proved the hyperplane conjecture. Notice that inequality (15) can be understood as a reverse of an inequality proved by Milman and Pajor [19, pp. 76–77].

Next we shall study the case $i \in (n-1, n) \cup (n, \infty)$, but for convenience we let $\alpha = i - n$.

**Theorem 3.2.** *Let $K \subseteq \mathbb{R}^n$ be a convex body. There exists an affine position of $K$, $t + TK$, such that* 0 *belongs to the interior of $t + TK$ and for every $\alpha \in (-1, 0) \cup (0, \infty)$*

(i) $\tilde{W}_{n+\alpha}(t + TK) \leqslant C(\alpha) \log(n)^\alpha |K|^{\frac{-\alpha}{n}} |D_n|^{\frac{n+\alpha}{n}}$, *if* $\alpha > 0$,

(ii) $\tilde{W}_{n+\alpha}(t + TK) \geqslant C(\alpha) \log(n)^\alpha |K|^{\frac{-\alpha}{n}} |D_n|^{\frac{n+\alpha}{n}}$, *if* $-1 < \alpha < 0$,

*where $C(\alpha)$ is a constant which only depends on $\alpha$.*

**Proof.** Suppose $K$ is symmetric with respect to the origin (0-symmetric). Let

$$M(K) = \frac{1}{\sigma(S^{n-1})}\int_{S^{n-1}} ||u||_K \, d\sigma(u),$$

where $||\cdot||_K$ is the norm on $\mathbb{R}^n$ whose unit ball is $K$. We use the well-known $MM^*$-estimate and so there exists a $T \in SL(n)$ such that

$$M(TK)M((TK)^\circ) \leqslant C \log n,$$

for some absolute constant $C > 0$ (this position is known as the $\ell$-*position* or mean width position, see for instance [6,23]). Since

$$M(TK) = |D_n|^{-1}\tilde{W}_{n+1}(TK),$$

$$M((TK)^\circ) = |D_n|^{-1}\tilde{W}_{n+1}((TK)^\circ)$$

and by using (3) and the Blaschke–Santaló inequality, we get that

$$\begin{aligned}\tilde{W}_{n+1}((TK)^\circ) &\geqslant |(TK)^\circ|^{\frac{-1}{n}}|D_n|^{\frac{n+1}{n}} \\ &\geqslant |K|^{\frac{1}{n}}|D_n|^{1-\frac{1}{n}},\end{aligned}$$

so we obtain

$$\tilde{W}_{n+1}(TK) \leqslant C \log n |K|^{-\frac{1}{n}}|D_n|^{1+\frac{1}{n}}. \tag{16}$$

Consider now the general case. Let $\alpha \in (-1,0)\cup(0,\infty)$. We use the same $\ell$-position and so

$$\begin{aligned}\tilde{W}_{n+\alpha}(TK) &= \frac{1}{n}\int_{S^{n-1}} ||u||_{TK}^{\alpha}\, d\sigma(u) \\ &\simeq C(\alpha)\frac{|D_n|}{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}\int_{\mathbb{R}^n} ||x||_{TK}^{\alpha}\, d\gamma_n(x),\end{aligned}$$

where $A \simeq B$ means here that the quotient $A/B$ is bounded from above and from below by absolute constants, $C(\alpha)$ is a constant depending on $\alpha$ and $d\gamma_n(x)$ is the canonical Gaussian probability on $\mathbb{R}^n$. Indeed, by using polar coordinates

$$\begin{aligned}\int_{\mathbb{R}^n} ||x||_{TK}^{\alpha}\, d\gamma_n(x) &= \frac{2^{\frac{\alpha-2}{2}}\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2}+1)|D_n|}\int_{S^{n-1}} ||u||_{TK}^{\alpha}\, d\sigma(u) \\ &\simeq C\frac{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}{e^{\frac{\alpha-2}{2}}\sigma(S^{n-1})}\left(\frac{n+\alpha-2}{n}\right)^{\frac{n}{2}}\int_{S^{n-1}} ||u||_{TK}^{\alpha}\, d\sigma(u) \\ &\simeq C(\alpha)\frac{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}{\sigma(S^{n-1})}\int_{S^{n-1}} ||u||_{TK}^{\alpha}\, d\sigma(u).\end{aligned}$$

It is well known that the canonical Gaussian probability is log-concave and the moments of order $\alpha \in (-1, \infty)$ of a norm with respect to log-concave measures are equivalent up to an absolute constant (see [12]), i.e. there exists an absolute constant $C > 0$ such that

$$C^{-1} \int_{\mathbb{R}^n} ||x|| \, d\gamma_n(x) \leqslant \left( \int_{\mathbb{R}^n} ||x||^\alpha \, d\gamma_n(x) \right)^{1/\alpha} \leqslant C \max\{1, \alpha\} \int_{\mathbb{R}^n} ||x|| \, d\gamma_n(x).$$

Then, if $\alpha > 0$

$$\begin{aligned}
\tilde{W}_{n+\alpha}(TK) &\leqslant C(\alpha) \frac{|D_n|}{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}} \left( \int_{\mathbb{R}^n} ||x||_{TK} \, d\gamma_n(x) \right)^\alpha \\
&\leqslant C(\alpha) \frac{|D_n|}{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}} \left( \frac{\sqrt{n}}{\sigma(S^{n-1})} \int_{S^{n-1}} ||u||_{TK} \, d\sigma(u) \right)^\alpha \\
&\leqslant C(\alpha) \left( \frac{n}{n+\alpha-2} \right)^{\frac{\alpha-1}{2}} |D_n|^{1-\alpha} (\tilde{W}_{n+1}(TK))^\alpha \\
&\leqslant C(\alpha) \left( \frac{n}{n+\alpha-2} \right)^{\frac{\alpha-1}{2}} (\log n)^\alpha |K|^{-\alpha/n} |D_n|^{1+\alpha/n} \\
&\leqslant C(\alpha) (\log n)^\alpha |K|^{-\alpha/n} |D_n|^{1+\alpha/n}.
\end{aligned}$$

The case $-1 < \alpha < 0$ is similar.

For a general convex body $K$, we can assume that its centroid is at the origin. Therefore $\emptyset \neq K \cap (-K) \subseteq K$ and since $K \cap (-K)$ is a symmetric convex body, there exist $T \in SL(n)$ such that

$$\begin{aligned}
\tilde{W}_{n+\alpha}(TK) \leqslant \tilde{W}_{n+\alpha}(T(K \cap (-K))) &\leqslant C(\alpha) \log(n)^\alpha |K \cap (-K)|^{-\frac{\alpha}{n}} |D_n|^{\frac{n+\alpha}{n}} \\
&\leqslant C(\alpha) \log(n)^\alpha |K|^{-\frac{\alpha}{n}} |D_n|^{\frac{n+\alpha}{n}},
\end{aligned}$$

since $|K|^{1/n} \leqslant 2 |K \cap (-K)|^{1/n}$ (see [20]). $\square$

If we want to study the case $i \in (1, n-1)$, a trivial calculation shows that, by using maximal volume positions, the dual Minkowski inequality can be reversed, but the inequality obtained is far from being sharp. Instead of doing such a straightforward computation, we will investigate the case of $B_p^n$ balls where we can go further and get sharper estimates as the following result shows.

**Proposition 3.3.** *Let $1 \leqslant p \leqslant \infty$. There exists $C_p > 0$ such that for every $n \in \mathbb{N}$ and every $0 < i < n$*

$$\tilde{W}_i(B_p^n) \geqslant C_p^{\min\{i, n-i\}} |D_n|^{i/n} \, |B_p^n|^{(n-i)/n}.$$

**Proof.** It is easy to check that for every star body $K \subseteq \mathbb{R}^n$, if $0 < i < n$

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i}\, d\sigma(u) = \frac{n-i}{n}\int_0^{+\infty} |\{x \in K;\ \ |x|^{-i} > t\}|\, dt$$
$$= \frac{(n-i)i}{n}\int_0^{+\infty} |K \cap sD_n|\, s^{-i-1}\, ds.$$

So we only have to give lower estimates for $|B_p^n \cap sD_n|$.

If $1 \leqslant p \leqslant 2$,

$$\tilde{W}_i(B_p^n) \geqslant \frac{(n-i)i}{n}\left[\int_0^{n^{1/2-1/p}} |D_n| s^{n-i-1}\, ds + \int_{n^{1/2-1/p}}^1 |B_p^n \cap sD_n| s^{-i-1}\, ds\right]$$
$$= \frac{(n-i)i}{n}\left[\frac{1}{n-i}|D_n|(n^{1/2-1/p})^{n-i} + \int_{n^{1/2-1/p}}^1 |B_p^n \cap sD_n| s^{-i-1}\, ds\right].$$

Notice that

$$\int_{n^{1/2-1/p}}^1 |B_p^n \cap sD_n| s^{-i-1}\, ds \geqslant \int_{2n^{1/2-1/p}}^1 |B_p^n \cap sD_n| s^{-i-1}\, ds$$
$$= \int_{2n^{1/2-1/p}}^1 |B_p^n| \left(1 - \frac{|\{x \in B_p^n;\ \ |x| > s\}|}{|B_p^n|}\right) s^{-i-1}\, ds,$$

hence, by using the estimates of the volume of the intersection of two $\ell_p^n$ balls (see [24]) we get that

$$\int_{n^{1/2-1/p}}^1 |B_p^n \cap sD_n| s^{-i-1}\, ds \geqslant \int_{2n^{1/2-1/p}}^1 |B_p^n| \left(1 - \frac{\exp(-cs^p n)}{|B_p^n|}\right) s^{-i-1}\, ds$$
$$\geqslant C_p |B_p^n| \int_{2n^{1/2-1/p}}^1 s^{-i-1}\, ds.$$

Therefore,

$$\tilde{W}_i(B_p^n) \geqslant |D_n|^{i/n} |B_p^n|^{(n-i)/n} \left[\frac{i}{n}(n^{1/2-1/p})^{n-i}\left(\frac{|D_n|}{|B_p^n|}\right)^{(n-i)/n}\right.$$
$$\left. + \ C_p \frac{n-i}{n}\left(\frac{|B_p^n|}{|D_n|}\right)^{i/n} \frac{n^{i/p-i/2}}{2^i}\right]$$
$$\geqslant |D_n|^{i/n} |B_p^n|^{(n-i)/n}\left[\frac{i}{n} C_p^{n-i} + \frac{n-i}{n} C_p^i\right].$$

Now it is easy to check that if $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and $0 < x < 1$, then

$$x^{\min\{\alpha,\beta\}} \geqslant \alpha x^{\beta} + \beta x^{\alpha} \geqslant \tfrac{1}{2} x^{\min\{\alpha,\beta\}},$$

hence

$$\tilde{W}_i(B_p^n) \geqslant |D_n|^{i/n} |B_p^n|^{(n-i)/n} \left[ \frac{i}{n} C_p^{n-i} + \frac{n-i}{n} C_p^i \right] \geqslant \frac{1}{2} |D_n|^{i/n} |B_p^n|^{(n-i)/n} C_p^{\min\{i,n-i\}}.$$

On the other hand, if $2 \leqslant p < \infty$,

$$\begin{aligned} \tilde{W}_i(B_p^n) &\geqslant \frac{(n-i)i}{n} \left[ \int_1^{n^{1/2-1/p}} |B_p^n \cap s D_n| \, s^{-i-1} \, ds + \int_{n^{1/2-1/p}}^{+\infty} |B_p^n| s^{-i-1} \, ds \right] \\ &= \frac{(n-i)i}{n} \left[ \int_1^{n^{1/2-1/p}} |B_p^n \cap s D_n| \, s^{-i-1} \, ds + \frac{1}{i} |B_p^n| (n^{1/p-1/2})^i \right]. \end{aligned}$$

Now, since

$$\begin{aligned} |B_p^n \cap s D_n| &= |\{x \in s D_n; \ \ ||x||_p \leqslant 1\}| \\ &= s^n |D_n| \left( 1 - \frac{|\{y \in D_n; \ \ ||y||_p > s^{-1}\}|}{|D_n|} \right), \end{aligned}$$

by using the estimates of the intersection of two $\ell_p^n$ balls given by Schechtman and Zinn [24] we can assert that if $n$ is big enough

$$\begin{aligned} \int_1^{n^{1/2-1/p}} |B_p^n \cap s D_n| \, s^{-i-1} \, ds &\geqslant \int_{\frac{1}{2} n^{1/2-1/p}}^{n^{1/2-1/p}} s^{n-i-1} |D_n| \left( 1 - \frac{\exp(-c \frac{n}{s^2})}{|D_n|} \right) ds \\ &\geqslant C |D_n| \int_{\frac{1}{2} n^{1/2-1/p}}^{n^{1/2-1/p}} s^{n-i-1} \, ds. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{W}_i(B_p^n) &\geqslant |D_n|^{i/n} |B_p^n|^{(n-i)/n} \left[ C_p \frac{i}{n} \left( \frac{|D_n|}{|B_p^n|} \right)^{(n-i)/n} (n^{1/2-1/p})^{n-i} \right. \\ &\quad \left. + \frac{n-i}{n} \left( \frac{|B_p^n|}{|D_n|} \right)^{i/n} n^{i/p-i/2} \right] \\ &\geqslant |D_n|^{i/n} |B_p^n|^{(n-i)/n} \left[ \frac{n-i}{n} C_p^i + \frac{i}{n} C_p^{n-i} \right] \\ &\geqslant \frac{1}{2} |D_n|^{i/n} |B_p^n|^{(n-i)/n} C_p^{\min\{i,n-i\}}. \end{aligned}$$

Note that if $n$ is not "*big enough*" (i.e. $1 \leqslant n \leqslant n_0$), we can obtain the same inequality as before simply by adjusting the constant $C_p$. The case $p = \infty$ can be proved as before but by considering $1/p = 0$. $\square$

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