
Analyzing Cost of Debt and Credit Spreads Using a Two Factor Model with Multiple Default Thresholds and Varying Covenant Protection

by

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Abstract: The cost of debt capital for corporations depends on credit spreads. Of course, the dramatically greater credit spreads of 2008 greatly increased the cost of debt for the majority of bond issuers. We analyze the shape of credit spread term structures paying special attention to the humps that have been observed by a number of researchers. The shape of credit spreads depends upon the shape of first passage default. Importantly, our work allows separation of default probability due to breach of barrier versus default probability due to assets being less than face value at maturity. We note that in some cases, first passage default has a hump but not in others. It is useful to see when and how first passage default humps contribute to a humped credit spread. The impact of recently popular weak covenants (covenant lite) is shown to play a major role in the shape of credit spreads. The implications of our study are important to such topics as measuring the riskiness of the banking system dependent upon credit spread slopes.

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Introduction

Pricing credit risk and default potential have always been important research topics and the credit crisis of 2007 and 2008 has generated even greater interest. Numerous empirical studies of credit spreads have been performed. For example, Krishnan, Ritchken and Thomson (2006) use credit spread slopes of bank debt to predict bank risk. As others, they find the shape of the credit spread can be positive or negative and is often humped. Furthermore, they find that over time the credit spread can change in different ways for individual banks.

Our purpose is to analyze the shape of credit spread term structures. More specifically, we analyze the term structure of first passage default and, in turn, its impact upon term structures of default risky yields and credit spreads. We analyze how these term structures depend on such things as covenants, the volatility of interest rates, volatility of firm value, correlation of firm value with interest rates, and other model parameters. Humps occur when the term structure slope changes from positive to negative. A hump in first passage default can encourage a hump in credit spread but is not necessary for a hump to occur. Conditions for a hump in term structures are particularly interesting and useful to explain. Furthermore, the impact of (weak) covenants upon these term structures is particularly timely given the recent credit crisis. As we show, it is important to separate default due to breach of barrier versus default due to assets being less than face value at maturity. Certain industries and firms may have a hump in term structure of default probability due to strength of covenants (barriers) where others do not. To our knowledge, we are the first to apply and analyze Giesecke's (2004) separation of default

probabilities. Banking regulators should find anything which helps predict default risk and the health of the banking system very useful in developing regulation policies.

Even though credit spread models have been found useful, a number of empirical studies have noted a lack of explanatory power. Collin-Dufresne, Goldstein and Martin (2001) find that some variables that should explain credit spread changes have only limited explanatory power.¹ In an effort to increase explanatory power, Driessen (2005), for example, focuses on adding an event risk premium but finds such a premium cannot be estimated very well. Christensen (2008) refers to the inability of empirical models to explain credit spreads as the ‘corporate bond credit spread puzzle’. The limitations of these empirical findings using extant models suggest a need for improved theoretical models that utilize firm specific or industry specific factors such as covenants and default barriers and, also, a need to segregate default probability into that due to barrier (covenants) versus classic Merton (1974) maturity default.²

The first structural model of credit spreads was developed by Merton (1974) where he assumed a constant short term interest rate.³ Leland (1994) and Leland and Toft (1996) also assumed a constant interest rate. It is unappealing to not allow interest rates to change in a model of bond valuation. Thus, in contrast, Longstaff and Schwartz (1995), Collin-Dufresne and

¹ Schaefer and Strebulaev (2008) suggest that although structural models may not predict spreads and prices very well, they provide good hedging information.

² Improved empirical modeling could also improve empirical results.

³ There are two broad classes of theoretical models: structural and reduced form. For a comprehensive analysis of alternative default models, see Duffie and Singleton (2003). Jarrow and Protter (2004) compare reduced form and structural models using an information based perspective. Jarrow, Lando,Turnbull (1997) provide reduced from model. We focus upon improvements to structural models.

Goldstein (2001) and Acharya and Carpenter (2002) models include both a value of the firm (V_t) process and, also, an interest rate (r_t) process. Eom, Helwege, and Huang (2004) empirically test alternative models and provide an excellent characterization of single factor versus two factor models in an appendix. Two factor models would certainly seem to be very appealing when, for example, interest rates are more volatile than average.

Longstaff and Schwartz (1995) develop a two factor model involving V_t and r_t where r_t is based on the Vasicek (1977) model. Using the standard framework for pricing a contingent claim, Longstaff and Schwartz (1995) define the probability of default as a solution of a partial differential equation. Also, using yet another known result, they then characterize the default density as solution of an integral equation. By invoking the standard discretization scheme, this latter equation is reduced to a system of linear equations which is then solved for the default probability.

Acharya and Carpenter (2002) also consider the two factor model involving V_t and r_t where, in contrast, r_t evolves according to the Cox, Ingersoll, Ross (1985) model. Using the classical (maturity) definition of default, they compute the default probability with numerical methods based upon a binomial tree approximation.

Our V_t and r_t models are similar to Longstaff and Schwartz (1995) and we make important enhancements to their model. We note that Lakshmivarahan, Qian, and Stock (2008) quantify the distribution of V_t and show that it has the lognormal form with time varying mean and variance. We use the definition of default due to Giesecke (2004) which, more

comprehensively, has two thresholds instead of one. By combining the lognormal distribution of V_t in Lakshmivarahan, Qian, and Stock (2008) and the newer, more realistic, definition of default of Giesecke (2004), we then explicitly express the default probability as the sum of two quantities. One quantity represents the classical (maturity) default of Merton (1974) and the other quantity represents barrier default which, critically, depends on covenants. Given work by Altman, Brady, Resti and Sironi (2005), we recognize that recovery in event of default is likely best modeled as dependent upon probability of default instead being assumed constant.

Furthermore, probability of default may be conveniently expressed as a function of expected value of V_t and variance of V_t . In effect, as maturity increases, there is a race between increasing $E(V_t)$ and increasing $Var(V_t)$ to determine first passage default probability and the shapes of the resulting term structures of default probability, default risky rates, and credit spreads.⁴ Thus, the impact of maturity upon credit spread is complex.

Interestingly, probability of default may or may not increase with maturity. If the increase in $E(V_t)$ due to greater maturity is stronger than the increase in $Var(V_t)$, the probability of default may decline along with the credit spread. This may be especially true when default barriers are low due to (recent) weak covenant protection (covenant lite). Such analysis helps explain humped credit spreads so frequently found by, among others, Merton (1974), Longstaff and Schwartz (1995) and Krishnan, Ritchken and Thomson (2006). Also, humps in credit

⁴ Default may mean renegotiation with lenders to restructure or, alternatively, foreclose. The restructure or foreclose decision is beyond the scope of our research and we refer interested readers to Brown, Ciochetti, and Riddough (2006) for more on this decision.

default swap term structures have been found by Bajlum and Larsen (2007) and Lando and Mortensen (2005) and others. We are the first to analyze first passage probabilities for both maturity and barrier default with a two factor model. An important observation is that greater passage of time suggests greater expected firm value but, at the same time, greater variance of V_t , so that the impact of maturity on default probability is unclear.

Recent events in the credit crisis make our analysis of barrier default especially useful. That is, a major determinant of barrier defaults are covenants which, if violated, may constitute default or suggest imminent default.⁵ The purpose of such covenants is to help control agency costs of debt.⁶ That is, bond buyers are concerned about the potential for stock holders to take actions that are to the disadvantage of bond holders at some time after issuance. More specifically, the firm may underinvest in projects that benefit bondholders but have little or no benefit for stock holders. Probably more relevant to our research is firm overinvestment in high risk projects that tend to produce benefits for stock holders but, at the same time, make the bond holder position more risky. As an example, Parrino and Weisbach (1999) find that firms may even invest in negative present value projects that tend to have high volatility. Billet, King and Maurer (2007) find strong evidence that bond holders require more covenant protection for firms

⁵ As given by Lando and Mortensen (2005), default may be viewed as triggered by covenants but can also be viewed as inability to cover required coupons due to liquidity constraints or inability to raise new equity capital

⁶ Chava and Roberts (2008) test the importance of covenant violation. They find that capital investment declines sharply following financial covenant violation as creditors use the threat of accelerating the loan to intervene in management.

with high growth opportunities, longer maturity debt, and greater leverage. Consistent with these findings, lower rated debt tends to have more covenant protection.

Curiously, there has been a recent trend in some debt markets to have much lower covenant protection than what agency theory and the Billet, King and Mauer (2007) findings would suggest. Such lack of covenant protection, termed “covenant lite”, has led to misleadingly low default rates even as the credit crisis of 2007 and 2008 continued and the economy weakened. Many fear that higher default rates are merely being delayed and these high delayed defaults will later be very damaging to a financial system that many thought had fully recovered from the credit crisis. One of our main contributions is explicit analysis of how default probability is a varying function of barriers determined by covenants.

Section 1 describes the probability density of default as dependent upon a two factor valuation process assuming the Vasicek (1977) process for the interest rate. Default occurs when first passage of firm value hits a barrier before maturity, or, at maturity if value of the firm is then less than face value of debt. In Section 2 we derive expressions for default risky spot rates and credit spreads. Probability of default depends upon such factors as leverage, barriers,⁷ and the volatility of firm value. Section 3 reports the computation for the term structures of risky spot rates, credit spreads and default probabilities. Then, in Section 4, we discuss the conversion of the risk neutral default probabilities to physical default probabilities. It turns out that the term

⁷ These barriers describe levels of V_t where debt holders lose patience with the firm or where covenants are violated.

structures of these physical default probabilities are very similar. Section 5 provides expressions for how credit spread slope is explicitly dependent upon default probability slope and suggests refinements in predicting future bank risk. Concluding observations are provided in the last section.

1. The Distribution of V_t and Expression for Default Probability

Let the value process V_t evolve according to the stochastic differential equation (SDE)

$$\frac{dV_t}{V_t} = r_t dt + \sigma_v dW_{v,t} \quad (1.1)$$

where the short-term interest rate r_t evolves according to the narrow sense linear model, due to Vasicek (1977), given by the SDE ⁸

$$dr_t = c \left[\frac{\theta}{c} - r_t \right] dt + \sigma_r dW_{r,t} . \quad (1.2)$$

The interest rate process has long run mean θ/c and reverts to the mean at the rate c . The driving innovations are correlated and

$$E \left[dW_{v,t} dW_{r,t} \right] = \rho dt \quad (1.3)$$

where $|\rho| \leq 1$. Here ρ is assumed constant. The model parameters σ_v , σ_r , θ , c , ρ and the initial conditions V_0 and r_0 are specified. ⁹

⁸ See Arnold (1974) for characterization of processes as narrow sense linear, general linear, and nonlinear.

⁹ The effect of including cash outflows, such as dividends and interest payments, as the factor γ is

This coupled system of SDEs can be solved explicitly and the solution is given by

$$\frac{V_t}{V_0} = e^{g_t} \quad \text{and} \quad g_t = g_t(\text{det}) + g_t(\text{ran}) \quad (1.4)$$

where

$$g_t(\text{det}) = mt + \left(\frac{r_0}{c} - \frac{\theta}{c^2} \right) (1 - e^{-ct}), \quad m = \left(\frac{\theta}{c} - \frac{1}{2} \sigma_v^2 \right) \quad (1.5)$$

and

$$g_t(\text{ran}) = \int_0^t [v_1(s) + (t-s)v_2(s)] dW_{1,s} + \int_0^t v_3(s) dW_2(s) \quad (1.6)$$

(Refer to Lakshmivaran, Qian, and Stock (2008) for details). Here $W_1(t)$ and $W_2(t)$ are two independent Wiener processes and

$$v_1(s) = \rho \sigma_v, \quad v_2(s) = \sigma_r e^{-(t-s)}, \quad v_3(s) = \sqrt{1 - \rho^2} \sigma_v \quad (1.7)$$

Since each of the Ito integrals on the RHS of (1.6) are martingales, their sum (Karatzas and Shreve (1991) and Shiryaev (1999)) can be equivalently represented by the “time change” of a standard Wiener process $B(t)$ as

$$g_t(\text{ran}) = B(\sigma^2(t)) \quad (1.8)$$

where

$$\sigma^2(t) = \int_0^t [v_1(s) + (t-s)v_2(s)]^2 ds + \int_0^t v_3^2(s) ds$$

$$\frac{dV_t}{V_t} = (r_t - \gamma) dt + \sigma_v(t) dW_v \quad .$$

This can be taken into account by replacing $\frac{1}{2} \sigma_v^2$ with $(\gamma + \frac{1}{2} \sigma_v^2)$ in all the expressions that follow.

$$= \sigma_v^2 t + \frac{\sigma_r^2}{4c^3} \left[1 - e^{-2ct} (2c^2 t^2 + 2ct + 1) \right] + \frac{2\rho\sigma_v\sigma_r}{c^2} \left[1 - e^{-ct} (ct + 1) \right]. \quad (1.9)$$

Consequently, with $\mu(t) = g_t(\text{det})$,

$$g_t = \mu(t) + B(\sigma^2(t)) \quad (1.10)$$

and $\left(\frac{V_t}{V_0}\right)$ has a lognormal distribution given by

$$P\left[\frac{V_t}{V_0}\right] = \frac{1}{\sqrt{2\pi}\sigma(t)\left(\frac{V_t}{V_0}\right)} \exp\left[-\frac{\left[\ln\left(\frac{V_t}{V_0}\right) - \mu(t)\right]^2}{2\sigma^2(t)}\right]. \quad (1.11)$$

Default is defined by the occurrence of the event

$$B(K, D) = \left\{ V_T < K \text{ or } \min_{0 \leq s \leq T} V_s < D \right\} \quad (1.12)$$

where the probability of default can be given as

$$P_d(T) = \text{Prob}[B(K, D)] = 1 - \text{Prob}\left[V_T > K \text{ and } \min_{0 \leq s \leq T} V_s > D\right] \quad (1.13)$$

As in Giescke (2004), Brockman and Turtle (2003), Reisz and Perlich (2007) and others, default occurs in two ways. First, in the classical case of Merton (1974), default occurs when the value of the firm falls below the face value of the debt (K) at time T . We assume $K = 100$. Additionally, default occurs before maturity, $t < T$, when the value of the firm falls below a *barrier* level, D . That is, frequently creditors have a right to pull the plug on the firm during financial distress. (Note that the Longstaff and Schwartz (1995) default is

prematurity only.) Various covenants in bank loans, bonds, and other debt may contain a right to effectively pull the plug. Similarly, bank regulators may close a financial institution when equity is below a certain level of assets. See Core and Schrand (1999) for analysis of this type of covenant. For our purposes, we assume, as given in Giesecke (2004) and assumed by Brockman and Turtle (2003), that D is below K . Thus, P_d is the probability the first passage is before T or at T . More formally, first passage to default is given as

$$\tau = \min (\tau_1 , \tau_2)$$

where τ_1 is first passage to barrier D and τ_2 is at maturity, T , if $V_T < K$. Please see Figure 1, as given in Giesecke (2004), for illustration.¹⁰

Referring to the first part of Appendix A, it turns out that the closed form expression for $P_d(T)$ is available only for the special case when

$$g_t = at + bB(t) \tag{1.14}$$

where a and b are constants and $B(t)$ is the standard Wiener process. See Elliott and Kopp (1999) and Giesecke (2004). Since (1.10) is not of the form (1.14), in the following we seek “good” linear approximations to $\mu(t)$ and $\sigma^2(t)$.

Referring to the second part of Appendix A, it can be verified that

$$\mu(t) \approx \bar{\mu}(t) = m_1 t \quad \text{and} \quad \sigma^2(t) \approx \bar{\sigma}^2(t) = m_2 t \tag{1.15}$$

¹⁰ Black and Cox (1976) were the first to suggest barriers but they assumed r_t was constant.

Where $m_1 = m_1(\alpha)$ and $m_2 = m_2(\alpha)$ are constants that depend on the real parameter α . Substituting (1.15) in (1.10) we get

$$g_t = m_1 t + \sqrt{m_2} B(t) \quad (1.16)$$

For this case, following Elliott and Kopp (1999) and Giesecke (2004) we readily obtain the expression for the default probability.

$$P_d(T) = \Phi \left[\frac{\ln \left(\frac{K}{V_0} \right) - m_1 T}{\sqrt{m_2} \sqrt{T}} \right] + \left(\frac{D}{V_0} \right)^{\frac{2m_1}{m_2}} \Phi \left[\frac{\ln \left(\frac{D^2}{KV_0} \right) + m_1 T}{\sqrt{m_2} \sqrt{T}} \right] \quad (1.17)$$

where

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^x \exp\left(-\frac{z^2}{2}\right) dz \quad (1.18)$$

The first term on the right hand side is “classical” default due to assets being less than K (face value) at maturity and the second is default due to breach of barrier. A sample plot of the variation $P_d(T)$ versus T is given in Figure 2. Appendix B gives expressions (derivatives) for the sensitivity of default to various parameters.

The flat barrier we use is similar to Longstaff and Schwartz (1995). As they do, we note that the barrier can alternatively be given as time varying with D equal to K at maturity. Alternative results using a time varying barrier (not shown here) are qualitatively unchanged from those reported below.

2. Risky Spot Rates and Credit Spreads As Dependent Upon First Passage Probability of Default

2.a. Derivation of Expressions for Spot Rates and Credit Spreads

Given distributions $\frac{V_t}{V_0}$ and $\log\left(\frac{V_t}{V_0}\right)$, we now develop compact expressions for default risky spot rates, $R_d(T)$, and the spot rate spread over a case with no default risk, $R_{df}(T)$. Here $R_d(T) - R_{df}(T)$ is denoted as $S(T)$. This combines the two processes for r_t and V_t where the level and volatility of V_t represent default risk. Debt with no default risk and maturity T has present value at T of $PV_{df}(T) = PAR$. For a bond with default risk, the present value at T is

$$PV_d(t|t=T) = PV_{df}(T)[1 - P_d(T)] + PV_{df}(T)P_d(T)RR(P_d(T)). \quad (2.1)$$

Here, $P_d(T)$ is the first time (first passage) the firm value hits a level, K or D, where default occurs.¹¹ RR is the recovery rate of principal in case the firm defaults and we express this as a function of P_d .

¹¹ Our first passage default probability should be distinguished from a hazard rate which is a default intensity measure typically starting at t and ending at t+1. Cumulative hazard rates form a series of nondecreasing rates from zero to a multi-period horizon. See Duffie, Saita and Wang (2007) for an example.

We can now derive the default risky bond price at maturity T , $PV_d(T)$, from the default-free bond present value, $PV_{df}(T)$, the par value of the bond. The risky spot rate and spread are derived from the default-free spot rate $R_{df}(T)$ and the default risky bond price at maturity, $PV_d(T)$. $PV_d(0)$ is the present value of the bond which may default at T.

$$PV_d(0) = PV_d(T) e^{-R_{df}(T)T} \quad (2.2)$$

$PV_d(0) \leq PV_{df}(0)$ due to the risk of default.

To define the risky spot rate, $R_d(T)$, we easily construct a default risky bond, which sells at $PV_d(0)$ at present, and is valued at PAR at maturity. Thus,

$$PV_d(0) e^{R_d(T)T} = PAR = PV_{df}(T) . \quad (2.3)$$

After substituting,

$$PV_d(T) e^{-R_{df}(T)T} e^{R_d(T)T} = PAR = PV_{df}(T) . \quad (2.4)$$

Then, dividing both sides by $PV_d(T)$ and simplifying we obtain

$$R_d(T) = \frac{1}{T} \ln \left[\frac{1}{1 - [1 - RR(P_d(T))] P_d(T)} \right] + R_{df}(T) \quad (2.5)$$

and

$$S(T) = R_d(T) - R_{df}(T) = \frac{1}{T} \ln \left[\frac{1}{1 - [1 - RR(P_d(T))] P_d(T)} \right] \quad (2.6)$$

As long as $P_d(T) > 0$ and $RR(P_d(T)) \leq 1$,

$$1 - [1 - RR(P_d(T))] P_d(T) < 1 \quad (2.7)$$

which implies that $R_d(T) > R_{df}(T)$.

Altman, Brady, Resti, and Sironi (2005) and Altman, Resti, and Sironi (2005) have modeled the recovery rate as dependent upon default probability and found a negative relation. The logic is that when default rates are high, the economy is typically weak and the value of assets is relatively depressed compared to cases where the economy is stronger. We use the below functional form that Altman, Resti, and Sironi (2005) found to be the best fit.

$$RR(P_d(T)) = a_{RR} (P_d(T))^{b_{RR}}, \quad (2.8)$$

where $a_{RR} = 0.1457$ and $b_{RR} = -0.2801$.

2.b. Computations of Risky Spot Rates and Credit Spreads

Given the above system, we may now analyze $P_d(T)$, $R_d(T)$, and $S(T)$ term structures as dependent upon various determining parameters where we stress the impact of the term structure of $P_d(T)$ upon $R_d(T)$ and $S(T)$. Our examination has an analytical advantage compared to

prior research in that we can readily compute expected values and variances for a *two factor* model from the above distribution for V_t . Assuming the popular Vasicek (1977) model of short term interest rates, we can very easily plot $E(V_T)$, $Var(V_T)$, and $P_d(T)$ for any parameters of a particular interest rate process. Then we analyze how parameter values such as the default barrier and complex interactions dictate various levels and shapes of $P_d(T)$, $R_d(T)$, and $S(T)$.

Again, $E(V_t)$ values that grow faster with time obviously tend to reduce default risk because, in the great majority of cases, the value of the firm is assumed to drift upward with the passing of time. In the above $E(V_t)$ expressions, the growth rate in firm value is obviously affected by the level of interest rates (higher interest rates, higher V_t growth) and the r_t process in the alternative interest rate models. However, as $E(V_t)$ grows with time, the variance of V_t also increases with time. That is, there is a race between the growth of expected value and variance to determine the precise impact of T on $P_d(T)$.

A large rate of increase in probability of default due to time passage tends to increase the relative slope of $R_d(T)$ and $S(T)$ even though the slopes may be gently positive or even negative. However, we note that the impact of a first passage probability that grows with time is complex. That is, first passage default probability may grow with time but, $S(T)$ and $R_d(T)$ may or may not increase with time because the present value of the expected loss is diminished with greater time. Also, the greater the level of interest rates, the more present value is

diminished. With respect to level and structure of interest rates, we note that Krishnan, Ritchken, and Thomson (2008) find that the shape of the riskless term structure can help predict credit spreads. In fact, they suggest that current credit spreads and risk free yields curves impound practically all necessary information for predicting credit spreads. It is quite interesting to demonstrate that the probability of default may not necessarily increase with time.

Let us consider how the distribution of V_t explains $P_d(T)$. Figure 3 is one example of how our analysis permits detailed analysis of default spreads. Here we assume $V_0 = 150$, $c = 0.3$, $\theta = 0.18$, $r_0 = 0.06$, $\sigma_r = 0.02$, $\sigma_v = 0.02$, $D = 60$, and no correlation between firm value and level of interest rates.¹² This is a flat $R_{df}(T)$ term structure as $\theta/c = r_0$. Expected value of the firm rises with maturity where it is 498 at a maturity of 20. $Var(V_T)$, in the second panel, increases to about 307,000 at maturity 20. $P_d(T)$ always grows with maturity in this case which is shown in the third panel. The fourth panel displays the derivative of $P_d(T)$ with respect to T which, in this case, is always positive although it consistently declines. The last two panels display $R_d(T)$ and $S(T)$. Note that $R_d(T)$ peaks at around $T=3$ which is in contrast to the flat $R_{df}(T)$. If $R_{df}(T)$ is flat, any $R_d(T)$ hump here is totally due to default risk and the $S(T)$ shape. Thus, even though $P_d(T)$ consistently increases (no hump) beyond $T=3$, its impact on the spot

¹² Different studies have found a wide variation in estimates of σ_r and other parameters need for computation. We use various parameter values for σ_r , θ and c estimated and used by Pritsker (1998), Zeytune and Gupta (2007) and Ait-Sahalia (1996). We use σ_v values estimated by Parrino, Poteshman, and Weisbach (2005). Appendices show our results are robust to a wide range of parameter values.

rate weakens in this case because the present value of the expected loss declines with time. $S(T)$ also peaks at around $T=3$ and then declines.

Figure 4 is another set of graphs depicting default spreads where θ is now 0.03. Thus, θ/c is 0.10 and greater than r_0 (0.06) such that the $R_d(T)$ term structure is positive. Here expected value of the firm grows much more rapidly than above but, at the same time, so does variance around expected value. For this particular case, the net effect is to generally reduce probability of default relative to the previous figure. Interestingly, the probability of default peaks and then falls and thus its derivative becomes negative. For this case, $R_d(T)$ always increases but $S(T)$ peaks a little bit earlier than our previous figure. The earlier $S(T)$ peak and subsequent steep decline is due to the hump in $P_d(T)$. We examine the functional relationship between $S(T)$ and $P_d(T)$ in more detail in a later section. The $P_d(T)$ hump is counterintuitive at first glance but greater $E(V_t)$ growth may dominate the increase in variance so that default declines with T . We explain the shape of $P_d(T)$ term structures in more detail below.

It is constructive to also consider graphs of the lognormal distribution of V_t and $\log\left(\frac{V_t}{V_0}\right)$. See Figure 5, panels a, b, c, and d, where the distributions for different maturities are given again assuming the initial value (V_0) is 150 and face value of debt is 100. Present value of debt will be much lower than 100 for long maturities. Panels a, b, and c use a ρ of zero. Here, we again use a flat $R_d(T)$ term structure as θ/c equals r_0 although the level is now 10%. For the moment, we

assume only the classic (maturity) default case where the barrier (D) is zero for this illustration. In fact, the role of alternative (pre-maturity) default barriers cannot be displayed in graphs of distributions of V_t and $\log\left(\frac{V_t}{V_0}\right)$. That is, distributions only address value at time T. At time T, if V_T is less than 100, default occurs as illustrated by the area to the left of the vertical line. One may compare the classic “default areas” for the different maturities to determine if classic maturity default probability increases or decreases with the assumed discrete maturities. That is, in Figure 5a we see that the area for V_T less than 100 (default) if $T = 20$ is less than for $T = 2, 5,$ and 10, thus yielding a hump. The maximum for these cases is at $T = 5$. A shortcoming of Figure 5a is that, given the dramatic asymmetry, one cannot see how $E(V_T)$ behaves with respect to T. Thus we give Figure 5b for $\log\left(\frac{V_T}{V_0}\right)$ where, given a symmetric distribution, one can see that the expected value of the distribution increases with T. The increase may seem small but realize that a small increase in $\log\left(\frac{V_T}{V_0}\right)$ represents a large increase in V_T . Figure 5c is the humped P_d curve associated with the 5a and b. Finally, Figure 5d, using a ρ of 0.5, shows similar behavior. We again note that Figure 5 does not consider barrier default but still illustrates behavior robust to later analysis including D values.

The figures of this section have used example parameter values that conveniently build upon our models for dV_t and dr_t and, also, tend to most easily illustrate shapes and properties we

wish to stress. We have found these qualitative results robust to alternative parameters where Appendix C gives some sample alternative parameter results.

3. Parameter Values That Determine Shape of $P_d(T)$

The above figures for V_t distributions and $P_d(T)$ are a start at understanding the general theory of $P_d(T)$ shape. We now more systematically analyze how parameters interact to create a hump as it may appear interesting and counterintuitive to many. Our analysis shows that the hump can appear or disappear by, for example, varying D , the slope of the term structure (R_{df}), σ_v , and leverage. For more analytics on the sensitivity of $P_d(T)$ to different parameters, please see Appendix B.

Sensitivity to Barriers (D)

Barrier values, unlike most other parameters, can be viewed as a decision variable that borrowers and lenders negotiate at issuance. Brockman and Turtle (2003) suggest the most common barriers are covenants such as debt/equity and times interest earned requirements. Furthermore, they contend that breaching any barrier can trigger debt recall, default, or bankruptcy where more than one can occur simultaneously. If lenders are inclined to have little patience for a firm with low credit quality, they may negotiate a high value for D whereas other lenders may be much more tolerant and thus allow a lower D . Lenders may require a relatively low D , if for example, assets to be claimed in default are tangible and tend to be relatively liquid.

In the years prior to the credit crisis of 2007-2008, a trend toward weaker covenant protection was one example of borrower friendly, liberal lending terms that may have led to an overleveraged financial system. One of the outstanding examples of liberal lending terms was the tendency for many leveraged loans to be “covenant lite” where the debt had weak or no covenants. One explanation is that bank demand to syndicate loans was so high they were willing to agree to any covenant structure.¹³ As another example of liberal lending terms, “toggle” high yield bonds became popular where, along with weak covenant protection, the issuer could choose to pay an interest in the form of newly issued bonds.

The effect of issuing debt with weak covenant protection earlier in the century has led to dangerously misleading low default rates early in the credit crisis. More specifically, weak covenant protection means that default barriers are very low thus permitting borrowers to continue operating even if ratios such as “times interest earned” are very low. Fitch has reported that defaults in 2006, 2007 and early 2008 were thus very low even though credit quality may have greatly deteriorated. Because low defaults (misleadingly) suggest low risk and tolerance for risk, more aggressive debt issuance is encouraged. In the long term, weak covenant protection may well *not reduce* default but merely *shift* (very high) default to later dates. Default rates are even more misleading low if, as is often suspected, weak covenants reduce the ability of

¹³ See A. Denitz, “Sponsors Will Not Allow the Feared Raft of Loan Defaults”, *Financial Times*, July 16, 2008.

lenders to restrict management from undertaking riskier projects (overinvestment) reflected in a shift toward greater σ_v a couple years after debt issuance.¹⁴

Delayed defaults could be bad for analysts and investors who may have predicted the worst of the credit crisis was over after initial credit shocks worked through the system within a typical period of time.¹⁵ Ratings agencies such as Fitch and Standard and Poor's now recognize such covenant risk and adjust their ratings for the degree of covenant protection.

Brockman and Turtle (2003) estimate implied D (as opposed to more narrowly defined explicit covenant D) values for numerous industries and find a very wide variation. Reisz and Perlich (2007) suggest improvements to BT barrier estimates. We accept the concept that there are both explicit (covenants) and implicit barriers at which lenders effectively “pull the plug” on borrowers and we thus assume various D values estimated from previous studies.¹⁶

¹⁴ See *Fitch Ratings, Credit Market Research* “U.S. Leveraged Loan Covenant Decline Accelerating in 2007”, and *Dealbreaker* “The Mystery of Low Defaults for Leveraged Loans”, January 10, 2008.

¹⁵ In one contrast to loans with weak covenants, Reuters (“Goldman Sees Faster Pickup in US Junk Bond Defaults,” June 23, 2008, www.Reuters.com) notes that many bank loans had reasonably strong covenants the effectively forced home builders to liquidate land and inventory to force covenant compliance.

¹⁶ Of course, if D is breached and technical default occurs, then lenders may permit subsequent reorganization of the firm or, alternatively, liquidation may occur. This choice is beyond the scope of our research. See, for example, Davydenko and Strebulaev (2007) for analysis of how renegotiation potential can affect credit spreads. Also, see Brown, Ciochetti, and Riddough (2006) for how financial distress may be resolved. Uhrig-Homburg (2005) describes conditions for bankruptcy as it varies across different countries. That is, in the United States, filing for bankruptcy is permitted under broader conditions (insolvency not required) than in Germany and Canada. Davydenko and Franks (2008) find differences in creditor rights across different countries where such differences could affect barrier values.

Our above figures for V_T distributions were useful to illustrate some basic issues but cannot fully illustrate first passage default at $t < T$ when a realistic D is imposed. The operative statement for default is: What is the probability that V_t will not decline to D before T and, also, finish above *both* D and K at T ? If growth in V_t dominates growth in $Var(V_T)$, then P_d may decline if the domination is strong enough.¹⁷ Of course a higher D increases $P_d(T)$ but we prefer to focus on the shape of default term structure instead of the level of $P_d(T)$.

Figure 6 helps analyze the situation by dividing $P_d(T)$ into default probability due to the sum of 1.) breaching barrier D before maturity and 2.) $V_T < K$ at maturity. We call these, respectively, barrier default and (classic) maturity default. To best visually illustrate the important relationships, we now assume base parameters of $r_0 = 0.06$, $\theta = 0.03$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.20$, $\rho = 0$, $V_0 = 150$, $K = 100$ where we note that this suggests an upward drift in interest rates and positive $R_{df}(T)$ as θ/c is 0.10 and $r_0 = 0.06$. In some cases, a lower D that is less likely to be breached shrinks barrier default to zero and simply makes $P_d(T)$ the classic Merton (1974) case. A greater D raises both barrier and total default probability. D values in the four different panels are 50, 60, 70 and 90. As T increases, barrier default always increases as more cumulative potential breaching events occur. Figure 6, panel a, $D = 50$, shows no significant potential for breaching the barrier until about $T = 12$. In contrast, panel d ($D = 90$) shows considerable barrier default potential much earlier, at $T = 2$. As D increases, barrier

¹⁷ Here we again assume $K > D$.

default tends to surpass classic maturity default at some maturity. This figure also shows that as D increases, the hump of classic maturity default tends to be neutralized by barrier default such that the (total) $P_d(T)$ hump disappears in the last two panels. For more details on the interest rate system for assumed parameters, the last panels of this figure include spot rates, consolidated $P_d(T)$ plots, and $P_d(T)$ derivatives with respect to T and D . A hump in credit spread occurs for all D values in the last panel.

The impact of varying D values can be sensitive to the assumed parameters. In Figure 7 we change θ to 0.018 such that there is no expected drift in rates ($R_{df}(T)$ is flat) because θ/c is 0.06 and equal to r_0 . Now the (total) $P_d(T)$ hump is gone although the classic maturity hump remains. Also note that $P_d(T)$ is much higher than the previous set of figures where this is likely because the drift term in V_t is lower. Unlike the previous figure, barrier default exceeds classic maturity default only for long maturities in panel d. For more details on the interest rate system for the assumed parameters, the last panels of this figure include spot rates, consolidated $P_d(T)$ plots, and the P_d derivatives with respect to T and D .

Sensitivity of $P_d(T)$ to σ_v , Leverage, and σ_r

Next we analyze the impact of σ_v on the shape of $P_d(T)$ for a positive term structure as used in Figure 6. In Figure 8 we increase σ_v to 0.25 (from 0.20) and again use increasing D values in the first four panels. Now the (total) $P_d(T)$ hump disappears. In each panel, $P_d(T)$ is

higher than in the earlier corresponding figure. Related to hump disappearance, barrier default becomes greater than classic maturity default at $T = 20$ in panel b and at even earlier maturities in later panels. The last three panels provide more details about interest rates for the assumed parameters.

Figure 9 shows how the hump varies for different levels of leverage as given by V_0 . If V_0 is only 110, the $P_d(T)$ curve is flat for a short while and then turns steeply negative. As V_0 increases up to 180, the hump occurs but tends to become more gentle for greater V_0 .

In Figure 10 we note that, given that the drift of the V_t process is dependent upon r_t , the basic relation is that greater volatility, σ_r , in interest rates will increase volatility of V_t and thus increase the probability of default, spreads and R_d values. However, we note that non-zero correlation in V_t and r_t processes can either enhance or reverse this effect. The more positive ρ is, the more volatility in the V_t process and the greater $P_d(T)$. However, if ρ is negative, the relation could be reversed. Figure 10 represents the $\rho = -0.5, 0$ and $+0.5$ cases and is consistent with the above. In these cases, all $P_d(T)$ curves are humped. When ρ is zero, greater σ_r values raise $P_d(T)$ but the relation is reversed when ρ is negative. Furthermore, the $P_d(T)$ spread between different σ_r values is much larger for positive ρ compared to $\rho=0$. Appendix B shows the derivative of $P_d(T)$ with respect to σ_r .

Sensitivity of $P_d(T)$ to ρ

The sign and strength of the correlation, ρ , between the r_t and V_t processes can have a very large impact on certain firms and their $P_d(T)$ shapes. Prime examples are firms in the banking, real estate, and construction industries. Low interest rates from (approximately) 2001-2006 fostered very strong growth and performance in these industries but subsequent changes in interest rates, along with the recent credit crisis, have often led to more recent disastrous performance for some firms in these industries.¹⁸ Higher ρ values generate greater levels of $P_d(T)$ because the variance of V_t depends on the correlation between returns on V_t and r_t changes. If the correlation is positive (negative), the correlation increases (decreases) the variance of V_t and increases (decreases) the level of $P_d(T)$.¹⁹

We can describe the relation in more detail with the below. Let $dW_{1,t}$ and $dW_{2,t}$ be two independent standard Wiener increment processes. The two correlated processes $dW_{v,t}$ and $dW_{r,t}$ can be expressed as linear functions of $dW_{1,t}$ and $dW_{2,t}$ given by $dW_{r,t} = dW_{1,t}$, $dW_{v,t} = \rho_t dW_{1,t} + \sqrt{1 - \rho_t^2} dW_{2,t}$. If ρ is -1, the stochastic part of the V_t process (before scaling for differing volatilities) is totally offset by the stochastic part of the r_t process. On other hand,

¹⁸ One outstanding example of a large negative ρ is Northern Rock of the UK where the bank failed to hedge against interest rate increases. As their cost of borrowing rose, the value of the bank plummeted. See “The Wreck of Northern Rock”, *Bloomberg Markets*, May 2008 by Richard Tomlinson and Ben Livesey.

¹⁹ Longstaff and Schwartz (1995) describe spread behavior similarly.

if ρ is 1, the stochastic parts of the two processes are equal and any volatility of one process is enhanced by the other. Appendix B provides more details on the relation between ρ and $P_d(T)$.

Figure 11 displays cases where ρ varies from strongly negative to strongly positive for different levels of σ_v . In panel a, $\rho = -0.5$, the level of $P_d(T)$ is lower than panels b, $\rho = 0$, and c, $\rho = +0.5$. In all three panels, note that the $P_d(T)$ shape is positive for the highest σ_v (0.30), essentially flat for the second highest σ_v (0.25) if maturity exceeds 5, and humped for all lower levels of σ_v . Thus, ρ has a strong effect on both level and shape of $P_d(T)$.

The figures of this section have used example parameter values that conveniently build upon our models for dV_t and dr_t and, also, tend to best illustrate shapes and properties we wish to stress. We have found the results robust to alternative parameters where Appendix D gives some sample alternative parameter results.

4. Physical probabilities

We have shown analysis of the term structure of first passage default in a two factor model where we suggest the most interesting aspects are the sensitivity of default to barriers and the hump in the $P_d(T)$ term structure. Researchers modeling default should find these results useful. Also, financial analysts developing investment strategies should note our findings. However, some may suggest that the usefulness is limited in that our $P_d(T)$ values are risk-

neutral probabilities, not physical probabilities. We respond in two ways. First, Chou and Wang (2006) maintain that risk neutral probabilities still provide a credible ranking of firms according to susceptibility to default. Second, we transform our risk-neutral probabilities to physical as given below.

The process to obtain physical probability of default is to multiply market price of risk by σ_v and add this product to the drift term. Reisz and Perlich (2007) use 0.15 for the market price of risk (λ). For completeness, we also use lower and higher values and compare to the risk neutral (zero λ) case.²⁰ The results are in Figure 12 where panel a is the risk-neutral case and panels b, c, d represent λ 's of 0.075, 0.15 and 0.30, respectively. In each panel we plot curves for different values of σ_v and show that greater σ_v values raise $P_d(T)$. As expected, $P_d(T)$ values decline with greater λ but the shape of P_d is robust to different λ . That is, if a hump occurred in the risk neutral case, it also occurs in the physical probability.

5. Credit spread slope as a function of $P_d(T)$ slope

The above analysis of default probability can be used to develop investment and hedging strategies. Furthermore, regulators can use our analysis to develop policy. In this context we note that many economists and regulators favor mandatory bank issuance of

²⁰ Reisz and Perlich (2007) get their estimate (λ) from Huang and Huang (2003). The computation is firm growth rate in excess of risk free rate divided by volatility of asset value.

subordinated debt as a way to enhance market discipline of banks. In fact, the Gramm-Leach-Bliley Act of 1999 requires large banks to have at least one subordinated debt issue outstanding at all times. Krishnan, Ritchken, and Thomson (2006) find that the shape of credit spread term structures may very well help predict bank risk where the basic idea is that credit spread slopes can predict forward spreads and risk. A strong positive slope tends to successfully predict greater forward credit spreads and greater risk.

Credit spread slopes can be computed for different discrete intervals where, for example, one may use the three year spread less the one year to estimate slope. Alternatively, as in Krishnan, Ritchken, and Thomson (2006), one could use the seven year spread less the three year spread. If a hump in credit spread occurs between the maturities used for computation, say at $T=2$, the estimated slope between one and three could be flat but this may be a misleading representation of future risk. In reality, for the hump case, the slope would be positive before $T=2$ and negative after. Regulators may interpret a zero slope as suggesting no need for action (no need for aggressive U. S. Treasury purchase of

low grade bank assets at high prices) whereas the reality is that action may be needed within a one year horizon.

We now explicitly express credit spread slope as a function of default probability slope. Our earlier expression for credit spread may be rewritten as

$$S(T) = R_d(T) - R_{rf}(T) = \frac{1}{T} S_1 \quad (5.1)$$

where

$$S_1 = -\ln \left[1 - \left[1 - RR(P_d(T)) \right] P_d(T) \right].$$

Thus the spread is S_1 (always positive) scaled by $1/T$. Because S_1 is dependent upon T through $P_d(T)$,

$$\frac{dS_1}{dT} = \frac{dS_1}{dP_d(T)} \frac{dP_d(T)}{dT} \quad (5.2)$$

Also,

$$\frac{dS_1}{dP_d(T)} = \frac{(1 - RR) - \frac{\partial(RR)}{\partial P_d(T)} P_d(T)}{(1 - RR) P_d(T)} \quad (5.3)$$

where $RR = RR(P_d(T))$ for simplicity. Since RR given in (2.8) is less than one and is also negatively related to $P_d(T)$, the derivative of S_1 with respect to $P_d(T)$ is clearly always positive. This also holds for a constant RR .

Furthermore, the slope of the credit spread term structure is

$$\frac{dS}{dT} = \frac{1}{T} \frac{dS_1}{dT} - \frac{1}{T^2} S_1 \quad (5.4)$$

Substituting, we obtain an alternative expression for credit spread slope as dependent upon the slope of $P_d(T)$.

$$\frac{dS}{dT} = \frac{1}{T} \left[\frac{dP_d(T)}{dT} \frac{dS_1}{dP_d(T)} \right] - \frac{1}{T^2} S_1 \quad (5.5)$$

$\frac{dP_d}{dT}$ can be found in Appendix B. As the derivative of S_1 with respect to $P_d(T)$ is always positive, the only sign indefinite term is $dP_d(T)/dT$. The last term, $(1/T^2)(S_1)$, always reduces the slope.

Consider the impact of $dP_d(T)/dT$. If this derivative is positive and strong, as often seen in our figures for short maturities, the slope of $S(T)$ tends to be positive. As T increases, $dP_d(T)/dT$ tends to either flatten or even change sign. If it merely flattens, the

slope of $S(T)$ may or may not remain positive depending the magnitude of changes relative to $(1/T^2)S_1$. If $dP_d(T)/dT$ becomes negative, the slope of $S(T)$ will clearly become negative and may well become strongly negative.

We may apply the above to numerous figures. First, consider Figure 3 where there is no hump in probability. Nonetheless, the spread has a hump due to the magnitude of $(1/T^2)S_1$ eventually becoming dominant. Figure 6 contains $P_d(T)$ curves that have a hump, say T^* , and others that do not. The hump in credit spreads for situations associated with humped $P_d(T)$ curves is due to both $P_d(T)$ humps and $(1/T^2)S_1$.

Given that location of the $S(T)$ hump can be important, we analyze how the location of the $S(T)$ hump may change with parameter values. That is, assuming everything else fixed, does the maturity at which the spread hump occurs, T^{**} , increase or decrease with various parameters? Figure 13 shows that T^{**} varies with D . For D values between 50 and 70, T^{**} gently increases and then increases much more rapidly until D is about 90 at which point it declines. Other figures, not shown, reveal that T^{**} is negatively related to σ_r , σ_v , and ρ but positively related to V_o .

6. Conclusion

Strong growth in credit derivatives and the recent credit crises have both increased the demand for improved models of credit risk. We note that popular structural models commonly utilize simultaneous processes for V_t and short term interest rates, r_t , where changes in V_t are dependent upon the particular r_t process assumed. Having the distribution for V_t permits us to more easily analyze first passage default probabilities and term structures of credit risk in detail. To our knowledge, no one else has analyzed first passage probabilities for both maturity and barrier default with a two factor model. Greater passage of time suggests greater expected firm value but, at the same time, greater variance of V_t so that the impact of maturity on default probability is complex. The behavior of default probability with passage of time has a clear impact on term structure of credit spreads where a strong growth in probability of default increases the slope of the term structure of credit spreads.

It is interesting to note that probability of default may display a hump with respect to maturity. This result holds for many combinations of parameters describing the simultaneous V_t and r_t processes. We find that the hump can be especially clear for smaller barrier values, smaller volatility of interest rates, larger volatility in V_t , and greater leverage. The impact of smaller barriers is especially interesting due to the recent trend toward weaker covenant protection in many recent debt issues. Weaker covenant protection reduces the near term probability of default but many fear that weaker covenants merely delay defaults. The applicability of the research is enhanced by the fact that the hump appears for both risk neutral

probabilities and conversions to physical probabilities. We provide explicit expressions for how credit spread slope is a function of default probability slope and suggest that, given our results, regulators can develop improved predictions of credit risk.

Appendix A

This appendix has two parts. In the first part we show that for the process g_t of the type (1.10), the standard method for computing the default probability does not lead to a closed form solution. Based on this, in part two we derive the best linear approximation to $\mu(t)$ and $\sigma^2(t)$ using a closed form expression for the default probability is obtained. The analysis in the main body of this paper is based on the latter closed form solution.

Need for approximation

Since $V_t = e^{g_t}$, from (1.13) it follows that

$$P_d(T) = 1 - \text{Prob} \left[g_T > \log K \text{ and } \min_{0 \leq s \leq T} g_s > \log D \right] \quad (\text{A.1})$$

Clearly, computing the second term on the right hand side of (A.1) involves computing the joint density of g_T and $\min_{0 \leq s \leq T} g_s$ where g_t is given by (1.10). By way of simplifying notation, let $\eta(t) = \sigma^2(t)$. Since $\eta(t)$ is an increasing function, setting $\eta(t) = \tau$, we get $t = \eta^{-1}(\tau)$ where $0 \leq \tau \leq \bar{T} = \eta(T)$ when $0 \leq t \leq T$. Hence (1.10) becomes

$$g(\eta^{-1}(\tau)) = \mu(\eta^{-1}(\tau)) + B(\tau) \quad (\text{A.2})$$

Defining $X(\tau) = g(\eta^{-1}(\tau))$ and $\lambda(\tau) = \mu(\eta^{-1}(\tau))$, (A.2) becomes, for all $0 \leq \tau \leq \bar{T}$,

$$X(\tau) = \lambda(\tau) + B(\tau) \quad (\text{A.3})$$

Define

$$\Lambda_\tau = \exp \left[-\int_0^\tau \dot{\lambda}(s) dB(s) - \frac{1}{2} \int_0^\tau \dot{\lambda}^2(s) ds \right] \quad (\text{A.4})$$

where $\dot{\lambda}(\tau) = \frac{d\lambda(\tau)}{d\tau}$. Define a new measure P^λ as

$$dP^\lambda = \Lambda_\tau dP \quad (\text{A.5})$$

Then, by Girsanov theorem, $X(\tau)$ is a standard Wiener process and the joint distribution of $X_{\bar{T}}$

and $\min_{0 \leq s \leq \bar{T}} X_s$ is given by

$$F(T, b, c) = E^\lambda \left[\Lambda_{\bar{T}} I \left\{ X_{\bar{T}} < b, \min_{0 \leq s \leq \bar{T}} X_s < c \right\} \right] \quad (\text{A.6})$$

where b and c and real constants, $I\{A\}$ is the standard indicator function, and E^λ is the expectation with respect to the new measure P^λ . Explicit computation of (A.6) involves the knowledge of the joint distribution of $\int_0^{\bar{T}} \dot{\lambda}(s) dB(s)$, $X_{\bar{T}}$ and $\min_{0 \leq s \leq \bar{T}} X_s$. To our knowledge this distribution is not known.

If $\dot{\lambda}(s) = a$ is a constant however, then $\int_0^{\bar{T}} \lambda(s) dB(s) = aB(\bar{T})$ and in this case the joint distribution is known (Giesecke (2004), Elliott and Kopp (1999)).

Fortunately, for the case when the r_t process follows the Vasicek model, the $\mu(t)$ in (1.5) and $\eta(t) = \sigma^2(t)$ in (1.9) are the sum of a linear and a “small” nonlinear term. So, we could effectively approximate $\mu(t)$ and $\sigma^2(t)$ by linear function.

Best linear approximations to $\mu(t)$ and $\sigma^2(t)$:

In this second part we seek best (in the sense of least squares) linear approximations to $\mu(t)$ and $\sigma^2(t)$. To this end, using (1.4) – (1.10), rewrite $\mu(t)$ and $\sigma^2(t)$ as

$$\mu(t) = mt + e_\mu(t) \tag{A.7}$$

and

$$\sigma^2(t) = \sigma_v^2 t + e_\sigma(t) \tag{A.8}$$

where $m = \left(\frac{\theta}{c} - \frac{1}{2} \sigma_v^2 \right)$,

$$e_\mu(t) = \frac{1}{c} \left(r_0 - \frac{\theta}{c} \right) (1 - e^{-ct}) \tag{A.9}$$

and

$$e_{\sigma^2}(t) = \frac{\sigma_r^2}{4c^3} \left[1 - e^{-2ct} (2c^2t^2 + 2ct + 1) \right] + \frac{2\rho\sigma_r\sigma_v}{c^2} \left[1 - e^{-ct} (ct + 1) \right]. \quad (\text{A.10})$$

From (A.7) – (A.10) it readily follow that both $\mu(t)$ and $\sigma^2(t)$ are expressed as a sum of a linear term and a nonlinear term. While both $\mu(t)$ and $\sigma^2(t)$ are defined for all $t \geq 0$ with $\mu(0) = 0 = \sigma^2(0)$, for purposes of application in the main body of this paper, we seek the best linear approximations $\bar{\mu}(t)$ and $\bar{\sigma}^2(t)$ for $\mu(t)$ and $\sigma^2(t)$ respectively, for t in a finite range, say $0 \leq t \leq \alpha < \infty$. Typically, the value of α is dictated by the maximum maturity of bonds, say $\alpha = 30$.

For the best linear approximation to $\mu(t)$, let $\bar{\mu}(t) = m_1 t$. Then we seek m_1 that minimize

$$f_1(m_1) = \int_0^\alpha (\mu(t) - m_1 t)^2 dt \quad (\text{A.11})$$

By routine computations it can be verified that the minimizing m_1 is given by

$$m_1(\alpha) = m + \frac{3}{c^3 \alpha^3} \left(r_0 - \frac{\theta}{c} \right) \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c\alpha + 1) - 1 \right] \quad (\text{A.12})$$

where $m_1(\alpha) \rightarrow m$ as $\alpha \rightarrow \infty$.

A plot of $m_1(\alpha)$ if given in Figure A.1 and an illustration of the best linear approximation of $\mu(t)$ by $\bar{\mu}(t) = m_1(\alpha)t$ for $0 \leq t \leq 30$ is given in Figure A.2.

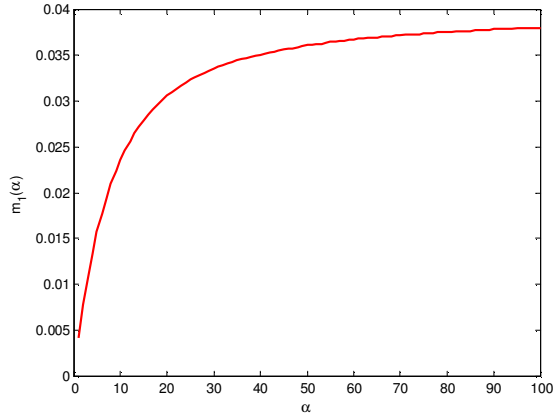


Figure A.1 Plot of $m_1(\alpha)$ vs. α . $r_0 = 0.06$, $\theta = 0.03$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.2$, $\rho = 0.5$

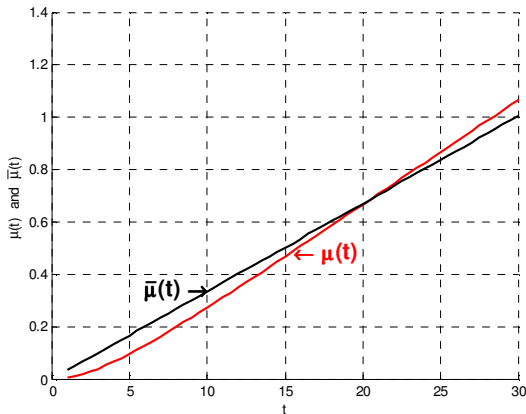


Figure A.2 An illustration of the best linear approximation $\bar{\mu}(t) = m_1(\alpha)t$ where $m_1(\alpha) = 0.033498$ for $\alpha = 30$. $r_0 = 0.06$, $\theta = 0.03$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.2$, $\rho = 0.5$

For the best linear approximation to $\sigma^2(t)$, let $\bar{\sigma}^2(t) = m_2 t$. We seek m_2 that minimizes

$$f_2(m_2) = \int_0^\alpha (\sigma^2(t) - m_2 t)^2 dt \quad (\text{A.13})$$

It can be verified that the minimizing m_2 is given by

$$m_2(\alpha) = \sigma_v^2 + \frac{3\sigma_r^2}{8c^5\alpha^3} \left[c^2\alpha^2 + e^{-2c\alpha} (2c^3\alpha^3 + 5c^2\alpha^2 + 6c\alpha + 3) - 3 \right] \\ + \frac{6\rho\sigma_r\sigma_v}{c^4\alpha^3} \left[\frac{1}{2}c^2\alpha^2 + e^{-c\alpha} (c^2\alpha^2 + 3c\alpha + 3) - 3 \right] \quad (\text{A.14})$$

and $m_2(\alpha) \rightarrow \sigma_v^2$ as $\alpha \rightarrow \infty$.

A plot of $m_2(\alpha)$ vs. α is given in Figure A.3. An illustration of the best approximation of $\sigma^2(t)$ by $m_2(\alpha)t$ for $0 \leq \alpha \leq 30$ is given in Figure A.4.

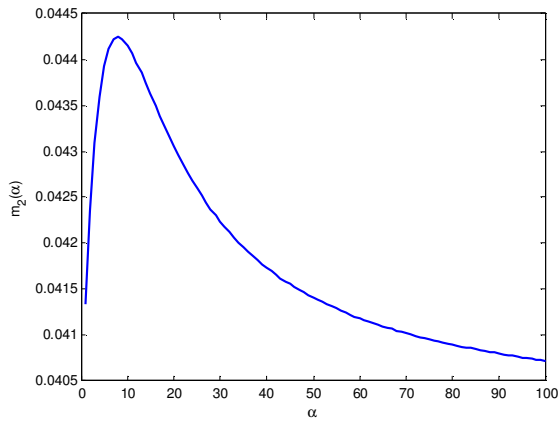


Figure A.3 Plot of $m_2(\alpha)$ vs. α . $r_0 = 0.06$, $\theta = 0.03$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.2$, $\rho = 0.5$

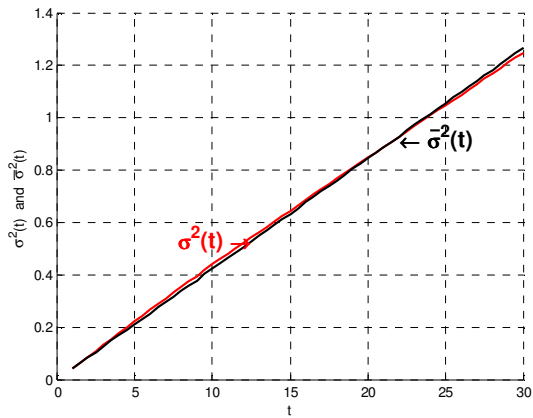


Figure A.4 An illustration of the best linear approximation $\bar{\sigma}^2(t) = m_2(t)$ where

$m_2(\alpha) = 0.0422$ for $\alpha = 30$. $r_0 = 0.06$, $\theta = 0.03$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.2$, $\rho = 0.5$

APPENDIX B

Sensitivity of Default Probability to Parameters

In this appendix we derive explicit expressions for the sensitivity of the default probability $P_d(T)$ in (1.17) with respect to various parameters. To simplify our analysis define

$$f_1 = f_1(K, V_0, m_1, m_2, \sigma_v, T) = \frac{\ln\left(\frac{K}{V_0}\right) - m_1 T}{\sqrt{m_2 T}}, \quad (\text{B.1})$$

$$f_2 = f_2(m_1, m_2, \sigma_v) = \frac{2m_1}{m_2}, \quad (\text{B.2})$$

and

$$f_3 = f_3(K, V_0, D, m_1, m_2, \sigma_v, T) = \frac{\ln\left(\frac{D^2}{KV_0}\right) + m_1 T}{\sqrt{m_2 T}} \quad (\text{B.3})$$

where $m_1 = m_1(\alpha)$ and $m_2 = m_2(\alpha)$ are given by (A.6) and (A.8) in Appendix A, respectively. Then, $P_d(T)$ in (1.17) takes the form

$$P_d(K, D, T) = \Phi(f_1) + \left(\frac{D}{V_0}\right)^{f_2} \Phi(f_3) \quad (\text{B.4})$$

where, recall from (1.18) that,

$$\Phi(f) = \int_{-\infty}^f \phi(z) dz \quad (\text{B.5})$$

and $\phi(z)$ is the standard normal density. Hence, if f is a function of x , then

$$\frac{d\Phi(f)}{dx} = \phi(f) \frac{\partial f}{\partial x}. \quad (\text{B.6})$$

From (B.4) and (B.6), we readily obtain that

$$\frac{\partial P_d(T)}{\partial x} = \phi(f_1) \frac{\partial f_1}{\partial x} + \left(\frac{D}{V_0}\right)^{f_2} \phi(f_3) \frac{\partial f_3}{\partial x} + \Phi(f_3) \left(\frac{\partial f_2}{\partial x}\right) \left(\frac{D}{V_0}\right)^{f_2} \log(f_2) \quad (\text{B.7})$$

The partial derivatives of f_1 , f_2 , and f_3 in (B.1) - (B.3) with respect to m_1 , m_2 , T , D , V_0 and K are given in Table B.1. By combining (B.7) with the appropriate entries in this table, we can obtain sensitivity of P_d with respect to six parameters of interest.

Table B.1 Derivative of f_1 , f_2 , f_3 in (B.1) - (B.3)

| x | $\frac{\partial f_1}{\partial x}$ | $\frac{\partial f_2}{\partial x}$ | $\frac{\partial f_3}{\partial x}$ |
|-------|---|-----------------------------------|---|
| m_1 | $-\frac{T}{\sqrt{m_2 T}}$ | $\frac{2}{m_2}$ | $\frac{T}{\sqrt{m_2 T}}$ |
| m_2 | $-\frac{\log \frac{K}{V_0} - m_1 T}{2(m_2 T)^{\frac{3}{2}}} T$ | $-\frac{2m_1}{m_2^2}$ | $-\frac{\log \frac{D^2}{KV_0} + m_1 T}{2(m_2 T)^{\frac{3}{2}}} T$ |
| T | $-\frac{m_1}{\sqrt{m_2 T}} - \frac{\log \frac{K}{V_0} - m_1 T}{2(m_2 T)^{\frac{3}{2}}}$ | 0 | $\frac{m_1}{\sqrt{m_2 T}} - \frac{\log \frac{D^2}{KV_0} + m_1 T}{2(m_2 T)^{\frac{3}{2}}}$ |
| D | 0 | 0 | $\frac{2}{D\sqrt{m_2 T}}$ |
| V_0 | $-\frac{1}{V_0\sqrt{m_2 T}}$ | 0 | $-\frac{1}{V_0\sqrt{m_2 T}}$ |
| K | $\frac{1}{K\sqrt{m_2 T}}$ | 0 | $-\frac{1}{K\sqrt{m_2 T}}$ |

Recall from (A.6) and (A.8) in Appendix A that m_1 and m_2 in turn depend on the parameters - r_0 , c , θ , σ_v , σ_r and ρ , of the model. In Table B.2, we provide expressions for

the derivative of m_1 and m_2 with respect to these six parameters. By combining (B.7) with entries in Table B.1 and B.2, we can derive expression for the sensitivity of $P_d(T)$ with respect to these parameters. Thus,

$$\frac{\partial P_d}{\partial y} = \left(\frac{\partial P_d}{\partial m_1} \right) \left(\frac{\partial m_1}{\partial y} \right) + \left(\frac{\partial P_d}{\partial m_2} \right) \left(\frac{\partial m_2}{\partial y} \right) \quad (\text{B.8})$$

where $y \in \{r_0, c, \theta, \sigma_v, \sigma_r, \rho\}$

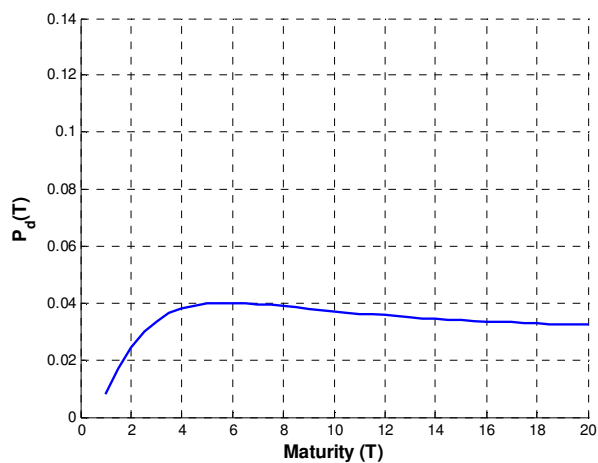
Table B.2 Derivative of m_1, m_2

| y | $\frac{\partial m_1}{\partial y}$ | $\frac{\partial m_2}{\partial y}$ |
|------------|---|--|
| r_0 | $\frac{3}{c^3 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c\alpha + 1) - 1 \right]$ | 0 |
| θ | $\frac{1}{c} - \frac{3}{c^4 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c\alpha + 1) - 1 \right]$ | 0 |
| c | $-\frac{\theta}{c^2} + \frac{3\theta}{c^5 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c\alpha + 1) - 1 \right]$ $+ \frac{3}{c^3 \alpha^3} \left(r_0 - \frac{\theta}{c} \right) \left[c\alpha^2 - c\alpha^2 e^{-c\alpha} \right]$ $-\frac{9}{c^4 \alpha^3} \left(r_0 - \frac{\theta}{c} \right) \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c\alpha + 1) - 1 \right]$ | $-\frac{15\sigma_r^2}{8c^6 \alpha^3} \left[c^2 \alpha^2 + e^{-2c\alpha} (2c^3 \alpha^3 + 5c^2 \alpha^2 + 6c\alpha + 3) - 3 \right]$ $+ \frac{3\sigma_r^2}{8c^5 \alpha^3} \left[2c\alpha^2 - e^{-2c\alpha} (4c^2 \alpha^3 + 4c^3 \alpha^4 + 2c\alpha^2) \right]$ $-\frac{24\rho\sigma_r\sigma_v}{c^5 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c^2 \alpha^2 + 3c\alpha + 3) - 3 \right]$ $+ \frac{6\rho\sigma_r\sigma_v}{c^4 \alpha^3} \left[c\alpha^2 - e^{-c\alpha} (c\alpha^2 + c^2 \alpha^3) \right]$ |
| σ_v | $-\sigma_v$ | $2\sigma_v + \frac{6\rho\sigma_r}{c^4 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c^2 \alpha^2 + 3c\alpha + 3) - 3 \right]$ |
| σ_r | 0 | $\frac{3\sigma_r}{8c^5 \alpha^3} \left[c^2 \alpha^2 + e^{-2c\alpha} (2c^3 \alpha^3 + 5c^2 \alpha^2 + 6c\alpha + 3) - 3 \right]$ $+ \frac{6\rho\sigma_v}{c^4 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c^2 \alpha^2 + 3c\alpha + 3) - 3 \right]$ |
| ρ | 0 | $\frac{6\sigma_r\sigma_v}{c^4 \alpha^3} \left[\frac{1}{2} c^2 \alpha^2 + e^{-c\alpha} (c^2 \alpha^2 + 3c\alpha + 3) - 3 \right]$ |

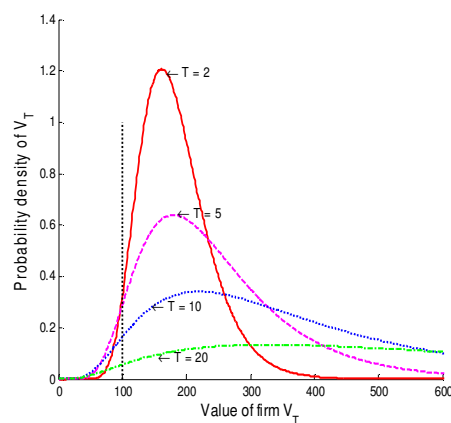
Appendix C

We performed extensive testing of sensitivity of behavior to different parameter values and found the results presented in the text are robust. For example, the results are robust to mean reversion, c , being 0.5 instead of 0.3. Please see examples below.

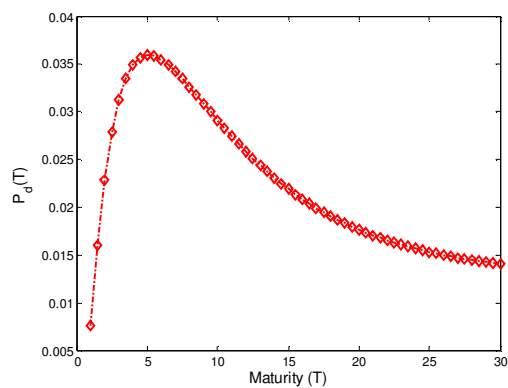
Counterpart to Figure 4c, $c = 0.5$



Counterpart to Figure 5a, $c = 0.5$



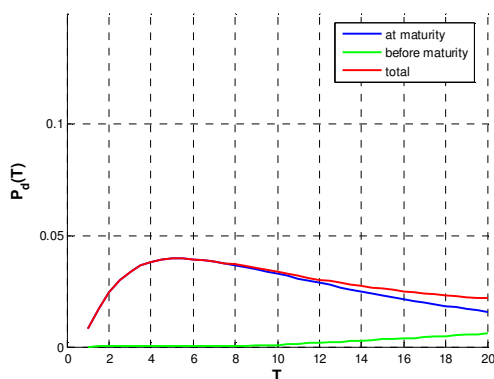
Counterpart to Figure 5c, $c=0.5$



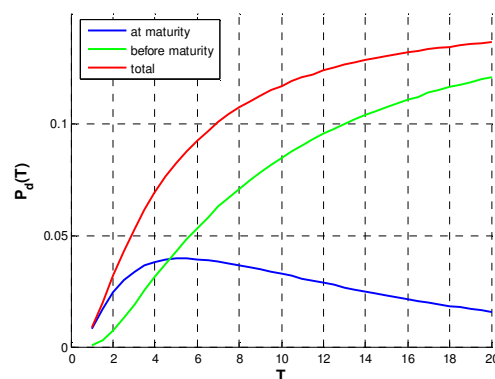
Appendix D

We performed extensive testing of sensitivity of behavior to different parameter values and found the results presented in the text are robust. For example, the results are robust to mean reversion, c , being 0.5 instead of 0.3. Also, results are robust to different σ_r values. Please see examples below.

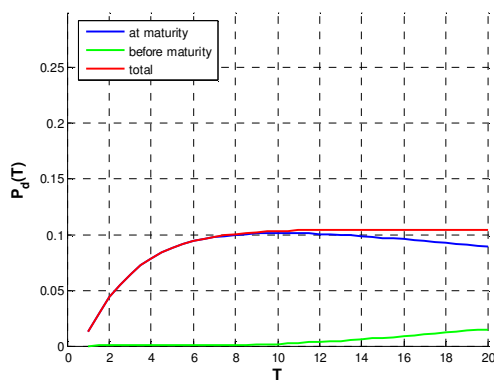
Counterpart to Figure 6a, $c = 0.5$



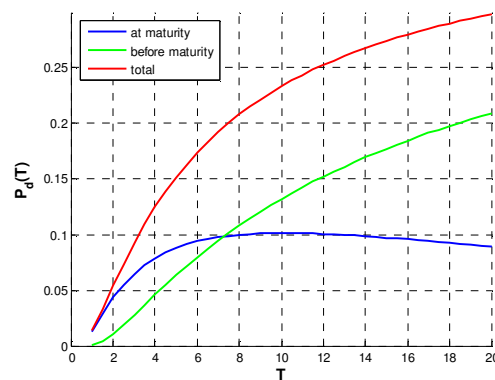
Counterpart to Figure 6d, $c = 0.5$



Counterpart to Figure 7a, $c = 0.5$



Counterpart to Figure 7d, $c = 0.5$



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Figure 1

First-passage Default Approach: Barrier or Classic Maturity

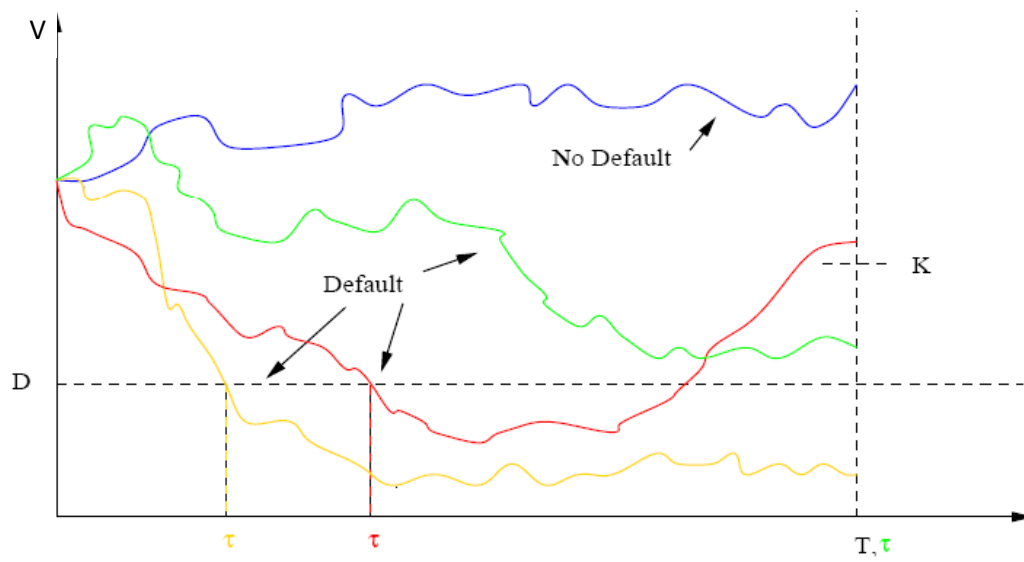
Default is defined by the occurrence of the event

$$B(K, D) = \left\{ V_T < K \text{ or } \min_{0 \leq s \leq T} V_s < D \right\}$$

where the probability of default can be given as

$$P_d(T) = \text{Prob}[B(K, D)] = 1 - \text{Prob}\left[V_T > K \text{ and } \min_{0 \leq s \leq T} V_s > D\right]$$

Default occurs in two ways. First, in the classical case of Merton (1974), default occurs when the value of the firm falls below the face value of the debt (K) at time T . Additionally, default occurs before maturity, $t < T$, when the value of the firm falls below a *barrier* level, D . We use a flat barrier as in Longstaff and Schwartz (1995). The barrier can alternatively be shown as time varying with D equal to K at maturity. Our results are qualitatively unchanged by using an alternative barrier shape.



Source: Giesecke (2004)

Figure 2

The variation in $P_d(T)$ for α of 30

The probability of default is shown to vary with α . A peak occurs at about a maturity of six.

$r_0 = 0.06$, $\theta = 0.03$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.2$, $\rho = 0.5$, $V_0 = 150$, $K = 100$, $D = 60$, $\lambda = 0$

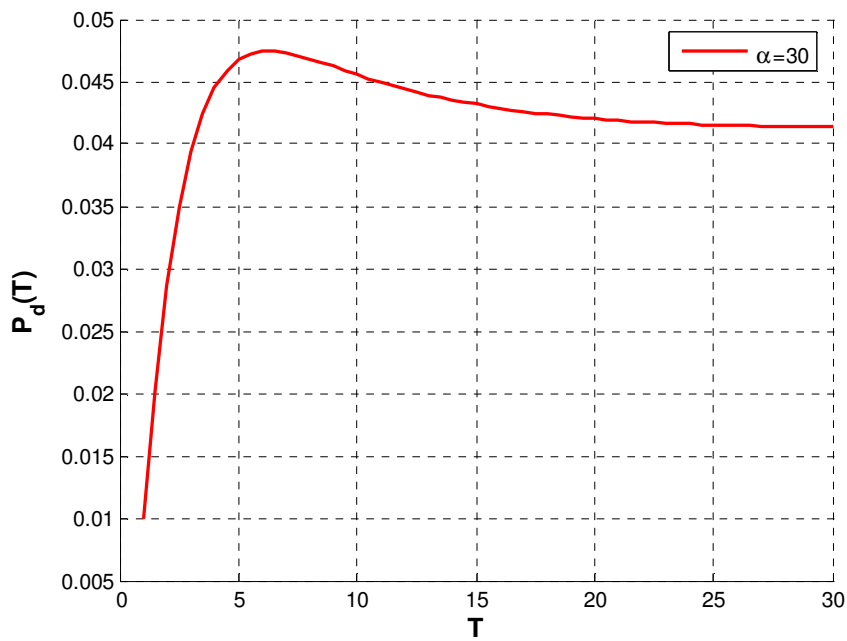


Figure 3

The Behavior of Default Risky Spot Rates and Spreads Dependent upon the Distribution of V_t : Probability of Default Increasing with Maturity (Flat $R_{df}(T)$ Term Structure)

Spreads are dependent upon the distribution of V_t and are computed assuming the below parameters.
 $r_0 = 0.06$, $\theta = 0.018$, $c = 0.3$, $\sigma_r = 0.02$, $\sigma_v = 0.2$, $\rho = 0$, $V_0 = 150$, $K = 100$, $D = 60$, $\lambda = 0$

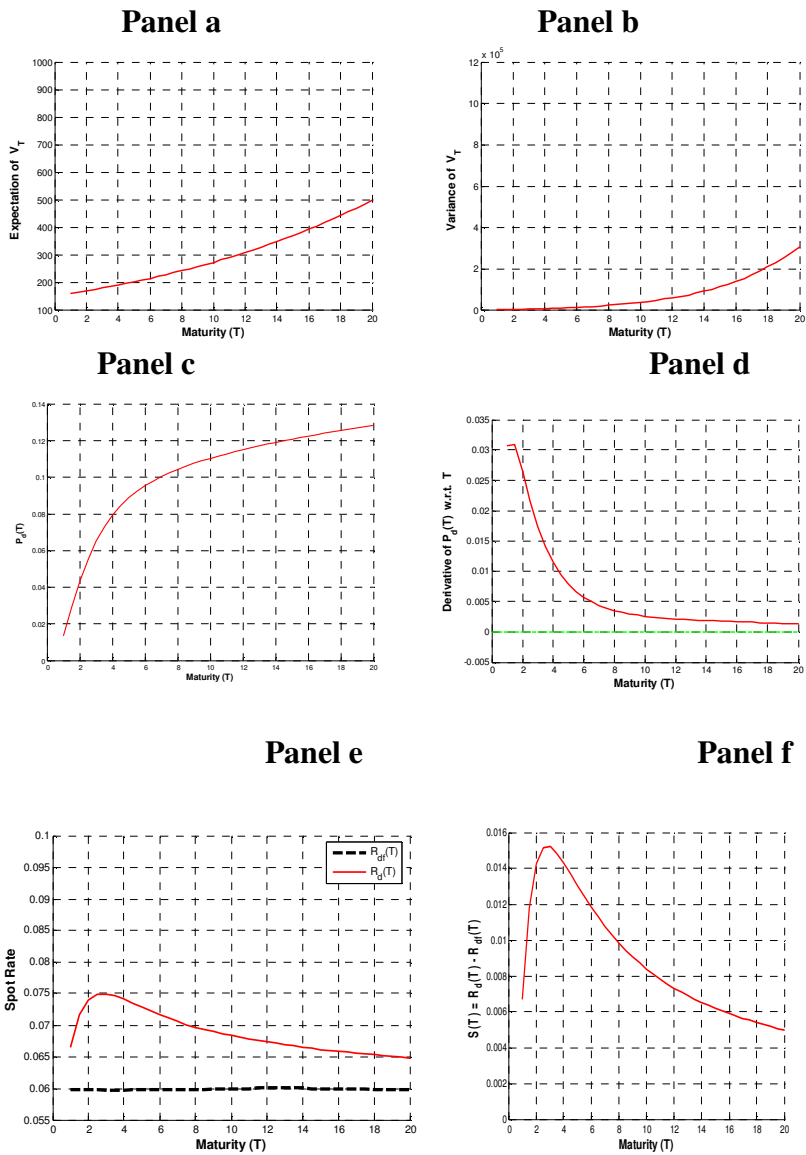


Figure 4

The Behavior of Default Risky Spot Rates and Spreads Dependent upon the Distribution of V_t : Probability of Default Decreasing with Maturity (Positive $R_{df}(T)$ Term Structure)

Spreads are dependent upon the distribution of V_t and are computed assuming the below parameters.

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \sigma_v = 0.2, \rho = 0, V_0 = 150, K = 100, D = 60, \lambda = 0$$

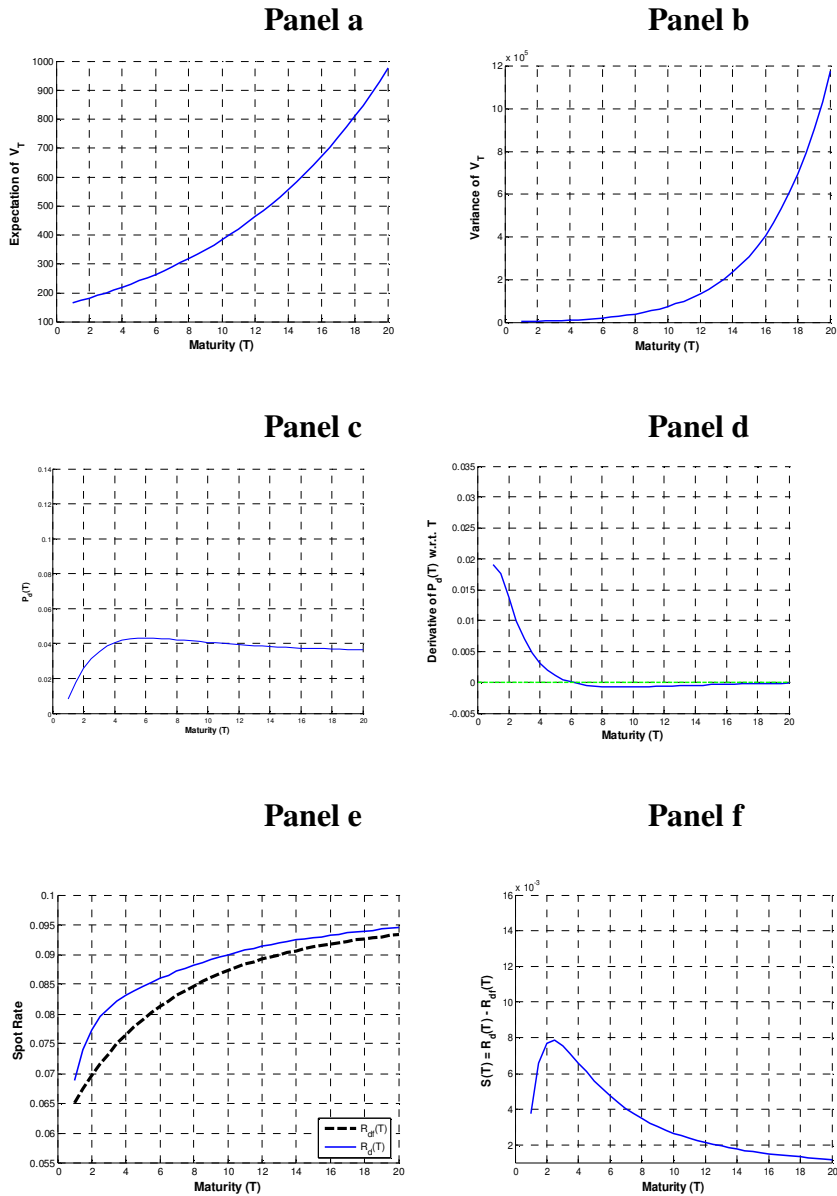


Figure 5a

Probability Density of V_T When $\rho = 0$

Assuming the Vasicek (1977) model, we compute the probability density of the value of the firm assuming an initial value of 150 and other given parameters.

$$r_0 = 0.1, \theta = 0.03, c = 0.3, V_0 = 150, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2,$$

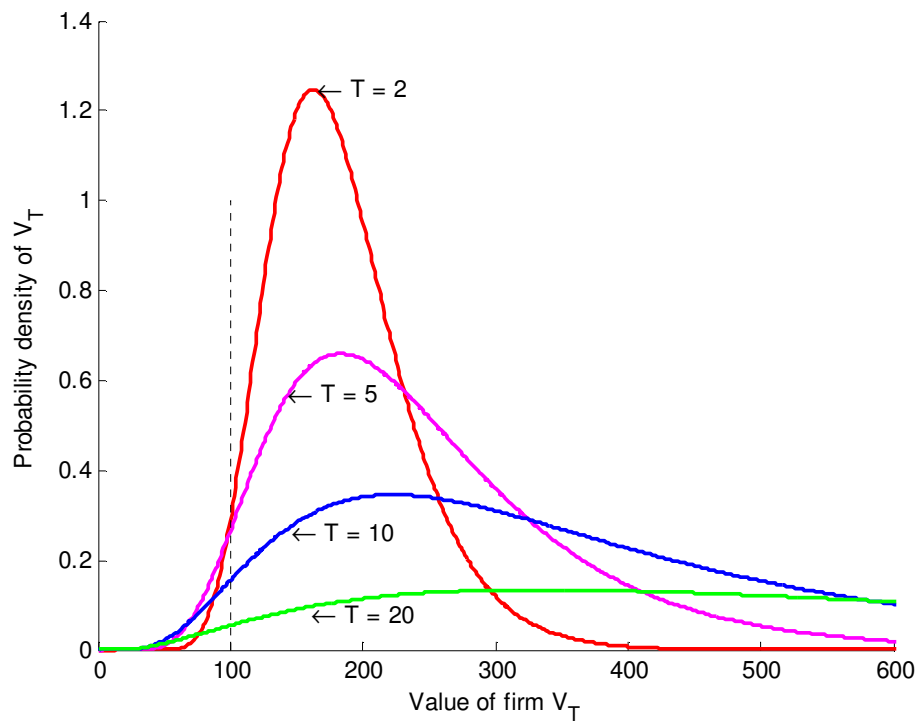


Figure 5b

Probability Density of $\log\left(\frac{V_T}{V_0}\right)$ When $\rho=0$

$r_0 = 0.1, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2, V_0 = 150$

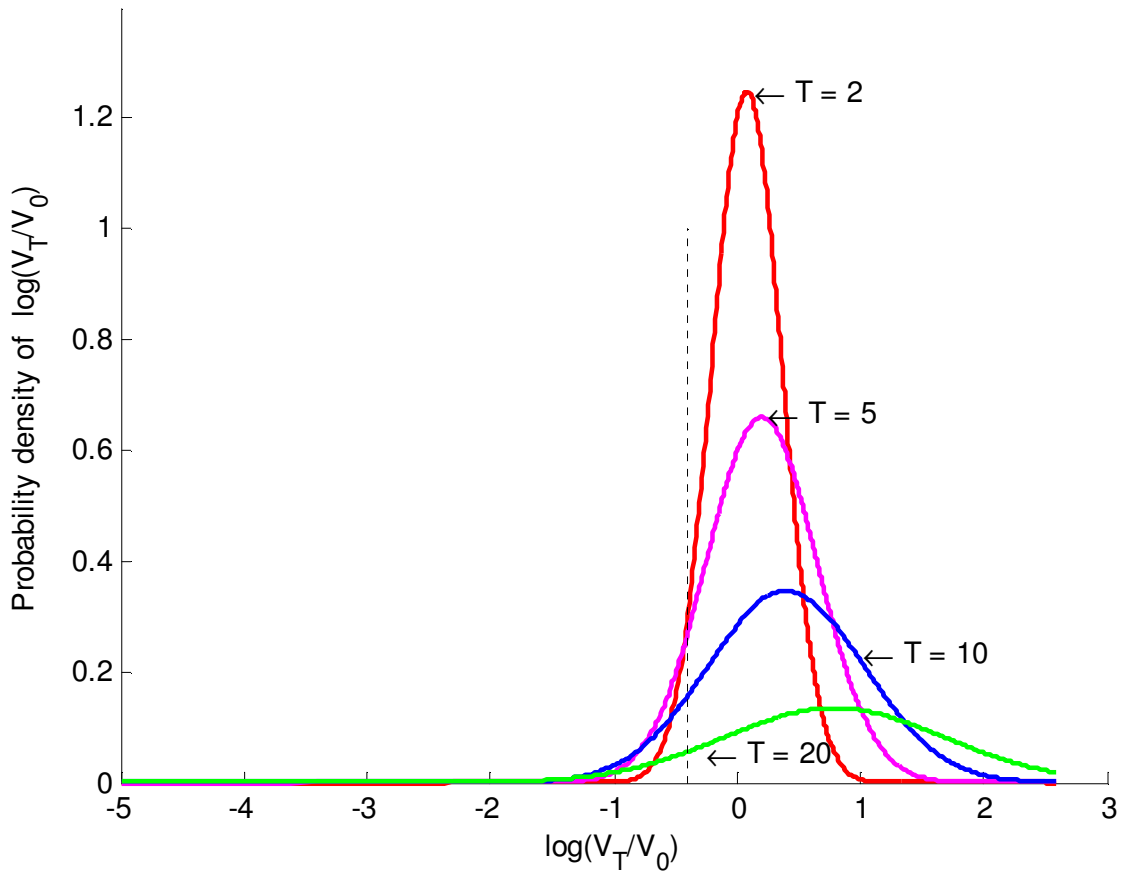


Figure 5c

First Passage Default Probability

This presents the probability density curve for earlier parts of this figure.

$$r_0 = 0.1, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2, V_0 = 150, K = 100, D = 50$$

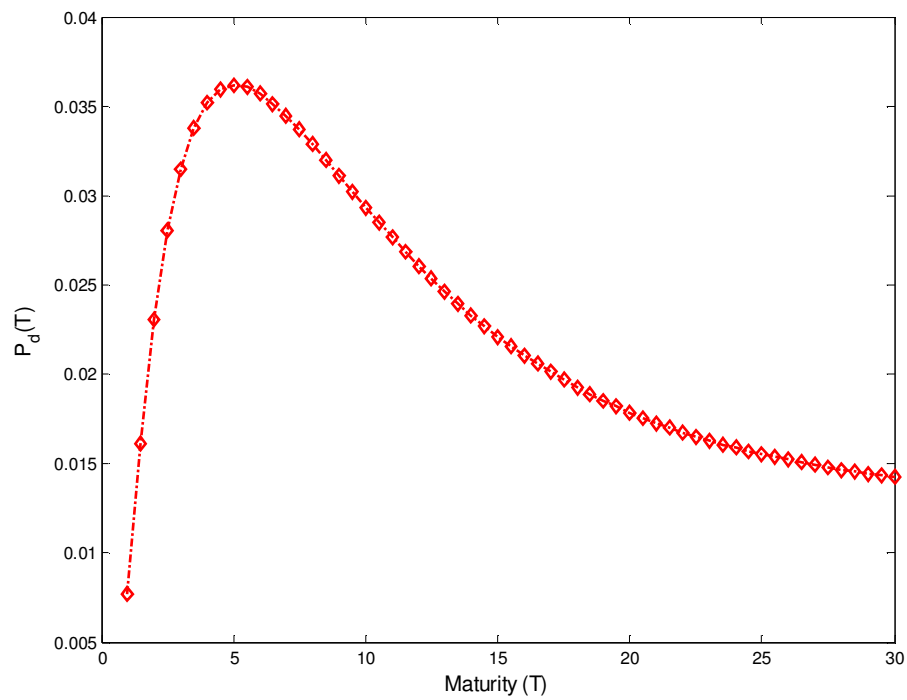


Figure 5d

Probability Density of V_T When $\rho=0.5$

Assuming the Vasicek (1977) model, we compute the probability density of the value of the firm assuming an initial value of 150 and other given parameters. ρ is assumed to be positive 0.5 instead of zero.

$$r_0 = 0.1, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \sigma_v = 0.2, V_0 = 150$$

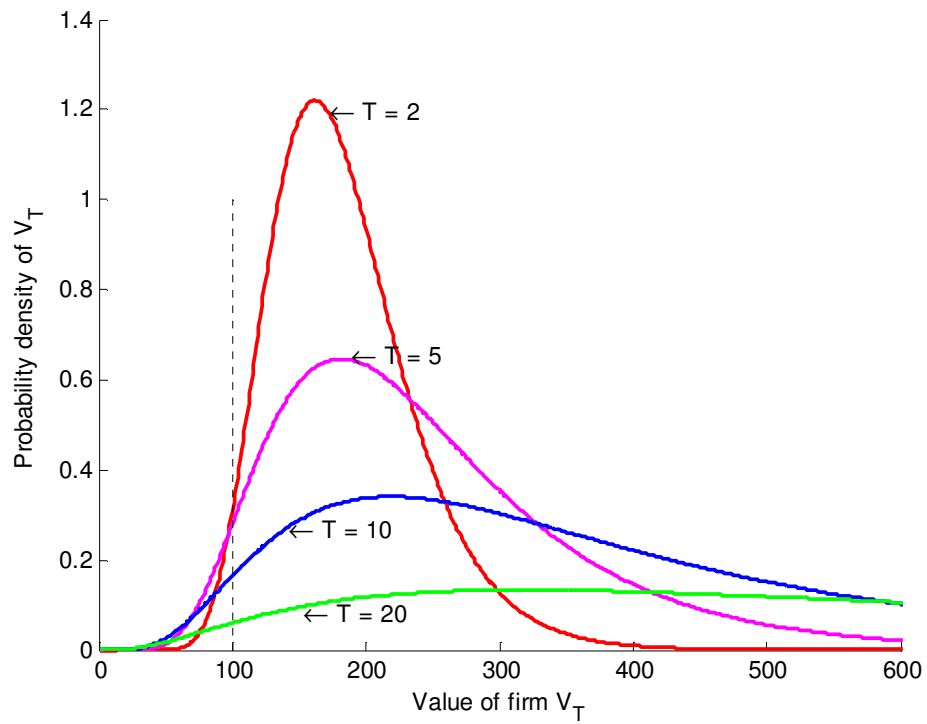


Figure 6

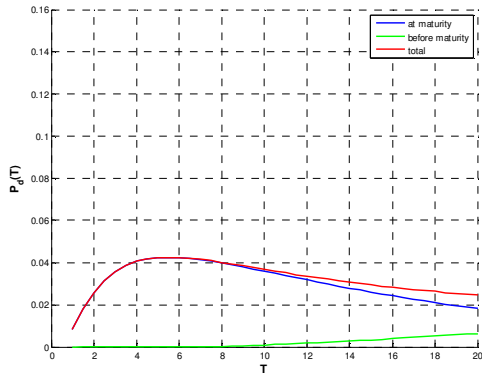
First Passage Default as Dependent Upon Alternative Barriers (D)
(Positive Risk- Free Term Structure)

Assuming parameters given below, we analyze the sensitivity of default and credit spreads to alternative barrier values. The shape of default probability varies widely.

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2, V_0 = 150, K = 100, \lambda = 0$$

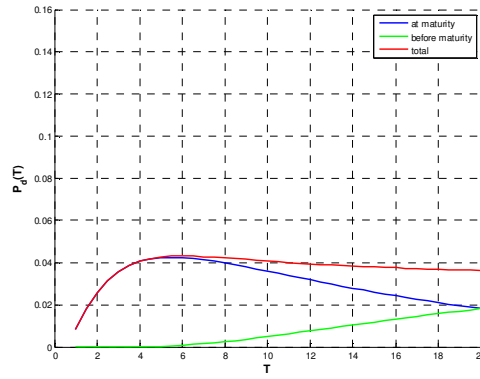
Panel a

D = 50



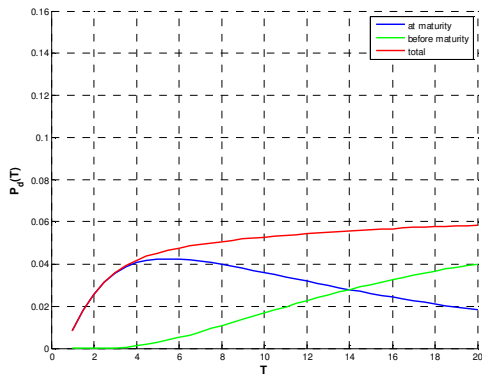
Panel b

D = 60



Panel c

D=70



Panel d

D = 90

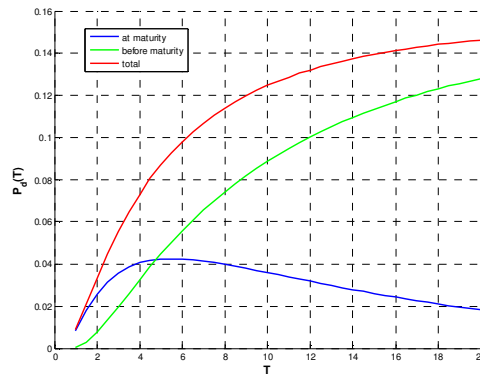
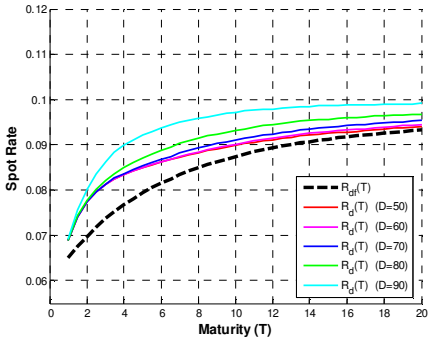
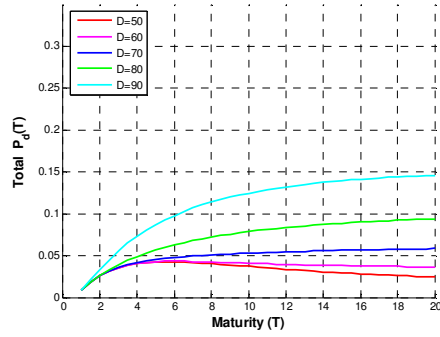


Figure 6 (continued)

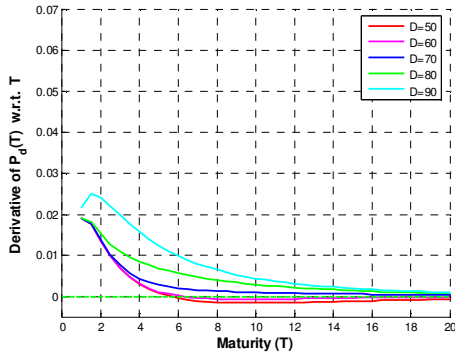
Panel e



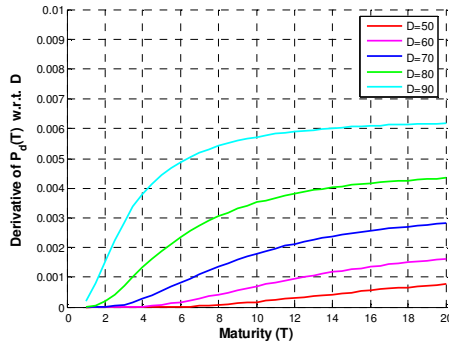
Panel f



Panel g



Panel h



Panel i

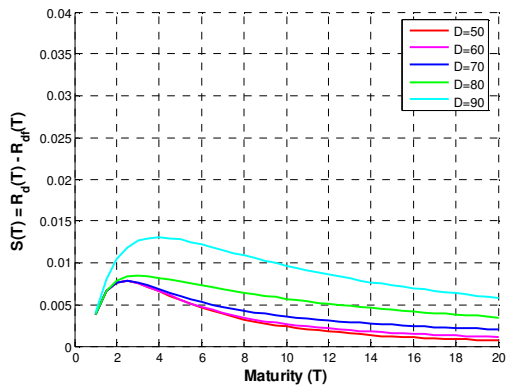


Figure 7

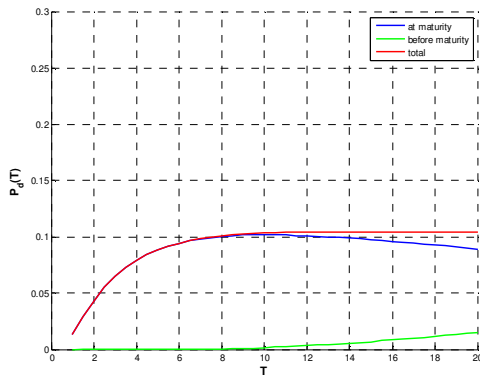
First Passage Default as Dependent Upon Alternative Barriers (D)
(Flat Risk Free Term Structure)

Assuming parameters given below, we analyze the sensitivity of default and credit spreads to alternative barrier values given, in contrast to a previous figure, a flat risk free term structure.

$$r_0 = 0.06, \theta = 0.018, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2, V_0 = 150, K = 100, \lambda = 0$$

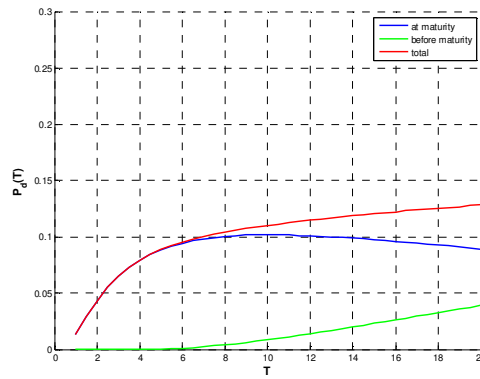
Panel a

D= 50



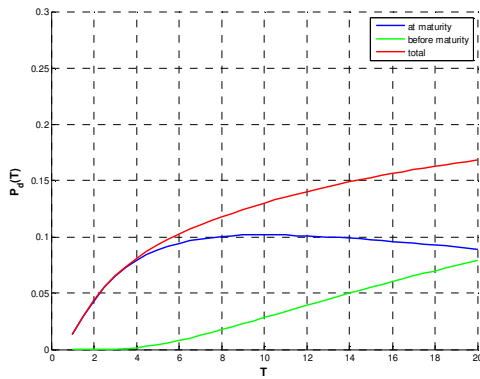
Panel b

D=60



Panel c

D = 70



Panel d

D = 90

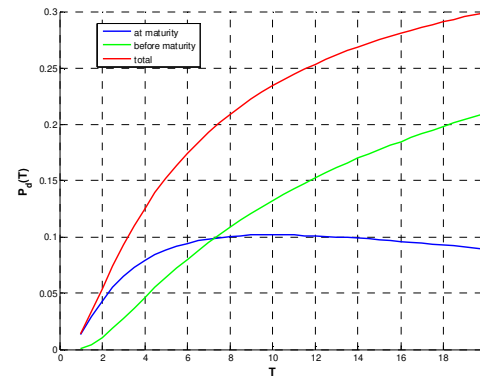
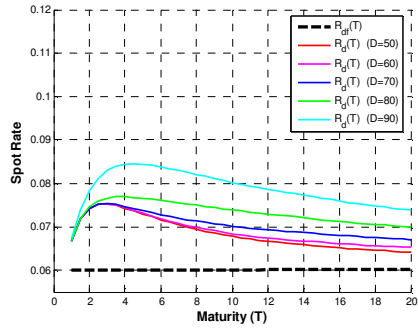
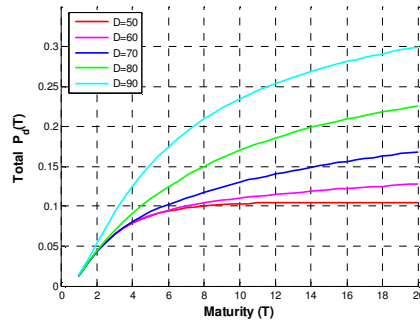


Figure 7 (continued)

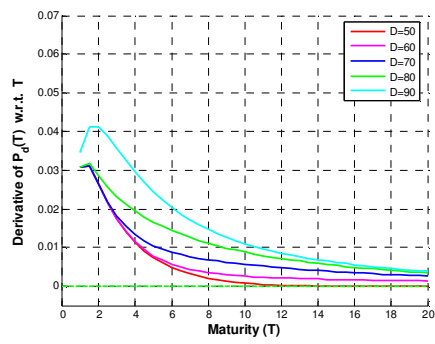
Panel e



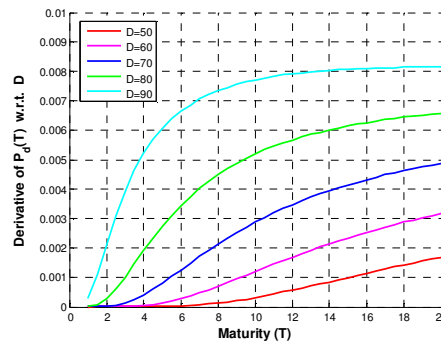
Panel f



Panel g



Panel h



Panel i

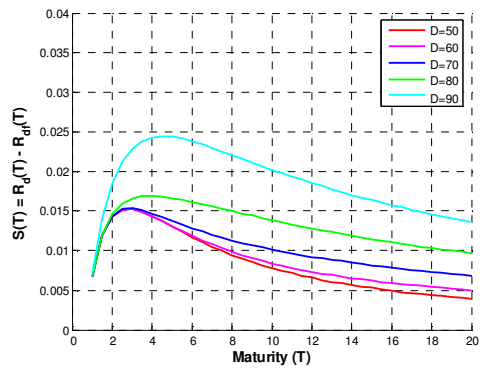


Figure 8

First Passage Default as Dependent Upon Barriers with Increased Volatility of Firm Value (σ_v), Positive Risk-Free Term Structure

Assuming an increased volatility of firm value and parameters given below, we analyze the sensitivity of default and credit spreads to alternative barrier values.

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.25, V_0 = 150, K = 100, \lambda = 0$$

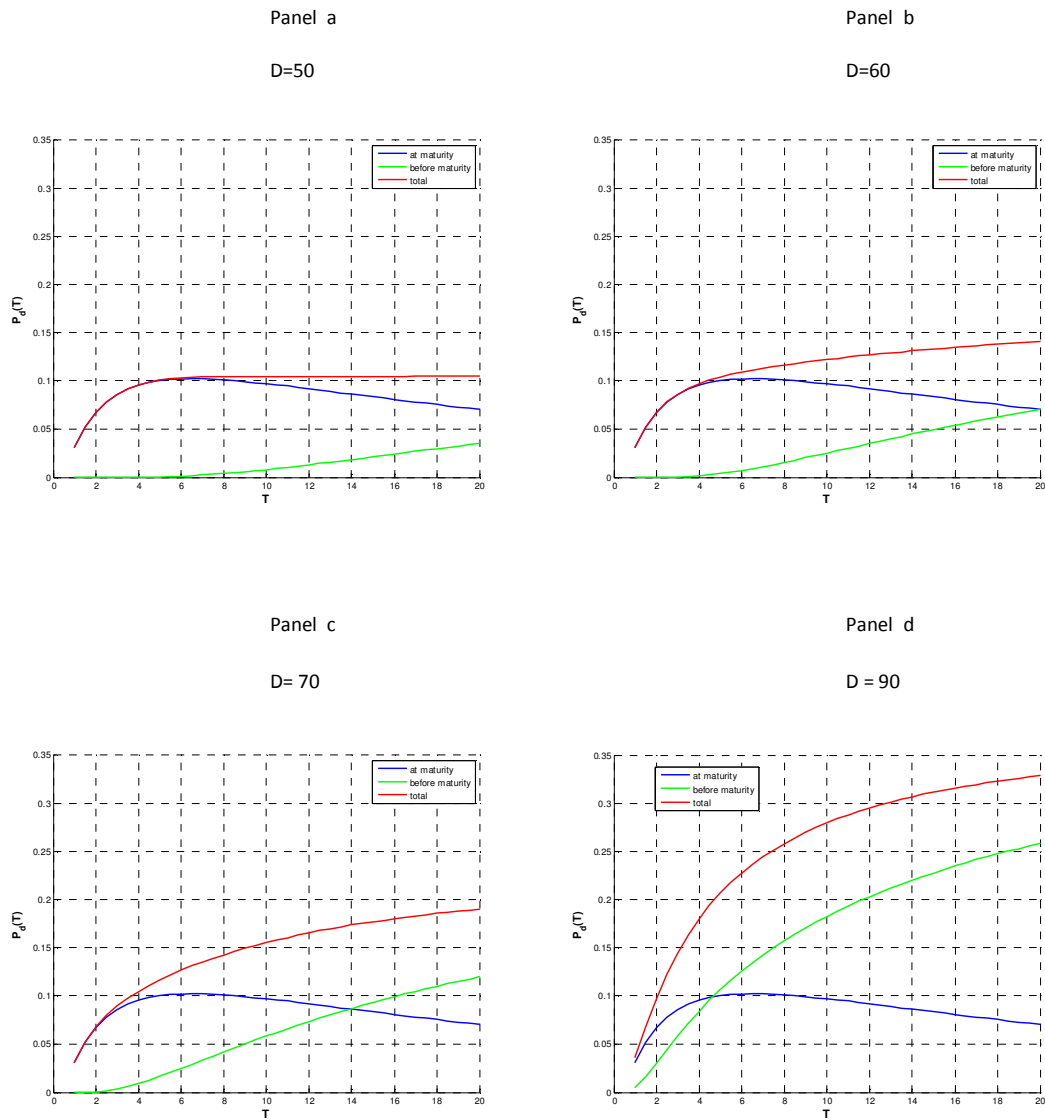
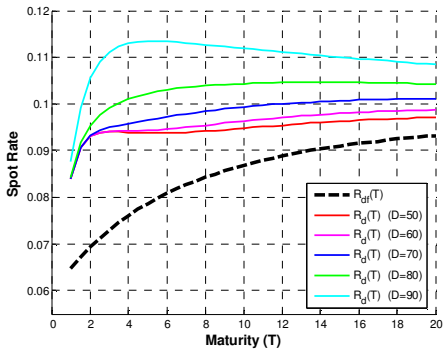
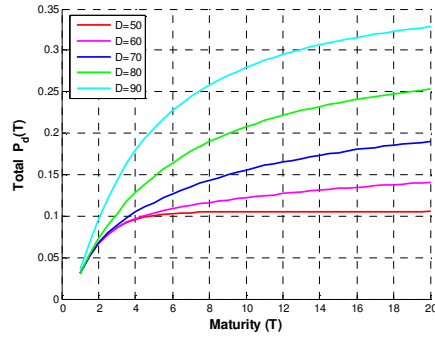


Figure 8 (continued)

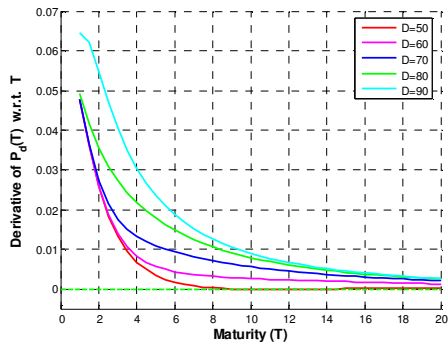
Panel e



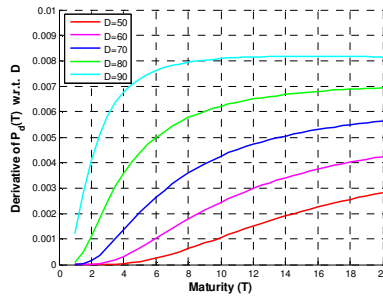
Panel f



Panel g



Panel h



Panel i

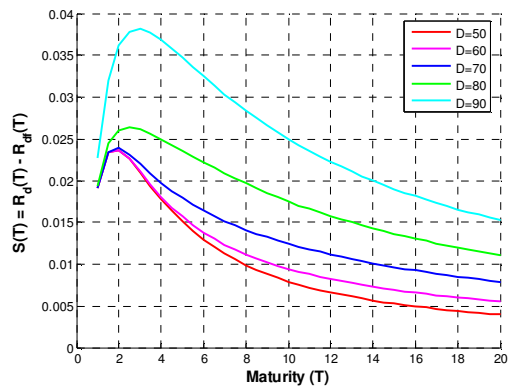


Figure 9

**Impact of Variation in Leverage Upon Probability Density
(Positive Risk-Free Term Structure)**

The shape of the $P_d(T)$ term structure can vary with leverage. The term structure can be practically negative throughout or have a quite pronounced hump.

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2, V_0 = 150, K = 100, D = 60, \lambda = 0$$

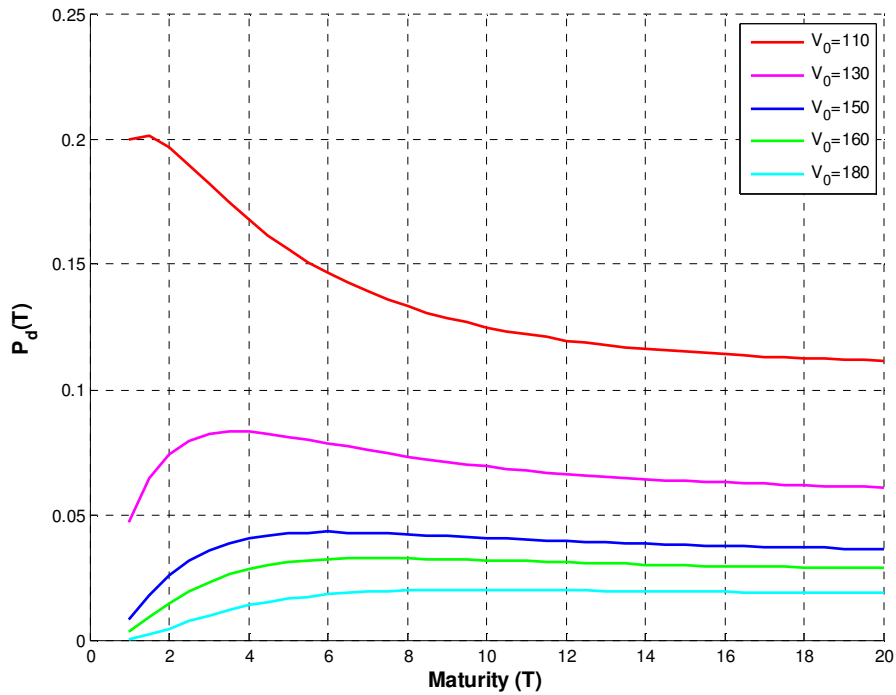


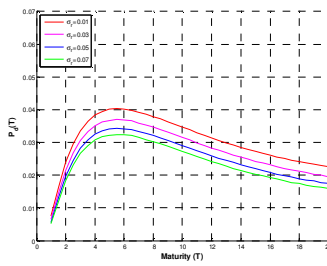
Figure 10

Impact of Variation in Interest Rate Volatility Upon Probability Density

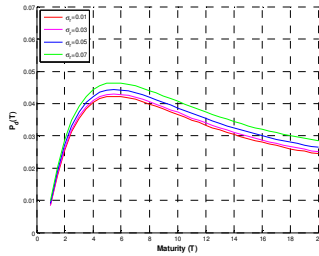
Using various interest rate volatilities and other parameters given below, term structures of default are computed. The hump is quite robust to the different volatilities.

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_v = 0.2, V_0 = 150, K = 100, D = 50$$

Panel a : $\rho = -0.5$



Panel b: $\rho = 0$



Panel c: $\rho = +0.5$

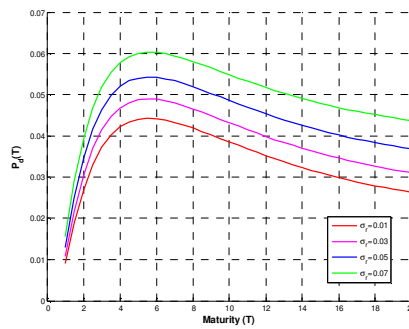
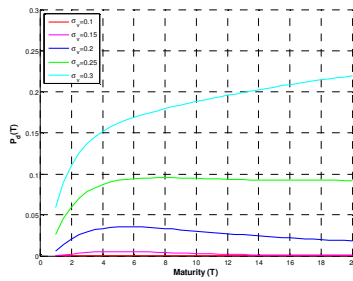


Figure 11 Impact of Variation in ρ Upon Probability Density

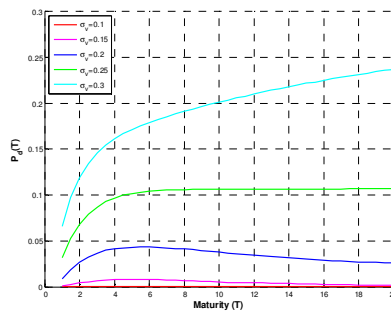
The correlation between the interest rate process and firm value process can have a large impact on the shape of term structure of default probability. A hump may or may not occur.

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, V_0 = 150, K = 100, D = 50, \lambda = 0$$

Panel a: $\rho = -0.5$



Panel b: $\rho = 0$



Panel c: $\rho = 0.5$

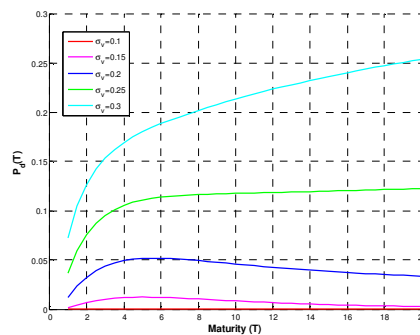


Figure 12

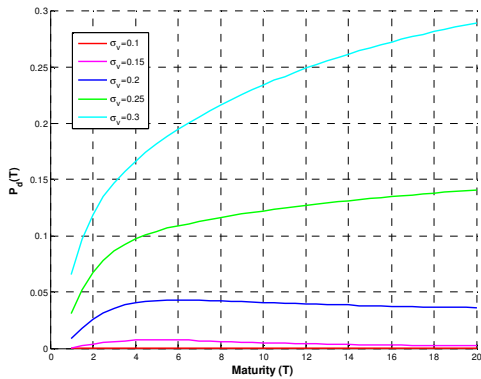
Conversion of Risk Neutral to Physical Probabilities

The level of physical probability of default depends upon the market price of risk (λ) assumed. Nonetheless, the shape of the curves is robust to all assumed λ .

$$r_0 = 0.06, \theta = 0.03, c = 0.3, \sigma_r = 0.02, \rho = 0, \sigma_v = 0.2, V_0 = 150, K = 100, D = 60$$

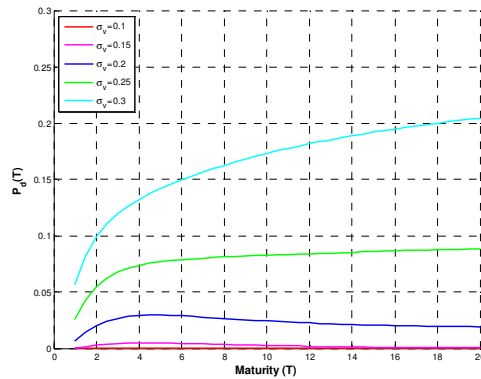
Panel a

$$\lambda = 0$$



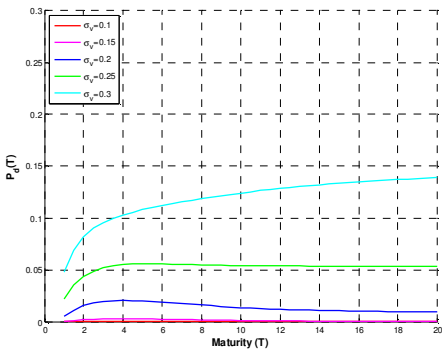
Panel b

$$\lambda = 0.075$$



Panel c

$$\lambda = 0.15$$



Panel d

$$\lambda = 0.3$$

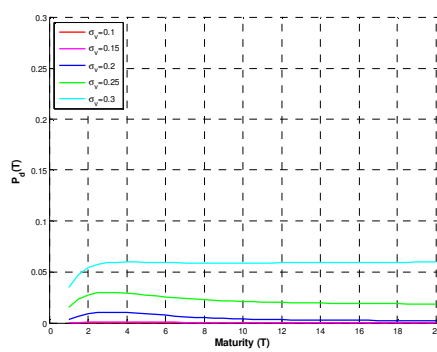


Figure 13

Sensitivity of S(T) Hump Location to Barrier Value (D)

The maturity at which the hump in credit spread occurs is sensitive to the parameters assumed to compute default probability and credit spread. Here the maturity at which the S(T) hump occurs, T^{**} , is shown to be related to barriers (D) in a complex way.

$$\sigma_r = 0.02, \sigma_v = 0.2, \rho = 0.5, D = 100, V_0 = 150, r_0 = 0.06, \theta = 0.03, c = 0.3$$

