IWAHORI-HECKE ALGEBRAS

THOMAS J. HAINES, ROBERT E. KOTTWITZ, AND AMRITANSHU PRASAD

Our aim here is to give a fairly self-contained exposition of some basic facts about the Iwahori-Hecke algebra H of a split p-adic group, including Bernstein's presentation and description of the center, Macdonald's formula, the Casselman-Shalika formula, and the Kato-Lusztig formula. There are no new results here, and the same is essentially true of the proofs. We have been strongly influenced by the notes [1] of a course given by Bernstein.

The reader may find in [21] another survey article which proves some of the results of the present paper by different methods.

The following notation will be used throughout this paper. We work over a p-adic field F with valuation ring $\mathcal{O}$ and prime ideal $P = (\pi)$. We denote by k the residue field $\mathcal{O}/P$ and by q the cardinality of k.

Consider a split connected reductive group G over F, with split maximal torus A and Borel subgroup $B = AN$ containing A. We write $\bar{B} = A\bar{N}$ for the Borel subgroup containing A that is opposite to B. We assume that G, A, N are defined over $\mathcal{O}$. We write K for $G(\mathcal{O})$ and I for the Iwahori subgroup of K defined as the inverse image under $G(\mathcal{O}) \to G(k)$ of $B(k)$. For $\mu \in X_*(A)$ we write π^μ for the element $\mu(\pi) \in A(F)$. Note that $\mu \mapsto \pi^\mu$ gives an isomorphism from $X_*(A)$ to $A/A_{\mathcal{O}}$. (We will often abbreviate $A(F)$ to A and $A(\mathcal{O})$ to $A_{\mathcal{O}}$, etc.)

1. Bernstein's presentation [17]

1.1. Extended affine Weyl group. The extended affine Weyl group $\tilde{W}$ is the quotient of $N_{G(F)}(A)$ by $A_{\mathcal{O}}$. Thus $\tilde{W}$ contains the translation subgroup $A/A_{\mathcal{O}} = X_*(A)$, as well as the finite Weyl group W, which we realize inside $\tilde{W}$ as the quotient of $N_K(A)$ by $A_{\mathcal{O}}$. Recall that $\tilde{W}$ is the semidirect product of W and $X_*(A)$. In the early part of the paper, when we are thinking about a cocharacter μ as an element of the translation subgroup of $\tilde{W}$, we denote it by π^μ; later on we often denote it instead by t_μ.

1.2. Iwahori-Hecke algebra H. We denote by H the Iwahori-Hecke algebra $C_c(I\backslash G/I)$. The convolution product is defined using the Haar measure giving I measure 1. The elements $T_x := 1_{IxI}$ $(x \in \tilde{W})$ form a $\mathbf{C}$-basis for H. (Throughout the paper we write the characteristic function of a subset S as 1_S.)

1.3. Double cosets $A_{\mathcal{O}}N\backslash G/I = \tilde{W}$. The obvious map $\tilde{W} \to A_{\mathcal{O}}N\backslash G/I$ is a bijection. How does this decomposition work? Write $g \in G$ as $g = \pi^\mu nk$. Then write $k = n_{\mathcal{O}}wi$ with $n_{\mathcal{O}} \in N_{\mathcal{O}}$, $i \in I$, $w \in W$, with w inside K. Thus $g = \pi^\mu nn_{\mathcal{O}}wi$, showing that g lies in the double coset of $\pi^\mu w \in \tilde{W}$. In short, we read off π^μ from the Iwasawa decomposition and then read off $w \in W$ by applying the Bruhat decomposition over the residue field to the element k produced from the Iwasawa decomposition.

1.4. **Definition of the module M.** Put $M := C_c(A_{\mathcal{O}}N\backslash G/I)$. Note that M is a right H-module (since it arises as the I-fixed vectors in the smooth G-module considered in (1.5.1) below).[1] For $x \in \tilde{W}$ we denote by v_x the characteristic function $1_{A_{\mathcal{O}}NxI}$. The elements v_x ($x \in \tilde{W}$) form a $\mathbf{C}$-basis for M. Of special importance (see Lemma 1.6.1 below) is the basis element $v_1 = 1_{A_{\mathcal{O}}NI}$. Let $R = C_c(A/A_{\mathcal{O}}) = \mathbf{C}[X_*]$, the group algebra of X_*, an abbreviation for $X_*(A)$. Thus the elements π^μ ($\mu \in X_*$) form a basis for the vector space R. We make M into a left R-module as follows. Let $\mu \in X_*$ and let $x \in \tilde{W}$. Then $\pi^\mu \cdot v_x := q^{-\langle\rho,\mu\rangle} v_{\pi^\mu \cdot x}$, where ρ is half the sum of the roots of A in $\mathrm{Lie}(N)$. (Note that the scalar $q^{-\langle\rho,\mu\rangle}$ is equal to $\delta_B(\pi^\mu)^{1/2}$, where for $a \in A$, $\delta_B(a)$ denotes the absolute value of the determinant of the adjoint action of a on $\mathrm{Lie}(N)$.) The actions of R and H commute, so that M is an (R,H)-bimodule.

1.5. **Second point of view on the module M.** Consider the representation (by right translations) of G on $C_c^\infty(A_{\mathcal{O}}N\backslash G)$. It is compactly induced from the trivial representation of $A_{\mathcal{O}}N$; doing the induction in stages, we see that

$$C_c^\infty(A_{\mathcal{O}}N\backslash G) = i_B^G(R). \tag{1.5.1}$$

Here we are using normalized induction and R is viewed as A-module via $\chi_{\mathrm{univ}}^{-1}$, where χ_{univ} is the tautological character $A/A_{\mathcal{O}} \to R^\times$ mapping π^μ to π^μ. Thus an element in this induced representation is a locally constant R-valued function ϕ on G satisfying

$$\phi(ang) = \delta_B(a)^{1/2} \cdot a^{-1} \cdot \phi(g)$$

for all $a \in A$, $n \in N$, $g \in G$, and the group G acts by right translations. The isomorphism (1.5.1) has the following explicit description. Let $\varphi \in C_c^\infty(A_{\mathcal{O}}N\backslash G)$. Then the corresponding element $\phi \in i_B^G(R)$ is defined by

$$\phi(g) = \sum_{a \in A/A_{\mathcal{O}}} \delta_B(a)^{-1/2} \varphi(ag) \cdot a \tag{1.5.2}$$

for $g \in G$. There is an obvious R-module structure on $i_B^G(R)$, with $r\phi$ given by $(r\phi)(g) = r(\phi(g))$. The isomorphism (1.5.1) induces an (R,H)-bimodule isomorphism from M to the Iwahori fixed vectors in $i_B^G(R)$.

Let χ denote a quasicharacter $\chi : A/A_{\mathcal{O}} \to \mathbf{C}^\times$. Then χ determines a $\mathbf{C}$-algebra homomorphism $R \to \mathbf{C}$. Using this homomorphism to extend scalars, we obtain the H-module

$$\mathbf{C} \otimes_R M = \mathbf{C} \otimes_R i_B^G(\chi_{\mathrm{univ}}^{-1})^I = i_B^G(\chi^{-1})^I.$$

1.6. **Structure of the module M.** The next result is due to Chriss and Khuri-Makdisi [8], who derived it from Bernstein's presentation of the Iwahori-Hecke algebra. Here we turn the logic around, first studying M directly, then using it to produce Bernstein's presentation.

Lemma 1.6.1. *The map $h \mapsto v_1 h$ is an isomorphism of right H-modules from H to M. In other words M is free of rank 1 as H-module with canonical generator v_1. In particular we have a canonical isomorphism $H \simeq \mathrm{End}_H(M)$, which identifies $h' \in H$ with the endomorphism $v_1 h \mapsto v_1 h' h$ of M.*

[1] More explicitly, the action of $h \in H$ on $m \in M$ is given by the convolution $m \cdot h$ of the two functions, where convolution is defined using the Haar measure on G giving I measure 1.

Proof. It suffices to show that the map $h \mapsto v_1 h$, written in terms of the bases $\{T_w\}_w$ and $\{v_w\}_w$, is "a triangular matrix with non-zero diagonal". This follows from the following claim.

Claim: $NxI \cap IyI \neq \emptyset \Rightarrow x \leq y$ in the Bruhat ordering.[2]

Proof of Claim: Suppose $nx \in IyI$, for $n \in N$. Choose μ so dominant that $\pi^\mu n \pi^{-\mu} \in I$. Then $(\pi^\mu n \pi^{-\mu})\pi^\mu x \in \pi^\mu IyI$, hence

$$I\pi^\mu xI \subset I\pi^\mu IyI \subset \coprod_{y' \leq y} I\pi^\mu y' I,$$

from which the claim follows.

□

The following three equalities (see [8], [23]) are also useful.

$$v_1 T_w = v_w, \text{ for every } w \in W, \tag{1.6.1}$$

$$v_{\pi^\mu} T_w = v_{\pi^\mu w}, \text{ for every } w \in W \text{ and } \mu \in X_*(A), \tag{1.6.2}$$

$$v_1 T_{\pi^\mu} = v_{\pi^\mu}, \text{ for } \mu \in X_*(A) \text{ dominant.} \tag{1.6.3}$$

Recall the Iwahori factorization $I = (I \cap N)A_{\mathcal{O}}(I \cap \bar{N})$. The first equality uses $A_{\mathcal{O}}NI \cdot IwI = A_{\mathcal{O}}NwI$ (a consequence of the Iwahori factorization) as well as $A_{\mathcal{O}}NI \cap wIw^{-1}I = I$ (a consequence of $A_{\mathcal{O}}NI \cap K = I$), and the second equality follows from the first (using the left R-module structure on M). The third equality uses $A_{\mathcal{O}}NI \cdot I\pi^\mu I = A_{\mathcal{O}}N\pi^\mu I$, a consequence of the Iwahori factorization and the dominance of μ, which implies that

$$\pi^\mu(I \cap N)\pi^{-\mu} \subset I \cap N \text{ and } \pi^{-\mu}(I \cap \bar{N})\pi^\mu \subset I \cap \bar{N},$$

and also uses $A_{\mathcal{O}}NI \cap \pi^\mu I\pi^{-\mu}I = I$, which we leave as an exercise for the reader.

1.7. **Rough structure of the algebra H.** The finite dimensional Hecke algebra $H_0 = C(I\backslash K/I)$ is a subalgebra of H. Moreover, elements in R can be viewed as endomorphisms of M, and hence by the previous lemma can be considered as elements in H. In this way we embed R as a subalgebra of H. We will denote by $\Theta_\lambda \in H$ the image of the basis element π^λ of R under the embedding $R \hookrightarrow H$. Unwinding the definitions, one finds the basic identity

$$v_1 \Theta_\lambda = \pi^\lambda v_1, \tag{1.7.1}$$

which says that v_1 is an eigenvector for the right action of the subalgebra R of H.

Lemma 1.7.1. *Multiplication in H induces a vector space isomorphism*

$$R \otimes_{\mathbf{C}} H_0 \xrightarrow{\approx} H,$$

sending $\pi^\mu \otimes h$ to $\Theta_\mu h$. Composing this isomorphism with the isomorphism $h \mapsto v_1 h$ considered above, we get a vector space isomorphism from $R \otimes_{\mathbf{C}} H_0$ to M, sending $\pi^\mu \otimes T_w$ to $q^{-\langle\rho,\mu\rangle} v_{\pi^\mu w}$.

---

[2]The Iwahori subgroup I determines in a canonical way the Bruhat order on $\tilde{W}$, but only when $\tilde{W}$ is viewed as $N_G(A)/A_{\mathcal{O}}$. When $\tilde{W}$ is viewed as the semidirect product of W and $X_*(A)$, the Bruhat order depends on the normalization of the isomorphism between $X_*(A)$ and $A/A_{\mathcal{O}}$. Our normalization is $\mu \mapsto \pi^\mu$, and therefore our Bruhat order on $X_*(A) \rtimes W$ is the one determined by the simple affine reflections about the walls of the unique alcove in $X_*(A)$ whose closure contains the origin and lies in the *negative* Weyl chamber. See section 7.

Proof. Using (1.6.1) and the definitions, one checks that the composition

$$R \otimes_{\mathbf{C}} H_0 \to H \to M$$

sends $\pi^\mu \otimes T_w$ to $q^{-\langle \rho, \mu \rangle} v_{\pi^\mu w}$ and is hence an isomorphism. Since $H \to M$ is an isomorphism by Lemma 1.6.1, the map $R \otimes_{\mathbf{C}} H_0 \to H$ is also an isomorphism. □

Remark 1.7.2. *It follows from* (1.6.3) *that* Θ_λ *agrees with the element denoted by this symbol in Lusztig's work: namely,* $\Theta_\lambda = q^{\langle \rho, -\lambda_1 + \lambda_2 \rangle} T_{\pi^{\lambda_1}} T^{-1}_{\pi^{\lambda_2}}$, *where* $\lambda = \lambda_1 - \lambda_2$, *and* λ_1, λ_2 *are dominant cocharacters.*

1.8. **Involutions on R and H.** Recall that in order to pass back and forth between left and right G-modules one uses the anti-isomorphism $g \mapsto g^{-1}$ from G to itself. The corresponding way of passing from left to right H-modules uses the standard anti-involution ι on H given by $\iota(h)(x) = h(x^{-1})$.

Moreover there is also an involution ι_A on R (which is the Iwahori-Hecke algebra for A); thus ι_A sends π^μ to $\pi^{-\mu}$.

1.9. **A sesquilinear form on M.** There is an R-valued pairing on $i_B^G(\chi_{\mathrm{univ}}^{-1})$, defined by

(1.9.1) $$(\phi_1, \phi_2) := \oint_{B \backslash G} \iota_A(\phi_1(g)) \phi_2(g).$$

What is the meaning of $\oint_{B\backslash G}$? Consider the induced representation $i_B^G(\delta_B^{1/2})$, which consists of locally constant functions F on G satisfying

$$F(ang) = \delta_B(a) F(g).$$

The space of G-invariant linear functionals on $i_B^G(\delta_B^{1/2})$ is 1-dimensional; we denote by $\oint_{B\backslash G}$ the unique such functional that takes the value 1 on the function $F_0 \in i_B^G(\delta_B^{1/2})$ defined by $F_0(ank) = \delta_B(a)$.

This pairing is sesquilinear, in the sense that

(1.9.2) $$(r_1\phi_1, r_2\phi_2) = \iota_A(r_1) r_2 \cdot (\phi_1, \phi_2).$$

Moreover it satisfies

(1.9.3) $$(\phi_2, \phi_1) = \iota_A(\phi_1, \phi_2)$$

and is G-invariant.

Note that $\phi \mapsto \iota_A \circ \phi$ is an ι_A-linear isomorphism from $i_B^G(\chi_{\mathrm{univ}}^{-1})$ to $i_B^G(\chi_{\mathrm{univ}})$. Therefore our sesquilinear form can also be thought of as an R-bilinear pairing

(1.9.4) $$i_B^G(\chi_{\mathrm{univ}}) \otimes_R i_B^G(\chi_{\mathrm{univ}}^{-1}) \to R.$$

After extending scalars $R \to \mathbf{C}$ using a quasicharacter $\chi : A/A_{\mathcal{O}} \to \mathbf{C}^\times$, the pairing (1.9.4) becomes the standard pairing

(1.9.5) $$i_B^G(\chi) \otimes_{\mathbf{C}} i_B^G(\chi^{-1}) \to \mathbf{C}.$$

Recall that we have identified M with the Iwahori fixed vectors in $i_B^G(\chi_{\mathrm{univ}}^{-1})$. Thus, by restriction, we get a perfect sesquilinear form on M, which we denote by (m_1, m_2). It satisfies the Hecke algebra analog of G-invariance, namely

(1.9.6) $$(m_1 h, m_2) = (m_1, m_2 \iota(h))$$

for all $h \in H$.

1.10. **Generalities on intertwiners.** For each $w \in W$ we would like to define an intertwiner I_w from one suitable completion of M to another. To this end it is best to let the Borel subgroup vary (and then recover I_w by bringing the second Borel back to the first by an element of the Weyl group). In this discussion the maximal torus A, the Iwahori subgroup I, and the maximal compact K will remain fixed. We let $\mathcal{B}(A)$ denote the set of Borel subgroups containing A. For $B = AN \in \mathcal{B}(A)$, put $M_B = C_c(A_{\mathcal{O}}N\backslash G/I)$.

First let's discuss the completions that will come up. Let J be a set of coroots which is a subset of some system of positive coroots. As usual R denotes the group algebra of $X_*(A)$. We denote by $\mathbf{C}[J]$ the $\mathbf{C}$-subalgebra of R generated by J, and by $\mathbf{C}[J]\hat{}$ the completion of $\mathbf{C}[J]$ with respect to the (maximal) ideal generated by J. Finally, we denote by R_J the R-algebra $\mathbf{C}[J]\hat{}\otimes_{\mathbf{C}[J]} R$, a completion of R that can be viewed as the convolution algebra of complex valued functions on $X_*(A)$ supported on a finite union of sets of the form $x + C_J$, with $x \in X_*(A)$ and where C_J is the submonoid of $X_*(A)$ consisting of all non-negative integral linear combinations of elements in J.

Given $B = AN \in \mathcal{B}(A)$ and given J as above, we then denote by $M_{B,J}$ the module $R_J \otimes_R M_B$, which can be thought of as consisting of functions f on $A_{\mathcal{O}}N\backslash G/I$ satisfying the following support condition: there exists a finite union S of sets of the form $x + C_J$ such that the support of f is contained in the union of the sets $A_{\mathcal{O}}N\pi^\nu K$ for $\nu \in S$. It is clear that $M_{B,J}$ is a left R_J-module and a right H-module.

Now let $B = AN$, $B' = AN'$ be two Borel subgroups in $\mathcal{B}(A)$. As usual we write $\bar{B} = A\bar{N}$ for the Borel subgroup in $\mathcal{B}(A)$ opposite to B. Let J be the set of coroots that are positive for B' and negative for B. We are going to define an intertwiner $I_{B',B} : M_{B,J} \to M_{B',J}$. This intertwiner is an (R_J, H)-bimodule map, and is defined as follows (viewing elements in completions as functions, as above). Let $\varphi \in M_{B,J}$. Then the intertwiner $I_{B',B}$ takes φ to the function φ' on $A_{\mathcal{O}}N'\backslash G/I$ whose value at $g \in G$ is defined by the integral

$$\varphi'(g) = \int_{N'\cap\bar{N}} \varphi(n'g)dn'.$$

The Haar measure dn' is normalized to give $N' \cap \bar{N} \cap K$ measure 1. Note that the integral makes sense since the integrand is a smooth and compactly supported function on the group $N' \cap \bar{N}$ (smoothness being trivial, compact support requiring justification, to be done in the lemma below). In fact things still work fine if we enlarge J in any way (but so that the enlarged set is still contained in some positive system, for instance, the positive system defined by B').

Now suppose that we have three Borel subgroups $B_1 = AN_1$, $B_2 = AN_2$, $B_3 = AN_3$ in $\mathcal{B}(A)$. Let J_{ij} be the set of coroots that are positive for B_i and negative for B_j, and assume that J_{31} is the disjoint union of J_{21} and J_{32}. Write I_{ij} as an abbreviation for the intertwiner I_{B_i,B_j}. Then I_{21}, I_{32} and I_{31} can all be defined using the biggest of the three sets J_{ij}, namely J_{31}, and when this is done we have the equality

$$I_{31} = I_{32}I_{21}.$$

In this formula we could also have taken J to be the set of all coroots that are positive for B_3.

Why do the integrals make sense? For this we need the following lemma, in which we return to B, B' as above.

Lemma 1.10.1. *For $\nu \in X_*(A)$ define a subset C_ν of the group $N' \cap \bar{N}$ by $C_\nu := N' \cap \bar{N} \cap \pi^\nu NK$. Then:*

(1) *If C_ν is non-empty, then ν is a non-negative integral linear combination of coroots that are positive for B and negative for B'.*
(2) *The subset C_ν is compact.*

Proof. We begin by recalling the definition of the retraction $r_B : G \to X_*(A)$. Let $g \in G$ and use the Iwasawa decomposition to write $g = \pi^\mu nk$ for some $\mu \in X_*(A)$, $n \in N$, $k \in K$; then put $r_B(g) := \mu$. It is well-known that $r_{B'}(g) - r_B(g)$ is a non-negative integral linear combination of coroots that are positive for B' and negative for B. (It is enough to prove this for adjacent B, B', for which a simple computation in $SL(2)$ does the job.)

To prove the first statement we consider an element $g \in C_\nu$. It is clear from the definition of C_ν that $r_{B'}(g) = 0$ and $r_B(g) = \nu$. Therefore $\nu = r_B(g) - r_{B'}(g)$ is a non-negative integral linear combination of coroots that are positive for B and negative for B'.

Now we turn to the second statement. It is enough to prove that $\bar{N} \cap NC$ is compact for any compact subset C of G, which is equivalent to proving that the map $\bar{N} \to N\backslash G$ is proper (in the topological sense). But in fact $\bar{N} \to N\backslash G$ is a closed immersion (in the algebraic sense), as follows from the fact that $N\bar{N}$ is closed in G. Recall the proof of this: For any dominant weight λ there exists a unique regular function f_λ on the algebraic variety G such that $f_\lambda(na\bar{n}) = \lambda(a)^{-1}$ for all $n \in N$, $a \in A$, $\bar{n} \in \bar{N}$. Then $N\bar{N}$ is the closed subvariety defined by the equations $f_\lambda = 1$ (one for every dominant λ). □

Next we need to understand how the intertwiners behave with respect to the sesquilinear form on M_B. Denote by $-J$ the set of negatives of the coroots in J. The involution ι_A on R extends to an isomorphism, still denoted ι_A, between R_J and R_{-J}, and the sesquilinear form $(\cdot,\cdot)$ on M_B extends to our completions in the following sense: given $m_1 \in M_{B,-J}$ and $m_2 \in M_{B,J}$ our old definition of (m_1, m_2) still makes sense and yields an element of R_J. The extended form $(\cdot,\cdot)$ still satisfies (1.9.2).

Consider the intertwiner $I_{B',B} : M_{B,J} \to M_{B',J}$, where J denotes (as before) the set of coroots that are positive for B' and negative for B. We also have the intertwiner $I_{B,B'} : M_{B',-J} \to M_{B,-J}$. Let $m \in M_{B,J}$ and $m' \in M_{B',-J}$. Then we claim that

$$(m', I_{B',B}m) = (I_{B,B'}m', m). \tag{1.10.1}$$

Indeed, let ϕ, ϕ' be the elements of $i_B^G(\chi_{\text{univ}}^{-1}) \otimes_R R_J$, $i_{B'}^G(\chi_{\text{univ}}) \otimes_R R_J$ corresponding to m, m' respectively. Put $H := A(N \cap N')$. Then one sees easily that both sides of the last equality are equal to

$$\oint_{H\backslash G} \phi'(g)\phi(g), \tag{1.10.2}$$

where $\oint_{H\backslash G}$ is the unique G-invariant linear functional on

$$\{f \in C^\infty(G) : f(hg) = \delta_H(h)f(g) \quad (\forall h \in H), \text{ compactly supported mod } H\}$$

that takes the value 1 on the function f_0 supported on HK whose values on HK are given by $f_0(hk) = \delta_H(h)$.

1.11. **Intertwiners I_w.** We return now to the earlier notation, where $B = AN$ is a fixed Borel subgroup. For each $w \in W$, we define an intertwiner

$$I_w : M_{B,w^{-1}J} \to M_{B,J}$$

as the composition $I_{B,wB}L(w)$. Here $L(w)$ is the isomorphism $M_{B,w^{-1}J} \xrightarrow{\sim} M_{wB,J}$ given by $(L(w)\phi)(g) = \phi(\dot{w}^{-1}g)$, where $\dot{w}$ is a representative for w taken in K. Thus I_w is defined by the integral

$$I_w(\varphi)(g) = \int_{N_w} \varphi(\dot{w}^{-1}ng)\,dn, \tag{1.11.1}$$

where N_w denotes $N \cap w\bar{N}w^{-1}$.

From the discussion above, the following properties are immediate.

Lemma 1.11.1. *We have*

(i) $I_w \circ \pi^\mu = \pi^{w\mu} \circ I_w$, $\forall \mu \in X_*(A)$,
(ii) $I_{w_1w_2} = I_{w_1} \circ I_{w_2}$, *if* $l(w_1w_2) = l(w_1) + l(w_2)$,
(iii) I_w *is a right* H*-module homomorphism.*

1.12. **Intertwiners in the rank 1 case.** We suppose for the moment that G has semisimple rank 1. We write α for the unique positive root of A, and s_α for the corresponding simple reflection, in this case the unique non-trivial element in W.

Now we compute $\varphi' = I_{s_\alpha}(\varphi)$ for $\varphi = v_1 = 1_{A_\mathcal{O}NI}$. We write $J(j,w)$ ($j \in \mathbf{Z}$, $w \in W$) for the value of φ' at the element $\pi^{j\alpha^\vee}w$. Note that other values of φ are 0 and also that $J(j,w) = 0$ unless $j \geq 0$, which we now assume. At this point we may as well take $G = SL(2)$. To simplify notation we temporarily write μ for $j\alpha^\vee$.

First suppose that $j = 0$. Note that $s_\alpha nw \in A_\mathcal{O}NK$ iff $n \in N_\mathcal{O}$. For $n \in N_\mathcal{O}$ the element $s_\alpha nw$ belongs to K and hence belongs to $A_\mathcal{O}NI$ iff its lower left entry is in the prime ideal in $\mathcal{O}$. We conclude that $J(0,1) = 0$ and that $J(0,s_\alpha) = q^{-1}$.

Suppose $j > 0$. We have $s_\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $n = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $\pi^\mu = \begin{bmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{bmatrix}$, so that

$$s_\alpha n \pi^\mu = \begin{bmatrix} 0 & -\pi^{-j} \\ \pi^j & x\pi^{-j} \end{bmatrix}.$$

For $s_\alpha n\pi^\mu w$ to lie in $A_\mathcal{O}NK$, we must have $x \in \pi^j\mathcal{O}^\times$. We now assume this and write $x = \pi^j u$ for some unit u. Then $s_\alpha n\pi^\mu = \begin{bmatrix} u^{-1} & -\pi^{-j} \\ 0 & u \end{bmatrix}\begin{bmatrix} 1 & 0 \\ u^{-1}\pi^j & 1 \end{bmatrix}$, the first factor lying in $A_\mathcal{O}N$, the second factor lying in K. Therefore $s_\alpha n\pi^\mu \in A_\mathcal{O}NI$ iff the second factor lies in I, which is always the case. Therefore $J(j,1)$ is the measure of $\pi^j\mathcal{O}^\times$, namely $q^{-j}(1-q^{-1})$.

Moreover $s_\alpha n\pi^\mu s_\alpha \in A_\mathcal{O}NI$ iff the product of the second factor and s_α, namely $\begin{bmatrix} 0 & -1 \\ 1 & -u^{-1}\pi^j \end{bmatrix}$, lies in I, which never happens. Therefore $J(j,s_\alpha) = 0$.

We have proved:

Lemma 1.12.1. $\varphi' = q^{-1}v_{s_\alpha} + (1-q^{-1})\sum_{j=1}^\infty q^{-j}v_{\pi^{j\alpha^\vee}}$.

Even easier:

Lemma 1.12.2. *The intertwiner sends* $1_{A_\mathcal{O}NK}$ *to*

$$q^{-1}1_{A_\mathcal{O}NK} + \sum_{j=0}^\infty q^{-j}(1-q^{-1})1_{A_\mathcal{O}N\pi^{j\alpha^\vee}K} = \frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}1_{A_\mathcal{O}NK}.$$

1.13. **Consequences of the calculations above.** Now we return to the general case. In the next lemma the calculations reduce easily to the rank 1 case treated above, so we just record the results.

Lemma 1.13.1. *Let α be a simple root and s_α the corresponding simple reflection. Then*

(i)
$$I_{s_\alpha}(v_1) = q^{-1}v_{s_\alpha} + (1-q^{-1})\sum_{j=1}^{\infty}\pi^{j\alpha^\vee}v_1.$$

(ii)
$$I_{s_\alpha}(v_1 + v_{s_\alpha}) = \Big(\frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\Big)(v_1 + v_{s_\alpha}).$$

(iii)
$$I_{s_\alpha}(1_{A_{\mathcal{O}}NK}) = \Big(\frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\Big)1_{A_{\mathcal{O}}NK}.$$

We now introduce the following notation. For $w \in W$ we denote by R_w the set of positive roots α such that $w^{-1}\alpha$ is negative.

Corollary 1.13.2 (Gindikin-Karpelevich formula)**.** *For $w \in W$ we have*

$$I_w(1_{A_{\mathcal{O}}NK}) = \Big(\prod_{w\in R_w}\frac{1-q^{-1}\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\Big)1_{A_{\mathcal{O}}NK}.$$

1.14. **Intertwiners J_w without denominators.** To eliminate denominators we define a new intertwiner J_w $(w \in W)$ by $J_w := \big(\prod_{\alpha\in R_w}(1-\pi^{\alpha^\vee})\big)\cdot I_w$. Note that J_w preserves the subspace M of $M_{B,w^{-1}J}$ and $M_{B,J}$ and hence can be regarded as an element of H, via our identification of H with $\mathrm{End}_H(M)$. For a simple root α, the element of H corresponding to J_{s_α} is (by Lemma 1.13.1(i)) equal to

(1.14.1)
$$(1-q^{-1})\pi^{\alpha^\vee} + q^{-1}(1-\pi^{\alpha^\vee})T_{s_\alpha}{}^{3}.$$

1.15. **Bernstein's relation.** Equation (1.14.1), together with the equality

(1.15.1)
$$J_w \circ \pi^\mu = \pi^{w(\mu)} \circ J_w$$

(for $w = s_\alpha$), yields Bernstein's relation:

(1.15.2)
$$T_{s_\alpha}\pi^\mu = \pi^{s_\alpha(\mu)}T_{s_\alpha} + (1-q)\frac{\pi^{s_\alpha(\mu)} - \pi^\mu}{1-\pi^{-\alpha^\vee}}.$$

Using Bernstein's relation one can calculate the square of J_{s_α}, viewed as element in H; it turns out to be the element $(1-q^{-1}\pi^{\alpha^\vee})(1-q^{-1}\pi^{-\alpha^\vee})$ in the subalgebra R of H.

Lemma 1.7.1 together with Bernstein's relation (1.15.2) gives Bernstein's presentation of H.

2. The center of H

2.1. **A preliminary result.** We are going to prove that the subalgebra R^W is the center of H, but we start by proving something weaker.

Lemma 2.1.1. *The subalgebra R^W is contained in the center of H.*

[3]Here we abuse notation and write π^λ in place of its image Θ_λ under our embedding $R \hookrightarrow H$. We will often do this (e.g. in (1.15.2) and again in section 2.1), leaving context to dictate what is really meant by π^λ.

Proof. Let $r \in R^W$. Then r commutes with all elements in R, so by Lemma 1.7.1 it suffices to show it commutes with T_{s_α} for all simple α. By the intertwining property (1.15.1) of J_{s_α}, it does commute with $(1-q^{-1})\pi^{\alpha^\vee} + q^{-1}(1-\pi^{\alpha^\vee})T_{s_\alpha}$. So r commutes with $(1-\pi^{\alpha^\vee})T_{s_\alpha}$ and hence the bracket of r and T_{s_α} is annihilated by $1-\pi^{\alpha^\vee}$. Since H is a free R-module, the bracket vanishes. □

2.2. **The normalized intertwiners K_w.** Let L denote the field of fractions of the integral domain R. Then L^W is the field of fractions of R^W. We now consider the algebra $H_{\text{gen}} := L^W \otimes_{R^W} H$ and the module $M_{\text{gen}} := L \otimes_R M = L^W \otimes_{R^W} M$, which is an (L, H_{gen})-bimodule.

We define the normalized intertwiners by

$$K_w := \Big(\prod_{\alpha \in R_w} \frac{1}{1-q^{-1}\pi^{\alpha^\vee}} \Big) \cdot J_w = \Big(\prod_{\alpha \in R_w} \frac{1-\pi^{\alpha^\vee}}{1-q^{-1}\pi^{\alpha^\vee}} \Big) \cdot I_w. \tag{2.2.1}$$

Each K_w is an endomorphism of the H_{gen}-module M_{gen} and fixes the spherical vector $1_{A_\mathcal{O}NK}$, as one sees from Corollary 1.13.2. For simple α we have $K_{s_\alpha}^2 = 1$. It follows from this and Lemma 1.11.1 that

$$K_{w_1 w_2} = K_{w_1} K_{w_2} \tag{2.2.2}$$

for all $w_1, w_2 \in W$.

The involution ι_A extends to L, and our sesquilinear pairing form $(\cdot,\cdot)$ on M extends to a sesquilinear L-valued form, still denoted $(\cdot,\cdot)$, on M_{gen}. It follows from (1.10.1) that

$$w\big(K_{w^{-1}}(m), m'\big) = \big(m, K_w(m')\big) \tag{2.2.3}$$

for all $m, m' \in M_{\text{gen}}$.

For later use we remark that it follows from (1.14.1) that for any $w \in W$ one has

$$K_w(v_1) = \sum_{w' \leq w} a_{ww'} \cdot v_{w'} \tag{2.2.4}$$

for certain elements $a_{ww'} \in L$, with the diagonal elements given by the simple formula

$$a_{ww} = \prod_{\alpha \in R_w} \frac{1-\pi^{-\alpha^\vee}}{1-q\pi^{-\alpha^\vee}}. \tag{2.2.5}$$

2.3. **Calculation of the center of H.** Since the endomorphism ring of the H_{gen}-module M_{gen} is H_{gen}, we can view the endomorphisms K_w as elements of H_{gen}. The map $w \mapsto K_w$ is a group homomorphism from W to $H_{\text{gen}}^\times$ and therefore induces an algebra homomorphism from the twisted group algebra $L[W]$ to H_{gen}.

Lemma 2.3.1. *The homomorphism $L[W] \to H_{\text{gen}}$ is an isomorphism. The center of H_{gen} is L^W. The center of H is R^W.*

Proof. The twisted group algebra is a matrix algebra over L^W, and is therefore simple, which implies our map is injective. Comparing dimensions, we see that the map is an isomorphism. Therefore H_{gen} is a matrix algebra over L^W, and its center is L^W. It follows easily that the center of H is R^W. (Use along the way the obvious fact that H is torsion-free as R^W-module.) □

3. Application: Restriction of two involutions to the center

3.1. **Restriction of ι to the center.** Recall from before the anti-involution $\iota : H \to H$ given by $\iota(h)(x) = h(x^{-1})$. We are going to see that the restriction of ι to the center of H is very simple.

Lemma 3.1.1. *There are two involutions on R^W, one obtained by restricting ι_A to R^W, the other obtained by restricting ι to the center of H, which we have identified with R^W. The two involutions on R^W coincide.*

Proof. This follows from (1.9.2), (1.9.6), the non-degeneracy of our sesquilinear form, and the basic identity

$$r\varphi = \varphi z_r \qquad \forall\, r \in R^W, \forall\, \varphi \in M$$

where z_r denotes the element of the center of H that corresponds to r. □

3.2. **Restriction of the Kazhdan-Lusztig involution to the center.** Now consider the affine Hecke algebra $\mathcal{H}$ associated to G. This is an algebra over the ring $\mathbb{Z}[v, v^{-1}]$ (v an indeterminate), generated by symbols T_w (w ranging over the extended affine Weyl group for G), which satisfy the usual braid and quadratic relations. If $q = p^n$ denotes the cardinality of the residue field of F, the map $v \mapsto q^{1/2}$ determines a ring homomorphism $\mathbb{Z}[v, v^{-1}] \to \mathbf{C}$. There is a canonical isomorphism $H = \mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbf{C}$ (see section 7.2).

The Kazhdan-Lusztig involution $h \mapsto \bar{h}$ of $\mathcal{H}$ is determined by $v \mapsto v^{-1}$ and $T_w \mapsto T_{w^{-1}}^{-1}$ (beware that this does not descend to an involution of H). There is also an anti-involution on $\mathcal{H}$ given by $v \mapsto v$ and $T_w \mapsto T_{w^{-1}}$. On specializing $v \to q^{1/2}$, this does descend to H and gives precisely the anti-involution of H denoted ι above; therefore we denote the anti-involution of $\mathcal{H}$ by the same symbol.

For each dominant coweight $\mu \in X_*(A)$, we let $z_\mu = \sum_{\lambda \in W\mu} \Theta_\lambda$, where Θ_λ is the element of $\mathcal{H}$ defined by $\Theta_\lambda = v^{\langle 2\rho, -\lambda_1+\lambda_2\rangle} T_{t_{\lambda_1}} T_{t_{\lambda_2}}^{-1}$, where $\lambda = \lambda_1 - \lambda_2$, and λ_i is dominant ($i = 1, 2$); see Remark 1.7.2. A result of Bernstein says that the elements z_μ form a $\mathbb{Z}[v, v^{-1}]$-basis for the center of $\mathcal{H}$, as μ ranges over dominant coweights in $X_*(A)$ (see also section 7.5).

These considerations yield a simple proof of Corollary 8.8 in [16]:

Lemma 3.2.1. $\overline{z_\mu} = z_\mu$.

Proof. First of all we can relate the two involutions by the easily-checked formula $\iota(\Theta_{-\lambda}) = \overline{\Theta}_\lambda$. It follows that $\iota(z_{-w_0\mu}) = \overline{z_\mu}$, where w_0 is the longest element of W. On the other hand, the previous lemma says that, at least after v is specialized to $q^{1/2}$, the elements $\iota(z_{-w_0\mu})$ and z_μ coincide as elements in H. Since this is true for every power of q by the same token, we must have the equality $\iota(z_{-w_0\mu}) = z_\mu$ in $\mathcal{H}$ as well, which proves that $\overline{z_\mu} = z_\mu$. □

4. Satake isomorphism [26]

4.1. **Definition of H_K and M_K.** Let e_K be the idempotent $1_K/\operatorname{meas}(K)$ in H. Put $H_K := C_c(K\backslash G/K)$, which we identify with the subring $e_K H e_K$ of H (so that $1_K \mapsto e_K$). We also put $M_K := C_c(A_{\mathcal{O}} N\backslash G/K)$, which we identify with the H_K-submodule Me_K of M. Then M_K is an (R, H_K)-bimodule, with R-module structure inherited from the one on M. Concretely, the action of the function $h \in H_K$ on $m \in M_K$ is given by $m * h$, where $*$ denotes convolution using the Haar measure on G giving K measure 1.

4.2. **The Satake transform.** Since M_K is free of rank 1 as R-module (with basis element the spherical vector $1_{A_\mathcal{O}NK}$), we get a $\mathbf{C}$-algebra homomorphism $H_K \to R$, denoted $h \mapsto h^\vee$ and called the Satake transform, characterized by the property that

$$m * h = h^\vee \cdot m \tag{4.2.1}$$

for all $h \in H_K$ and all $m \in M_K$.

Taking m to be the spherical vector, we get the equation

$$1_{A_\mathcal{O}NK} * h = h^\vee \cdot 1_{A_\mathcal{O}NK}. \tag{4.2.2}$$

In fact $h^\vee$ lies in the subalgebra R^W, as one sees by applying the normalized intertwiners (which fix the spherical vector) to equation (4.2.2). Thus the Satake transform actually maps H_K into R^W.

Recall that π^ν (for ν ranging through X_*) form a $\mathbf{C}$-basis for R. Evaluating both sides of equation (4.2.2) on the element π^ν and using the usual $G = ANK$ integration formula (see [4]), one sees that the coefficient of π^ν in $h^\vee$ is equal to

$$\delta_B(\pi^\nu)^{-1/2} \int_N h(n\pi^\nu)\, dn,$$

where the Haar measure dn is normalized so that $N_\mathcal{O}$ has measure 1.

4.3. **Satake transform is an isomorphism.** The elements $h_\mu := 1_{K\pi^\mu K}$, with μ a dominant coweight, form a $\mathbf{C}$-basis for H_K. The elements $s_\nu := \sum_{\lambda \in W\nu} \pi^\lambda$, with ν a dominant coweight, form a $\mathbf{C}$-basis for R^W. The coefficients $c_{\mu\nu}$ of $h_\mu^\vee$ in the basis s_ν are given by

$$c_{\mu\nu} = \delta_B(\pi^\nu)^{-1/2} \int_N 1_{K\pi^\mu K}(n\pi^\nu) dn. \tag{4.3.1}$$

The real number $c_{\mu\nu}$ is non-negative and is non-zero if and only if $K\pi^\mu K$ meets $N\pi^\nu$. It follows from [3, 4.4.4] that $c_{\mu\nu}$ is 0 unless $\nu \le \mu$ (by which we mean that $\mu - \nu$ is a non-negative integral linear combination of simple coroots), and it is obvious that $c_{\mu\mu}$ is non-zero. Therefore a standard upper-triangular argument shows that the Satake transform is an isomorphism from H_K to R^W. In particular H_K is commutative.

We remark that in [19, Théoremè 5.3.17], [22], [12], it is shown that if $\nu \le \mu$ (both dominant), then $c_{\mu\nu}$ is non-zero.

4.4. **Compatibility of two involutions.** Recall from 1.8 the involutions ι, ι_A on H, R respectively. It is clear that ι preserves the subring H_K and that ι_A preserves the subring R^W. One sees easily (imitate the proof of Lemma 3.1.1) that the Satake isomorphism is compatible with these involutions, in the sense that

$$(\iota(h))^\vee = \iota_A(h^\vee). \tag{4.4.1}$$

4.5. **Further discussion of the Satake transform.** Consider a quasicharacter $\chi : A/A_\mathcal{O} \to \mathbf{C}^\times$. Then χ determines a $\mathbf{C}$-algebra homomorphism $R \to \mathbf{C}$. Using this homomorphism to extend scalars, we obtain an H_K-module $\mathbf{C} \otimes_R M_K$ which can be identified with the (1-dimensional) space of K-fixed vectors in the unramified principal series representation $i_B^G(\chi^{-1})$. It is customary to work with left G-modules (and hence left modules over Hecke algebras) rather than right modules, and to switch back and forth between right and left one uses $g \mapsto g^{-1}$ on G (and hence the involution ι on H_K). Bearing these remarks in mind, one sees that for any $h \in H_K$ and any K-fixed vector $v \in i_B^G(\chi)$ there is an equality

$$hv = h^\vee(\chi)v, \tag{4.5.1}$$

where for $r \in R$ we write $r(\chi)$ for the image of r under the homomorphism $R \to \mathbf{C}$ determined by χ. (We used that $\iota_A(r)(\chi^{-1}) = r(\chi)$.)

4.6. **Compatibility of the Satake and Bernstein isomorphisms** [9, 11, 16]**.** We now have canonical isomorphisms (Satake and Bernstein)

$$H_K \simeq R^W \simeq Z(H),$$

where $Z(H)$ denotes the center of H. Let $h \in H_K$, $r \in R^W$, $z \in Z(H)$ be elements that correspond to each other under these isomorphisms. We have

$$mh = me_K h = rme_K = me_K z$$

for all $m \in M$. It follows that

$$h = e_K z, \tag{4.6.1}$$

which is the compatibility referred to in the heading of this section.

5. Macdonald's formula [6, 18, 19]

5.1. **Preliminary remarks about unramified matrix coefficients.** The contragredient of the induced representation $i_B^G(\chi)$ is $i_B^G(\chi^{-1})$. (Recall that $i_B^G(\chi^{-1})$ has as usual a left G-action given by right translations.) Now choose K-fixed vectors $v \in i_B^G(\chi)$ and $\tilde{v} \in i_B^G(\chi^{-1})$ such that $\langle v, \tilde{v}\rangle = 1$, and put

$$\Gamma_\chi(g) := \langle gv, \tilde{v}\rangle, \tag{5.1.1}$$

an unramified matrix coefficient, otherwise known as a zonal spherical function. Clearly Γ_χ is a $\mathbf{C}$-valued function on $K\backslash G/K$, and we have

$$\Gamma_\chi(1) = 1. \tag{5.1.2}$$

Let $h \in H_K$. It follows from the definition of Γ_χ that $(h * \Gamma_\chi)(g) = \langle gv, h\tilde{v}\rangle$, which by (4.5.1) is equal to $h^\vee(\chi^{-1})\Gamma_\chi(g)$. Thus we have

$$h * \Gamma_\chi = h^\vee(\chi^{-1})\Gamma_\chi. \tag{5.1.3}$$

Similarly we have

$$\Gamma_\chi * h = h^\vee(\chi^{-1})\Gamma_\chi. \tag{5.1.4}$$

The function Γ_χ is uniquely determined by (5.1.2) and either of (5.1.3), (5.1.4); indeed, taking $h = 1_{K\pi^{-\mu}K} = \iota(K\pi^\mu K)$ in (5.1.3) and then evaluating both sides at the identity element, we see that

$$\text{meas}(K\pi^\mu K) \cdot \Gamma_\chi(\pi^\mu) = (1_{K\pi^\mu K})^\vee(\chi), \tag{5.1.5}$$

where the measure is taken with respect to the Haar measure on G that gives K measure 1. In other words, knowing the values of unramified matrix coefficients is essentially the same as knowing the Satake transforms of the elements $1_{K\pi^\mu K} \in H_K$.

5.2. **Definition of Γ.** It is more convenient to work with the R-valued matrix coefficient Γ defined by

$$\Gamma(g) := (1_{A_O NK}, 1_{A_O NK} \cdot g), \tag{5.2.1}$$

where $(\cdot,\cdot)$ is our sesquilinear form on $i_B^G(\chi_{\text{univ}}^{-1})$ (regarded as a right G-module).

Of course Γ is a function on $K\backslash G/K$ with values in R; applying the homomorphism $R \to \mathbf{C}$ determined by χ to the values of Γ, we get the $\mathbf{C}$-valued function Γ_χ. Therefore computing Γ is the same as computing Γ_χ for all χ.

We can rewrite (5.2.1) as

$$\Gamma(g) := (1_{A_O NK}, 1_{A_O NK} * e_{KgK}), \tag{5.2.2}$$

where e_{KgK} denotes $\text{meas}(KgK)^{-1} \cdot 1_{KgK}$, from which it follows that

$$\Gamma(g) = (e_{KgK})^\vee, \tag{5.2.3}$$

in agreement with (5.1.5). Equation (5.2.3) shows that Γ actually takes values in R^W and hence that $\Gamma_{w\chi} = \Gamma_\chi$ for all $w \in W$.

Macdonald's formula [6, 18, 19] is an explicit formula for Γ_χ, which we will now derive, following Casselman's method [6]. As mentioned above, it is the same to give an explicit formula for Γ, and this is what we will do.

5.3. **Decomposition of the spherical vector as a sum of eigenvectors.** As a first step towards Macdonald's formula, we are going to decompose the spherical vector $1_{A_O NK} \in M$ as a sum of eigenvectors for the action of the commutative subalgebra R of H. This can only be done in M_{gen}.

Recall that v_1 denotes the element $1_{A_O NI} \in M$. The vector v_1 is an eigenvector for the subalgebra R of H by the very definition of that subalgebra; more precisely we have the formula

$$v_1 \Theta_\lambda = \pi^\lambda \cdot v_1, \tag{5.3.1}$$

where (as before) Θ_λ is a notation for the image of $\pi^\lambda \in R$ under $R \hookrightarrow H$. Applying the normalized intertwiner K_w to this equation, we see that

$$K_w(v_1)\Theta_\lambda = \pi^{w\lambda} \cdot K_w(v_1), \tag{5.3.2}$$

which shows that $K_w(v_1)$ is an eigenvector for R with character $w^{-1}(\chi_{\text{univ}})$.

Lemma 5.3.1. *In M_{gen} we have the formula*

$$1_{A_O NK} = \sum_{w \in W} w\Big(\prod_{\alpha>0} \frac{1 - q\pi^{\alpha^\vee}}{1 - \pi^{\alpha^\vee}}\Big) \cdot K_w(v_1).$$

Proof. Let w_0 denote the longest element of W. Recall the standard basis elements v_x for M. Then v_w ($w \in W$) form an R-basis for M, hence an L-basis for M_{gen}. From (2.2.4) it is clear that the vectors $K_w(v_1)$ also form an L-basis for M_{gen}. Write the spherical vector in this second basis:

$$1_{A_O NK} = \sum_{w \in W} d_w \cdot K_w(v_1). \tag{5.3.3}$$

We can also write the spherical vector in the first basis; since the spherical vector is equal to $\sum_{w\in W} v_w$, it is clear that the coefficient of the basis element v_{w_0} in the spherical vector is 1; on the other hand, from (2.2.4) and (5.3.3), it is clear that this same coefficient is also equal to $d_{w_0} a_{w_0 w_0}$; equating the two expressions for the coefficient and using the explicit formula (2.2.5) for $a_{w_0 w_0}$, we see that

$$d_{w_0} = \prod_{\alpha>0} \frac{1 - q\pi^{-\alpha^\vee}}{1 - \pi^{-\alpha^\vee}}.$$

Moreover, since the normalized intertwiners K_w fix the spherical vector, we have

$$d_{w_1 w_2} = w_1(d_{w_2})$$

for all $w_1, w_2 \in W$, from which it follows that

$$d_w = w\Big(\prod_{\alpha>0} \frac{1 - q\pi^{\alpha^\vee}}{1 - \pi^{\alpha^\vee}}\Big).$$

This completes the proof. □

5.4. **Partial information about some more matrix coefficients.** We see from Lemma 5.3.1 that in order to calculate Γ it would be enough to calculate the matrix coefficients $(1_{A_\mathcal{O} NK}, K_w(v_1) \cdot g)$. Now this new matrix coefficient is a function on $I\backslash G/K = X_*$ rather than $K\backslash G/K$, and it is difficult to calculate all its values. Fortunately it easy to calculate them for elements g of the form π^μ for dominant coweights μ, and in the end this is enough since Γ is K-bi-invariant and hence determined by its values on such elements.

Lemma 5.4.1. *For group elements of the form $g = \pi^\mu$ with μ dominant, we have*

$$(1_{A_\mathcal{O} NK}, K_w(v_1) \cdot g) = [K : I]^{-1} \cdot \delta_B(\pi^\mu)^{1/2} \cdot \pi^{w\mu}.$$

Proof. For $f \in M$, the function $f \cdot g$ need not be right I-invariant, and so need not have a simple form. However, letting δ_g denote the Dirac measure concentrated at g, we have an equality of measures $e_I \cdot \delta_g \cdot e_I = e_{IgI}$, where e_X is the characteristic function of a set X divided by its measure. Since $g = \pi^\mu$ with μ dominant, we have $e_{IgI} = \delta_B(\pi^\mu) T_{\pi^\mu} = \delta_B(\pi^\mu)^{1/2} \Theta_\mu$. Using these considerations (and the fact that the idempotent e_I fixes both $1_{A_\mathcal{O} NK}$ and $K_w(v_1)$), we see that

$$(1_{A_\mathcal{O} NK}, K_w(v_1) \cdot g) = \delta_B(\pi^\mu)^{1/2} \cdot (1_{A_\mathcal{O} NK}, K_w(v_1)\Theta_\mu).$$

From (5.3.2) we have $K_w(v_1)\Theta_\mu = \pi^{w\mu} K_w(v_1)$, and therefore

$$(1_{A_\mathcal{O} NK}, K_w(v_1) \cdot g) = \delta_B(\pi^\mu)^{1/2} \cdot \pi^{w\mu} \cdot (1_{A_\mathcal{O} NK}, K_w(v_1)).$$

Using (2.2.3), we see that

$$(1_{A_\mathcal{O} NK}, K_w(v_1)) = w(K_{w^{-1}}(1_{A_\mathcal{O} NK}), v_1) = w(1_{A_\mathcal{O} NK}, v_1).$$

Moreover $(1_{A_\mathcal{O} NK}, v_1) = [K : I]^{-1}$, as follows immediately from the definitions. This completes the proof.

□

5.5. **Macdonald's formula.** Combining Lemmas 5.3.1 and 5.4.1, we get

Theorem 5.5.1 (Macdonald)**.** *For any dominant coweight μ we have*

$$\Gamma(\pi^\mu) = [K : I]^{-1} \cdot \sum_{w\in W} w\Big(\prod_{\alpha>0} \frac{1 - q\pi^{\alpha^\vee}}{1 - \pi^{\alpha^\vee}}\Big) \cdot \delta_B(\pi^\mu)^{1/2} \cdot \pi^{w\mu}$$

and

$$\Gamma_\chi(\pi^\mu) = [K:I]^{-1} \cdot \sum_{w\in W} \Big(\prod_{\alpha>0} \frac{1-q(w\chi)(\pi^{\alpha^\vee})}{1-(w\chi)(\pi^{\alpha^\vee})}\Big) \cdot \delta_B(\pi^\mu)^{1/2} \cdot (w\chi)(\pi^\mu).$$

5.6. **Alternative version of Macdonald's formula.** For any finite subset $X \subset \tilde{W}$, define the polynomial $X(t) := \sum_{w\in X} t^{l(w)}$, where the length function $l(\cdot)$ is defined using the set of reflections for the $\bar{B}$-positive simple affine roots, as in section 7.1. Let W_μ denote the stabilizer of μ in W. We write t_μ for the element π^μ of the translation subgroup of $\tilde{W}$.

Theorem 5.6.1. *For any dominant coweight* μ*,*

$$(1_{K\pi^\mu K})^\vee = \frac{q^{\langle\rho,\mu\rangle}}{W_\mu(q^{-1})} \sum_{w\in W} w\Big(\prod_{\alpha>0} \frac{1-q^{-1}\pi^{-\alpha^\vee}}{1-\pi^{-\alpha^\vee}}\Big) \cdot \pi^{w\mu}.$$

Proof. Using Theorem 5.5.1, this follows easily from the identities $[K:I] = W(q)$, $W(q) = q^{l(w_0)}W(q^{-1})$, $\mathrm{meas}(K\pi^\mu K) = Wt_\mu W(q)/W(q)$, and

$$Wt_\mu W(q) = \frac{W(q)q^{l(t_\mu)}W(q^{-1})}{W_\mu(q^{-1})}. \tag{5.6.1}$$

To prove (5.6.1), note that any element in $Wt_\mu W$ has a unique decomposition of the form $w^\mu t_\mu w$, where $w \in W$ and w^μ is a minimal length representative for a coset in W/W_μ. Furthermore, such an element has length $l(w) + l(t_\mu) - l(w^\mu)$ (as may be seen by induction on $l(w^\mu)$; note that μ is anti-dominant for $\bar{B}$). □

6. Casselman-Shalika formula [7, 24, 27]

We are going to give an exposition of Casselman-Shalika's proof of their formula [7] for unramified Whittaker functions.

6.1. **Unramified characters ψ on $\bar{N}$.** Let Δ denote the set of simple roots. The abelian group $\prod_{\alpha\in\Delta} N_{-\alpha}$ is a quotient of $\bar{N}$. Here $N_{-\alpha}$ is the root subgroup for $-\alpha$, which we identify with the additive group $\mathbf{G}_a$ over $\mathcal{O}$. Given characters $\psi_\alpha : N_{-\alpha} \to \mathbf{C}^\times$, their product defines a character on $\prod_{\alpha\in\Delta} N_{-\alpha}$ and hence a character ψ on $\bar{N}$. We say that ψ is *principal* if all the ψ_α are non-trivial. We say that ψ is *unramified* if all the characters ψ_α are trivial on $\mathcal{O}$ but non-trivial on $\mathfrak{p}^{-1}$.

6.2. **Whittaker functionals.** Let ψ be a principal character on $\bar{N}$. Let S be a commutative $\mathbf{C}$-algebra. The inclusion of $\mathbf{C}$ in S lets us view ψ as a character with values in $S^\times$.

Let $\chi : A \to S^\times$ be an S-valued character, and form the induced representation $i_B^G(\chi)$, which is both a G-module and an S-module. A *Whittaker functional* on $i_B^G(\chi)$ is an S-module map

$$L : i_B^G(\chi) \to S$$

such that $L(\bar{n}\phi) = \psi(\bar{n})L(\phi)$ for all $\bar{n} \in \bar{N}$ and all $\phi \in i_B^G(\chi)$.

In case $S = \mathbf{C}$ Rodier [25] (see also [7]) proved that the space of Whittaker functionals is 1-dimensional and that there exists a unique Whittaker functional W whose restriction to the subspace of functions ϕ in $i_B^G(\chi)$ supported on the big cell $B\bar{N}$ is given by the integral

$$W(\phi) = \int_{\bar{N}} \phi(\bar{n})\psi(\bar{n})^{-1}\, d\bar{n} \tag{6.2.1}$$

(the integrand of which is compactly supported by our assumption on the support of ϕ). Here $d\bar{n}$ denotes the Haar measure on $\bar{N}$ that gives measure 1 to $\bar{N} \cap K$. For general S the same proof shows that there again exists a unique Whittaker functional W given by (6.2.1) for functions supported on the big cell, and that the S-module of all Whittaker functionals is free of rank 1 with W as basis element.

Now let us consider the case in which S is R and χ is χ_{univ}^{-1}. We let J denote the set of negative coroots and consider the completion R_J of R defined in 1.10. It follows from Lemma 1.10.1 that the integral (6.2.1) makes sense as an element of R_J for all $\phi \in i_B^G(\chi_{\text{univ}}^{-1})$, and even for $\phi \in i_B^G(\chi_{\text{univ}}^{-1}) \otimes_R R_J$. Using the uniqueness of W for R_J, we see that for $\phi \in i_B^G(\chi_{\text{univ}}^{-1})$ the integral (6.2.1) actually takes values in the subring R of R_J. (In other words the presence of the principal character ψ causes all but finitely many of the coefficients of the Laurent power series $W(\phi)$ to vanish.) Therefore we will now regard the Whittaker functional W on $i_B^G(\chi_{\text{univ}}^{-1})$ as being defined by the integral (6.2.1).

Recall that we have identified the module M with the Iwahori-fixed vectors in $i_B^G(\chi_{\text{univ}}^{-1})$, and thus we also have the (restricted) Whittaker functional $W : M \to R$. It is necessary to calculate W for a few very special vectors in M.

From now on, we assume the character ψ is principal and unramified.

Lemma 6.2.1. *Let w_0 denote the longest element in W, and let α be a simple root with corresponding simple reflection s_α. Then*

(i) $$W(v_1) = q^{-l(w_0)}$$

(ii) $$W(v_1 + v_{s_\alpha}) = q^{1-l(w_0)} \cdot (1 - q^{-1}\pi^{-\alpha^\vee}).$$

Proof. The first statement follows from the fact that $\bar{N} \cap BI = \bar{N} \cap I$, which has measure $q^{-l(w_0)}$. Similarly, the second statement reduces to a calculation in $SL(2)$, which we leave to the reader. Note that for $SL(2)$ the second statement gives the value (namely $1 - q^{-1}\pi^{-\alpha^\vee}$) of the Whittaker functional on the spherical vector. □

6.3. **Effect of intertwiners on the Whittaker functional.** Earlier we defined normalized intertwiners K_w, normalized in the sense that they preserve the spherical vector $1_{A_\mathcal{O}NK}$. Now we normalize them differently. Put

$$K'_w := \Big(\prod_{\alpha \in R_w} \frac{1 - q^{-1}\pi^{\alpha^\vee}}{1 - q^{-1}\pi^{-\alpha^\vee}} \Big) \cdot K_w = \Big(\prod_{\alpha \in R_w} \frac{1 - \pi^{\alpha^\vee}}{1 - q^{-1}\pi^{-\alpha^\vee}} \Big) \cdot I_w. \tag{6.3.1}$$

Lemma 6.3.1 ([13],[7])**.** *The newly normalized intertwiners K'_w preserve the Whittaker functional W in the sense that $W \circ K'_w = w \circ W$ for all $w \in W$. On the right side of this equality w stands for the automorphism of R determined by w. Moreover $K'_{w_1 w_2} = K'_{w_1} K'_{w_2}$.*

Proof. One sees directly from the definition that K'_w is multiplicative in w. Therefore to prove the first statement of the lemma, it is enough to treat the case $w = s_\alpha$ for a simple root α. By uniqueness of W there exists $c \in L^\times$ such that

$$W \circ K'_{s_\alpha} = c(s_\alpha \circ W).^4 \tag{6.3.2}$$

[4]Here, we are implicitly using normalized intertwiners $K_w : L \otimes_R i_B^G(\chi_{\text{univ}}^{-1}) \to L \otimes_R i_B^G(\chi_{\text{univ}}^{-1})$. The reader may derive the existence and basic properties of such intertwiners following the method of sections 1.10-2.2. Although the discussion there was limited to the theory of intertwiners on the

To prove that $c = 1$ we evaluate both sides of (6.3.2) on $v_1 + v_{s_\alpha}$, using Lemma 1.13.1(ii) and Lemma 6.2.1(ii). □

Lemma 6.3.2. *In M_{gen} we have the formula*

$$1_{A_{\mathcal{O}}NK} = q^{l(w_0)} \cdot \Big(\prod_{\alpha>0}(1 - q^{-1}\pi^{-\alpha^\vee})\Big) \cdot \sum_{w\in W} w\Big(\prod_{\alpha>0} \frac{1}{1-\pi^{-\alpha^\vee}}\Big) \cdot K'_w(v_1).$$

Proof. This follows from Lemma 5.3.1 and the definition of K'_w. □

6.4. **Whittaker functions.** We continue with S, ψ and χ as in 6.2. For any $\phi \in i_B^G(\chi)$ we define the corresponding Whittaker function $\mathcal{W}_\phi : G \to S$ by

$$\mathcal{W}_\phi(g) := W(g\phi). \tag{6.4.1}$$

Then $\mathcal{W}_\phi$ is an S-valued function satisfying the transformation law

$$f(\bar{n}g) = \psi(\bar{n})f(g) \qquad \forall\ \bar{n} \in \bar{N}, \tag{6.4.2}$$

and $\phi \mapsto \mathcal{W}_\phi$ is a G-map from $i_B^G(\chi)$ to the space of functions satisfying (6.4.2).

6.5. **Unramified Whittaker functions.** From now on we assume that both ϕ and χ are unramified. Then inside $i_B^G(\chi)$ we have the normalized spherical vector ϕ_χ defined by

$$\phi_\chi(ank) = \chi(a)\delta_B(a)^{1/2}.$$

The Casselman-Shalika formula is an explicit formula for the Whittaker function $\mathcal{W}_\chi := \mathcal{W}_{\phi_\chi}$ corresponding to the spherical vector ϕ_χ. It is enough to consider the case in which S is R and χ is χ_{univ}^{-1}, in which case we abbreviate $\mathcal{W}_{\chi_{\text{univ}}^{-1}}$ to $\mathcal{W}$.

Since $\mathcal{W}$ is right K-invariant and satisfies (6.4.2), it is determined by its values on elements $g \in G$ of the form $\pi^{-\mu}$ for $\mu \in X_*$. In fact $\mathcal{W}(\pi^{-\mu}) = 0$ unless μ is dominant. Indeed, for $x \in N_{-\alpha} \cap K$ we have

$$\mathcal{W}(\pi^{-\mu}) = \mathcal{W}(\pi^{-\mu}x) = \psi_\alpha(\pi^{-\mu}x\pi^{\mu})\mathcal{W}(\pi^{-\mu}),$$

which implies that $\mathcal{W}(\pi^{-\mu})$ vanishes unless ψ_α is trivial on $\mathfrak{p}^{\langle\alpha,\mu\rangle}$, which, since ψ is unramified, implies in turn that $\langle\alpha,\mu\rangle \geq 0$. Therefore it is enough to find the values of $\mathcal{W}(g)$ for g of the form $\pi^{-\mu}$ for dominant μ.

Theorem 6.5.1 (Casselman-Shalika)**.** *Let μ be a dominant coweight. Then*

$$\mathcal{W}(\pi^{-\mu}) = \big(\prod_{\alpha>0}(1 - q^{-1}\pi^{-\alpha^\vee})\big) \cdot \delta_B(\pi^\mu)^{1/2} \cdot E_\mu,$$

where $E_\mu \in R^W$ is the character of the irreducible representation of the Langlands dual group $G^\vee$ having highest weight μ.

Proof. As usual (see 1.5) we identify $i_B^G(\chi_{\text{univ}}^{-1})$ with $C_c^\infty(A_{\mathcal{O}}N\backslash G)$. The spherical vector in the induced representation corresponds to $1_{A_{\mathcal{O}}NK}$. We begin by noting that

$$\mathcal{W}(\pi^{-\mu}) = W(1_{A_{\mathcal{O}}NK\pi^\mu}) = W(1_{A_{\mathcal{O}}NK} \cdot e_{I\pi^\mu I}) \tag{6.5.1}$$

where $e_{I\pi^\mu I}$ denotes the characteristic function of $I\pi^\mu I$ divided by its measure. Here we used that $I\pi^\mu I = I\pi^\mu(I \cap \bar{N})$ (a consequence of the Iwahori factorization $I = (I \cap B)\cdot(I \cap \bar{N})$ and the dominance of μ), as well as the right I-invariance of

---

Iwahori-invariants in the induced modules in question, it is possible to develop a similar theory on the induced modules themselves.

$1_{A_{\mathcal{O}}NK}$ and the fact that ψ is trivial on $I \cap \bar{N}$. Since $e_{I\pi^\mu I} = \delta_B(\pi^\mu)^{1/2} \cdot \Theta_\mu$, we can rewrite the equation above as

$$\mathcal{W}(\pi^{-\mu}) = \delta_B(\pi^\mu)^{1/2} W(1_{A_{\mathcal{O}}NK} \cdot \Theta_\mu). \tag{6.5.2}$$

It then follows from Lemmas 6.2.1(i), 6.3.1, and 6.3.2 that

$$\mathcal{W}(\pi^{-\mu}) = \delta_B(\pi^\mu)^{1/2} \cdot \Big(\prod_{\alpha>0}(1 - q^{-1}\pi^{-\alpha^\vee})\Big) \cdot \sum_{w \in W} w\Big(\prod_{\alpha>0} \frac{1}{1-\pi^{-\alpha^\vee}}\Big) \cdot \pi^{w\mu}.$$

The Casselman-Shalika formula now follows from the Weyl character formula:

$$E_\mu = \sum_{w\in W} w\Big(\prod_{\alpha>0} \frac{\pi^\mu}{1-\pi^{-\alpha^\vee}}\Big).$$

□

7. The Kato-Lusztig formula [14, 16]

Following the strategy of [14], we derive the formula of Kato-Lusztig and, as a corollary, another result of Lusztig [16]. The Kato-Lusztig formula relates the Satake transforms of the functions $1_{K\pi^\lambda K}$ to the character E_μ of the highest weight module of the Langlands dual group corresponding to μ. It is the function-theoretic counterpart of the geometric Satake isomorphism [10, 20], and it can also be formally deduced from that statement by using the function-sheaf dictionary.

The proof requires us to give v-analogs of several objects studied above (here v is an indeterminate which can be specialized to $q^{1/2}$). Most importantly, we need the v-analog of Theorem 5.6.1. The reader willing to accept that on faith may skip directly to section 7.7.

7.1. **Preliminaries about affine roots.** Write T for the group $X_*(A)$, viewed as the group of translations in the extended affine Weyl group $\widetilde{W}$; thus $\widetilde{W} = T \rtimes W$. We denote by t_μ the element of T corresponding to the cocharacter μ. For simplicity, we assume here that the root system underlying G is irreducible. Let $\alpha_1, \dots, \alpha_r$ denote the B-positive simple roots, and let $\tilde{\alpha}$ denote the B-highest root. Let $s_0 = t_{-\tilde{\alpha}^\vee} s_{\tilde{\alpha}}$, and $S_{\text{aff}} = S \cup \{s_0\}$. Here $S = \{s_{\alpha_i} = s_{-\alpha_i}\}_{i=1}^r$ is the set of simple reflections corresponding to the B-positive (or $\bar{B}$-positive) simple roots, but our definition of s_0 means that S_{aff} is the set of simple affine reflections corresponding to the $\bar{B}$-positive affine roots.

We have $\widetilde{W} = W_{\text{aff}} \rtimes \Omega$, where W_{aff} is the Coxeter group generated by S_{aff}, and Ω is the subgroup of $\widetilde{W}$ which preserves the set of $\bar{B}$-positive simple affine roots under the usual left action (an affine-linear automorphism acts on a functional by precomposition with its inverse). The set S_{aff} induces a length function and a Bruhat order on $\tilde{W}$ (the same as that mentioned in Lemma 1.6.1). The elements $\sigma \in \Omega$ are of length zero, and the algebra generated by the functions $1_{I\sigma I}$ is naturally isomorphic to $\mathbf{C}[\Omega]$. We have a twisted tensor product decomposition $H = H_{\text{aff}} \otimes \mathbf{C}[\Omega]$, where H_{aff} is the algebra generated by the functions 1_{IxI}, $x \in W_{\text{aff}}$ (this follows from the remarks following Lemma 7.2.1 below).

Recall our convention for embedding $X_*(A)$ into A: $\lambda \mapsto \pi^\lambda = \lambda(\pi)$. We also regard each $w \in W$ as an element in K, fixed once and for all. These conventions tell us how we view elements of $\tilde{W}$ as elements in G. For example, for SL(2), we are identifying s_0 with the element $\begin{bmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{bmatrix}$. It is important to bear these conventions in mind in this section.

7.2. More about the H-action on M.

Lemma 7.2.1. *Fix $\varphi \in M$ and $\pi^\lambda w \in \tilde{W}$, where $w \in W$. Suppose $\sigma \in \Omega$, and that $s = s_\alpha \in S$ corresponds to a B-positive simple root α. Then we have*

(i) $$\varphi T_{s_\alpha}(\pi^\lambda w) = \begin{cases} q \cdot \varphi(\pi^\lambda ws), & \text{if } w(\alpha) \text{ is } B\text{-positive} \\ \varphi(\pi^\lambda ws) + (q-1)\cdot\varphi(\pi^\lambda w), & \text{if } w(\alpha) \text{ is } B\text{-negative} \end{cases}$$

(ii) $$\varphi T_{s_0}(\pi^\lambda w) = \begin{cases} q \cdot \varphi(\pi^\lambda ws_0), & \text{if } w(\tilde\alpha) \text{ is } B\text{-negative.} \\ \varphi(\pi^\lambda ws_0) + (q-1)\cdot\varphi(\pi^\lambda w), & \text{if } w(\tilde\alpha) \text{ is } B\text{-positive.} \end{cases}$$

(iii) $\varphi T_\sigma(\pi^\lambda w) = \varphi(\pi^\lambda w\sigma^{-1})$.

Proof. To illustrate the method, we prove (ii). Let $\{x_i\}_{i=0}^{q-1}$ denote a set of representatives for $\mathcal{O}/P$ taken in $\mathcal{O}^\times \cup \{0\}$. For a root β, let $u_\beta : \mathbf{G}_a \to G$ denote the associated homomorphism. Then we have the decomposition

$$Is_0I = \coprod_i u_{-\tilde\alpha}(\pi x_i)s_0I.$$

We therefore have

$$\varphi T_{s_0}(\pi^\lambda w) = \sum_i \varphi(\pi^\lambda w\, u_{-\tilde\alpha}(\pi x_i)s_0).$$

If $w(\tilde\alpha)$ is B-negative, then each term in the sum is $\varphi(\pi^\lambda ws_0)$. If $w(\tilde\alpha)$ is B-positive, then the term for $x_i = 0$ is $\varphi(\pi^\lambda ws_0)$. If $x_i \neq 0$, then using the identity

$$u_{-\tilde\alpha}(\pi x_i)s_0I = u_{\tilde\alpha}(\pi^{-1}x_i^{-1})I$$

(which holds whenever $x_i \in \mathcal{O}^\times$), we see the term indexed by x_i is $\varphi(\pi^\lambda w)$.

Part (i) can be proved in a similar way; alternatively it can be derived from (1.6.2) together with the usual relations in the Hecke algebra for W. □

The proof of (ii) above parallels the standard proof of the Iwahori-Matsumoto relations in H, which state that for $x \in \tilde{W}$, $s \in S_{\text{aff}}$, and $\sigma \in \Omega$

$$T_xT_s = \begin{cases} T_{xs}, & \text{if } x < xs \\ q\cdot T_{xs} + (q-1)\cdot T_x, & \text{if } xs < x \end{cases}$$
$$T_xT_\sigma = T_{x\sigma},$$

where $<$ denotes the Bruhat order determined by S_{aff}. If $\mathcal{H}$ denotes the affine Hecke algebra over $\mathbb{Z}_v := \mathbb{Z}[v, v^{-1}]$ associated to our root system, this means we have a canonical isomorphism $H = \mathcal{H} \otimes_{\mathbb{Z}_v} \mathbf{C}$.

Note that R is the Iwahori-Hecke algebra for the group A, and hence it also has a v-analog over $\mathbb{Z}_v$, which we denote by $\mathcal{R}$. Concretely, we have $\mathcal{R} = \mathbb{Z}_v[X_*(A)]$.

We will use Lemma 7.2.1 as the starting point in defining v-analogs

$$\mathcal{M},\ \ i_T^{\tilde W}(\chi_{\text{univ}}^{-1}),\ \ (\mathcal{R},\mathcal{H}) - \text{actions},\ \ (\cdot|\cdot),\ \ K_w,\ \ \mathcal{M}'e_W,\ \ h^\vee$$

of the objects we have already studied

$$M,\ \ i_B^G(\chi_{\text{univ}}^{-1})^I,\ \ (R,H) - \text{actions},\ \ (\cdot,\cdot),\ \ K_w,\ \ M_K,\ \ h^\vee.$$

7.3. **v-analogs of M and $i_B^G(\chi_{\text{univ}}^{-1})^I$.** Let us define $\mathcal{M}$ to be the set of functions $\varphi : \tilde{W} \to \mathbb{Z}_v$ which are supported on a finite subset. This is a free $\mathbb{Z}_v$-module with basis given by the characteristic functions 1_x, $x \in \tilde{W}$.

Next we define $\delta : T \to \mathbb{Z}_v^\times$ by $\delta(t_\lambda) := v^{-2\langle\rho,\lambda\rangle}$. This is the v-analog of the function δ_B. By $\delta^{1/2}$ we will mean the obvious square root of δ, namely the character $t_\lambda \mapsto v^{-\langle\rho,\lambda\rangle}$.

The left action of $\mathcal{R}$ on $\mathcal{M}$ is given by the formula $t \cdot 1_x := \delta^{1/2}(t) 1_{tx}$. The right $\mathcal{H}$-action is given by defining (following Lemma 7.2.1), for $\varphi \in \mathcal{M}$,

(i) $$\varphi T_{s_\alpha}(t_\lambda w) = \begin{cases} v^2 \cdot \varphi(t_\lambda ws), & \text{if } w(\alpha) \text{ is } B\text{-positive} \\ \varphi(t_\lambda ws) + (v^2-1)\cdot\varphi(t_\lambda w), & \text{if } w(\alpha) \text{ is } B\text{-negative} \end{cases}$$

(ii) $$\varphi T_{s_0}(t_\lambda w) = \begin{cases} v^2 \cdot \varphi(t_\lambda ws_0), & \text{if } w(\tilde\alpha) \text{ is } B\text{-negative.} \\ \varphi(t_\lambda ws_0) + (v^2-1)\cdot\varphi(t_\lambda w), & \text{if } w(\tilde\alpha) \text{ is } B\text{-positive.} \end{cases}$$

(iii) $$\varphi T_\sigma(t_\lambda w) = \varphi(t_\lambda w\sigma^{-1}).$$

These rules determine a right $\mathcal{H}$-module structure on $\mathcal{M}$ [5], and moreover $\mathcal{M}$ is an $(\mathcal{R},\mathcal{H})$-bimodule. Indeed, it suffices to observe that by Lemma 7.2.1 this statement holds after every specialization $v \mapsto q^{1/2}$. Specialization arguments like this will be used repeatedly below to prove v-analogs of statements known for M.

Now define $i_T^{\tilde W}(\chi_{\text{univ}}^{-1})$ to be the set of functions $\phi : \tilde{W} \to \mathcal{R}$ which satisfy

$$\phi(tx) = \delta^{1/2}(t)\phi(x)\cdot t^{-1},$$

for $t \in T$ and $x \in \tilde{W}$. As in section 1.5, there is a canonical isomorphism

$$\mathcal{M} = i_T^{\tilde W}(\chi_{\text{univ}}^{-1}). \tag{7.3.1}$$

Explicitly, we associate to $\varphi \in \mathcal{M}$ the function ϕ given by

$$\phi(x) = \sum_{t\in T} \delta^{-1/2}(t)\ \varphi(tx)\cdot t.$$

The left action of $\mathcal{R}$ on $i_T^{\tilde W}(\chi_{\text{univ}}^{-1})$ is defined by $(t\cdot\phi)(x) = t(\phi(x))$. The right action of $\mathcal{H}$ is defined by requiring the isomorphism $\mathcal{M} \to i_T^{\tilde W}(\chi_{\text{univ}}^{-1})$ to be $\mathcal{H}$-linear (one could also write out an explicit rule, again in the spirit of Lemma 7.2.1). Then $\mathcal{M} = i_T^{\tilde W}(\chi_{\text{univ}}^{-1})$ is an isomorphism of $(\mathcal{R},\mathcal{H})$-bimodules.

7.4. **v-analog of the sesquilinear pairing.** We will define an $\mathcal{R}$-valued sesquilinear pairing $(\cdot|\cdot)$ on $i_T^{\tilde W}(\chi_{\text{univ}}^{-1})$ (thus on $\mathcal{M}$) which is *almost* the v-analog of $(\cdot,\cdot)$ (they differ by a constant). Hence, it will automatically satisfy the analogs of (1.9.2), (1.9.3), and (1.9.6). We write $\iota_{\mathcal{R}}$ for the v-analog of ι_R, namely the involution on $\mathcal{R} = \mathbb{Z}_v[X_*(A)]$ induced by the identity on $\mathbb{Z}_v$ and the map $\mu \mapsto -\mu$ on $X_*(A)$.

For $\phi_1, \phi_2 \in i_T^{\tilde W}(\chi_{\text{univ}}^{-1})$, define

$$(\phi_1|\phi_2) = \sum_{w\in W} v^{2l(w)}\ \iota_{\mathcal{R}}\phi_1(w)\,\phi_2(w). \tag{7.4.1}$$

Lemma 7.4.1. *The pairing $(\cdot|\cdot)$ on $\mathcal{M}$ induces the pairing $W(q)\,(\cdot,\cdot)$ on $M = \mathcal{M}\otimes_{\mathbb{Z}_v}\mathbf{C}$.*

[5] Matsumoto gives a similar definition for a left $\mathcal{H}$-action in [19, section 4.1.1].

Proof. Consider the $\mathcal{R}$-basis $\{1_w\}_{w\in W}$ for $\mathcal{M}$, and the corresponding R-basis $\{v_w\}_{w\in W}$ for M. From the definitions, we easily see

$$(1_w \,|\, 1_{w'}) = v^{2l(w)}\, \delta_{w,w'}.$$

It is therefore enough to prove

$$(v_w, v_{w'}) = q^{l(w)}\, W(q)^{-1}\, \delta_{w,w'}.$$

The orthogonality is clear, and then one can easily check that

$$(v_w, v_w) = (1_{A_{\mathcal{O}}NK}, v_w) = (1_{A_{\mathcal{O}}NK} T_{w^{-1}}, v_1) = q^{l(w)} W(q)^{-1}.$$

□

7.5. **v-analogs of normalized intertwiners.** For a simple reflection $s = s_\alpha$, define $J_s : \mathcal{M} \to \mathcal{M}$ by setting

$$J_s(1_1) = v^{-2}(1 - t_{\alpha^\vee}) \cdot 1_s + (1 - v^{-2}) t_{\alpha^\vee} \cdot 1_1,$$
$$J_s(1_1 h) = J_s(1_1) h, \ \text{ for } h \in \mathcal{H}.$$

This makes sense because $\mathcal{M}$ is the free $\mathcal{H}$-module generated by 1_1 (by the same upper-triangular argument we used to prove that M is the free H-module generated by v_1). Further, for any $w \in W$, choose a reduced expression $w = s_1 \cdots s_n$, and set

$$J_w := J_{s_1} \circ \cdots \circ J_{s_n}.$$

(The usual specialization argument shows that the right hand side is independent of the choice of reduced expression.)

Next define $\mathcal{L}$ to be the fraction field of $\mathcal{R}$; note that $\mathcal{L}^W$ is the fraction field of $\mathcal{R}^W$ and that $\mathcal{L} = \mathcal{L}^W \otimes_{\mathcal{R}^W} \mathcal{R}$. Imitating what we did before, we see that $\mathcal{R}^W$ embeds into the center of $\mathcal{H}$, so that we can form the algebra $\mathcal{H}_{\text{gen}} := \mathcal{L}^W \otimes_{\mathcal{R}^W} \mathcal{H}$ and the right $\mathcal{H}_{\text{gen}}$-module $\mathcal{M}_{\text{gen}} := \mathcal{L}^W \otimes_{\mathcal{R}^W} \mathcal{M} = \mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}$. Finally we define the normalized intertwiner $K_w : \mathcal{M}_{\text{gen}} \to \mathcal{M}_{\text{gen}}$ by

$$K_w := \Big(\prod_{\alpha \in R_w} \frac{1}{1 - v^{-2} t_{\alpha^\vee}} \Big) \cdot J_w.$$

It is clear that

(i) K_w is $\mathcal{H}_{\text{gen}}$-linear;
(ii) $K_w \circ t_\lambda = t_{w\lambda} \circ K_w$;
(iii) K_w fixes 1_W,

where $1_W := \sum_{w\in W} 1_w$ is the v-analog of $1_{A_{\mathcal{O}}NK}$. It is also clear that $w \mapsto K_w$ defines a homomorphism $W \to \mathcal{H}_{\text{gen}}^\times$, and that the v-analogs of (2.2.3-2.2.5) hold (using $(\cdot|\cdot)$ in (2.2.3)). Moreover, the v-analog of Lemma 5.3.1 holds. Finally, we recover Bernstein's result that $\mathcal{R}^W$ is the center of $\mathcal{H}$ by going through all the same steps we did before.

7.6. **v-analog of the Satake isomorphism.** Let $\mathbb{Z}'_v$, $\mathcal{R}'$, $\mathcal{H}'$, and $\mathcal{M}'$ denote the localizations of $\mathbb{Z}_v$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{M}$ at the element $W(v^2) \in \mathbb{Z}_v$. Let $T_W = \sum_{w\in W} T_w$ and $e_W = W(v^2)^{-1} T_W$, an element in $\mathcal{H}'$. Further, define $\mathcal{H}_0 = e_W \mathcal{H}' e_W$, and $\mathcal{M}_0 = \mathcal{M}' e_W$. Then $\mathcal{H}_0$ is a $\mathbb{Z}'_v$-algebra with product $* := W(v^2)^{-1} \cdot$ and identity element T_W, where $\cdot$ denotes the usual product in $\mathcal{H}$. Similarly $\mathcal{M}_0$ is an $\mathcal{H}_0$-module with product $* := W(v^2)^{-1} \cdot$, where $\cdot$ now denotes the usual $\mathcal{H}$-action on $\mathcal{M}$. It is clear that $*$ makes $\mathcal{M}_0$ an $(\mathcal{R}', \mathcal{H}_0)$-bimodule.

The $\mathcal{R}'$-module $\mathcal{M}_0$ is free of rank 1, so there is a homomorphism

$$\vee : \mathcal{H}_0 \to \mathcal{R}'$$

characterized by

$$m_0 * h_0 = h_0^\vee m_0,$$

for all $h_0 \in \mathcal{H}_0$ and all $m_0 \in \mathcal{M}_0$.

We have the formula

$$h_0^\vee = W(v^2)^{-1}(1_W \mid 1_W * h_0) = (1_1 \mid 1_1 h_0). \tag{7.6.1}$$

We can now easily derive the v-analog of Theorem 5.6.1. We apply the first equality of (7.6.1) to the function

$$h_\mu := \sum_{w \in W t_\mu W} T_w = \frac{W(v^2)W(v^{-2})}{W_\mu(v^{-2})}\, e_W T_{t_\mu} e_W \tag{7.6.2}$$

to get

$$\begin{aligned}(h_\mu)^\vee &= \frac{W(v^{-2})}{W(v^2)W_\mu(v^{-2})}\,(1_W \mid 1_W \cdot e_W T_{t_\mu} e_W)\\ &= \frac{v^{-2l(w_0)} v^{2(l(t_\mu)/2)}}{W_\mu(v^{-2})}\,(1_W \mid 1_W \Theta_\mu).\end{aligned}$$

Now using the v-analog of Lemma 5.3.1 as in the proof of Theorem 5.5.1, we find

$$(h_\mu)^\vee = \frac{v^{2(l(t_\mu)/2)}}{W_\mu(v^{-2})} \sum_{w \in W} w\Big(\prod_{\alpha>0} \frac{1 - v^{-2} t_{-\alpha^\vee}}{1 - t_{-\alpha^\vee}}\Big) \cdot t_{w\mu}. \tag{7.6.3}$$

7.7. **The Satake isomorphism commutes with the Kazhdan-Lusztig involution.** The compatibility of the Bernstein and Satake isomorphisms (4.6) is the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{R}'^W & \xleftarrow{\;\;b\;\;} & e_W \mathcal{H}' e_W \\ {\scriptstyle B}\big\downarrow & \nearrow{\scriptstyle -\cdot e_W} & \\ Z(\mathcal{H}'), & & \end{array}$$

where $b(h_0) := W(v^2) h_0^\vee$ and where the Bernstein isomorphism B sends $\sum_{\lambda \in W\mu} t_\lambda$ to z_μ. By Lemma 3.2.1, B commutes with the Kazhdan-Lusztig involution. Since $\overline{e_W} = e_W$ [6], the diagonal map does as well. We thus have the following lemma which is implicit in [16], section 8.

Lemma 7.7.1. *For every $h_0 \in \mathcal{H}_0$,*

$$b(\overline{h_0}) = \overline{b(h_0)}.$$

Equivalently,

$$(\overline{h_0})^\vee = v^{-2\,l(w_0)}\,\overline{h_0^\vee}.$$

[6] Via the function-sheaf dictionary, the Kazhdan-Lusztig involution corresponds to taking the Verdier dual. The equality can then be derived from the fact that if the constant sheaf on the smooth variety G/B is placed in degree $-l(w_0)$ and Tate-twisted by $l(w_0)/2$, the resulting complex is Verdier self-dual.

Note that the Kazhdan-Lusztig involution on the commutative ring $\mathcal{R}'$ is simply the map sending $\sum_\lambda z_\lambda(v, v^{-1})t_\lambda$ to $\sum_\lambda z_\lambda(v^{-1}, v)t_\lambda$.

7.8. **The Kato-Lusztig formula.** We now switch notation and let $q^{1/2}$ play the role of the indeterminate v used in sections 7.1-7.7. In this section we will use some elementary properties of the Kazhdan-Lusztig polynomials $P_{x,y}(q)$ attached to $x, y \in \tilde{W}$, all of which may be found in [15].

Recall that throughout this article, the Bruhat order $\leq$ and the length function $l(\cdot)$ on $\tilde{W}$ are defined using the $\bar{B}$-positive affine reflections $S_{\rm aff} := S \cup \{s_0\}$. For any dominant coweight λ, the element $w_\lambda := t_\lambda w_0$ is the unique longest element in $Wt_\lambda W$, and $l(t_\lambda w_0) = l(w_0) + l(t_\lambda) = l(w_0) + 2\langle\rho, \lambda\rangle$. It is known that

$$\{x \leq w_\mu\} = \cup_{\lambda \preceq \mu} Wt_\lambda W,$$

where λ ranges over dominant coweights such that $\mu - \lambda$ is a sum of B-positive coroots.

Theorem 7.8.1 (Kato, Lusztig)**.** *For any dominant coweight μ, let E_μ denote the character of the corresponding highest weight module of the Langlands dual group $G^\vee$. Let h_μ denote the function $\sum_{w \in Wt_\mu W} T_w$. Then we have*

$$E_\mu = \sum_{\lambda \preceq \mu} q^{-l(t_\mu)/2} P_{w_\lambda, w_\mu}(q)\ (h_\lambda)^\vee.$$

Proof. We have the identity

$$\overline{q^{-l(y)/2} \sum_{x \leq y} P_{x,y}(q) T_x} = q^{-l(y)/2} \sum_{x \leq y} P_{x,y}(q) T_x.$$

Applying this to $y = w_\mu$ and using $P_{ww_\lambda w', w_\mu}(q) = P_{w_\lambda, w_\mu}(q)$ for every $w, w' \in W$, we get

$$\overline{q^{-\langle\rho,\mu\rangle - l(w_0)/2} \sum_{\lambda \preceq \mu} P_{w_\lambda, w_\mu}(q)\ h_\lambda} = q^{-\langle\rho,\mu\rangle - l(w_0)/2} \sum_{\lambda \preceq \mu} P_{w_\lambda, w_\mu}(q)\ h_\lambda.$$

Applying the Satake isomorphism to both sides and using Lemma 7.7.1, we have

$$q^{\langle\rho,\mu\rangle} \sum_{\lambda \preceq \mu} P_{w_\lambda, w_\mu}(q^{-1})\ \overline{h_\lambda^\vee} = q^{-\langle\rho,\mu\rangle} \sum_{\lambda \preceq \mu} P_{w_\lambda, w_\mu}(q)\ h_\lambda^\vee.$$

By (7.6.3), this gives

$$\sum_{\lambda \preceq \mu} q^{\langle\rho,\mu-\lambda\rangle} P_{w_\lambda, w_\mu}(q^{-1}) W_\lambda(q)^{-1} \sum_{w \in W} t_{w\lambda} \prod_{\alpha > 0} \frac{1 - q\, t_{-w\alpha^\vee}}{1 - t_{-w\alpha^\vee}}$$
$$= \sum_{\lambda \preceq \mu} q^{-\langle\rho,\mu-\lambda\rangle} P_{w_\lambda, w_\mu}(q) W_\lambda(q^{-1})^{-1} \sum_{w \in W} t_{w\lambda} \prod_{\alpha > 0} \frac{1 - q^{-1} t_{-w\alpha^\vee}}{1 - t_{-w\alpha^\vee}}.$$

Now $\deg P_{w_\lambda, w_\mu}(q) \leq \langle\rho, \mu - \lambda\rangle - 1/2$ if $\lambda < \mu$ and so Lemma 7.8.2 below implies that the right hand side is a polynomial in q^{-1} (with coefficients in $\mathbb{Z}[X_*]^W$). Similarly, the left hand side is a polynomial in q. The result now follows since the constant terms are equal to

$$\sum_{w \in W} t_{w\mu} \prod_{\alpha > 0} (1 - t_{-w\alpha^\vee})^{-1} = E_\mu.$$

□

Lemma 7.8.2. *We have*

$$W_\lambda(q^{-1})^{-1}\sum_{w\in W} t_{w\lambda}\prod_{\alpha>0}\frac{1-q^{-1}t_{-w\alpha^\vee}}{1-t_{-w\alpha^\vee}} \in \mathbb{Z}[q^{-1}][X_*]^W.$$

Proof. It is obvious that the expression belongs to $\mathbb{Z}[[q^{-1}]][X_*]^W$, so it is enough by (7.6.3) to show that $h^\vee_\lambda$ belongs to $\mathcal{R}$. But this follows from (7.6.1). □

Taking $q=1$ we immediately recover Theorem 6.1 of [16]:

Theorem 7.8.3 (Lusztig)**.** *For any dominant coweight μ,*

$$E_\mu = \sum_{\lambda\preceq\mu} P_{w_\lambda,w_\mu}(1)\ (\sum_{w\in W/W_\lambda} t_{w\lambda}).$$

References

[1] J. Bernstein, *Representations of p-adic groups*, Notes taken by K. Rumelhart of lectures by J. Bernstein at Harvard in the Fall of 1992.

[2] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259.

[3] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[4] P. Cartier, *Representations of $\mathfrak{p}$-adic groups: a survey* In: *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math., vol. **33**, part 1, Amer. Math. Soc., Providence, RI, 1979, pp. 111–155.

[5] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, unpublished notes.

[6] W. Casselman, *The unramified principal series of $\mathfrak{p}$-adic groups I. The spherical function*, Compositio Math. **40** (1980), 387–406.

[7] W. Casselman and J. Shalika, *The unramified principal series of $\mathfrak{p}$-adic groups II. The Whittaker function*, Compositio Math. **41** (1980), 207–231.

[8] N. Chriss and K. Khuri-Makdisi, *On the Iwahori-Hecke algebra of a p-adic group*, Internat. Math. Res. Notices (1998), 85–100.

[9] J.-F. Dat, *Caractères à valeurs dans le centre de Bernstein*, J. Reine Angew. Math. **508** (1999), 61–83.

[10] V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality*, Preprint (1996). alg-geom/9511007.

[11] T. J. Haines, *The combinatorics of Bernstein functions*, Trans. Amer. Math. Soc. **353** (2001), 1251–1278.

[12] T. J. Haines, *On matrix coefficients of the Satake isomorphism: complements to the paper of Rapoport*, manuscripta math. **101** (2000), 167–174.

[13] H. Jacquet, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France **95** (1967), 243–309.

[14] S. Kato, *Spherical functions and a q-analogue of Kostant's weight multiplicity formula*, Invent. Math. **66** (1982), 461-468.

[15] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.

[16] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities* In: *Analyse et topologie sur les espaces singuliers, I-II (Luminy, 1981)*, Soc. Math. France, Paris, 1983, pp. 208–229.

[17] G. Lusztig, *Affine Hecke algebras and their graded versions*, J. Amer. Math. Soc. **2** (1989), 599–635.

[18] I. G. Macdonald, *Spherical functions on a group of $\mathfrak{p}$-adic type*, Ramanujan Institute, University of Madras Publ., 1971.

[19] H. Matsumoto, *Analyse Harmonique dans les Systèmes de Tits Bornologiques de Type Affine*, Springer Lecture Notes 590, Berlin, 1977.

[20] I. Mirkovic, K. Vilonen, *Perverse sheaves on affine Grassmannians and Langlands duality*, Math. Res. Lett. **7** (2000), no.1, 13-24.

[21] K. Nelsen, A. Ram, *Kostka-Foulkes polynomials and Macdonald spherical functions*, Surveys in combinatorics, 2003 (Bangor), 325–370, London Math. Soc. Lecture Note Ser., **307**, Cambridge Univ. Press, Cambridge, 2003.
[22] M. Rapoport, *A positivity property of the Satake isomorphism*, manuscripta math. **101** (2000), 153–166.
[23] M. Reeder, *On certain Iwahori invariants in the unramified principal series*, Pacific J. Math. **153** (1992), 313–342.
[24] M. Reeder, $\mathfrak{p}$-*adic Whittaker functions and vector bundles on flag manifolds*, Compositio Math. **85** (1993), 9–36.
[25] F. Rodier, *Whittaker models for admissible representations of reductive* $\mathfrak{p}$-*adic split groups*, In: *Harmonic Analysis on Homogeneous Spaces*, Proc. Symp. Pure Math. XXVI, Amer. Math. Soc., 1973.
[26] I. Satake, *Theory of spherical functions on reductive algebraic groups over* $\mathfrak{p}$-*adic fields*, Inst. Hautes Études Sci. Publ. Math. **18** (1963), 1–69.
[27] T. Shintani, *On an explicit formula for class-*1 *Whittaker functions*, Proc. Japan Acad. **52** (1976), 180–182.

University of Maryland
Mathematics Department
College Park, MD 20742-4015
tjh@math.umd.edu

University of Chicago
Department of Mathematics
5734 S. University Ave.
Chicago, IL 60637
kottwitz@math.uchicago.edu

IHES
Le Bois-Marie
35, route de Chartres
F-91440 Bures-sur-Yvette, FRANCE.
prasad@ihes.fr