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32-vertex model on the triangular lattice†

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Abstract. The 32-vertex model on a triangular lattice is solved exactly in two special cases. In one case the model is reducible to a three-spin Ising model and has the exponents $\alpha = \alpha' = \frac{2}{3}$ The other soluble case is the free-fermion model which exhibits the Ising behaviour but may possess multiple transitions.

1. Introduction

The vertex model in statistical mechanics plays an important role in the study of phase transitions in lattice systems. A case of current interest is the 8-vertex model of a square lattice (Fan and Wu 1970, Baxter 1972), which is characterized by continuously varying critical exponents and a unique transition point. Very little is known, however, for vertex models on other lattices.

In a recent paper, one of us (Wu 1974) has studied a general vertex model on a honeycomb lattice. The situation is more complicated for the triangular lattice, since there are a total of 64 possible vertex configurations. To begin with, one may consider the 20-vertex triangular ice-rule problem, and some progress has been made toward its solution (Baxter 1969, Kelland 1974a, b). In this paper we consider the next step in line, the 32-vertex model on the triangular lattice. This is the counterpart of the 8-vertex model of a square lattice. Some duality relations are already known for this model (Runnels L K 1975, preprint). As we shall see, an interesting feature of the 32-vertex model is the possible existence of multiple phase transitions. Thus, determination of its critical point from duality should be made with caution.

2. Model definition

We place bonds on the lattice edges of a triangular lattice of N sites subject to the constraint that there is always an even number (0, 2, 4 or 6) of bonds incident at each site. Thus, globally the bonds form closed graphs. There are 32 allowed vertex configurations as shown in figure 1. Note that the 20 configurations in the first two rows in figure 1 which have equal numbers of bonds incident on the right and left are the ice-rule configurations‡. Number the edges surrounding a vertex as shown in figure 1. To each

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‡ The usual ice rule (three arrows leaving and entering each vertex) is recovered by attaching to each edge with (without) a bond an arrow pointing towards the right (left).

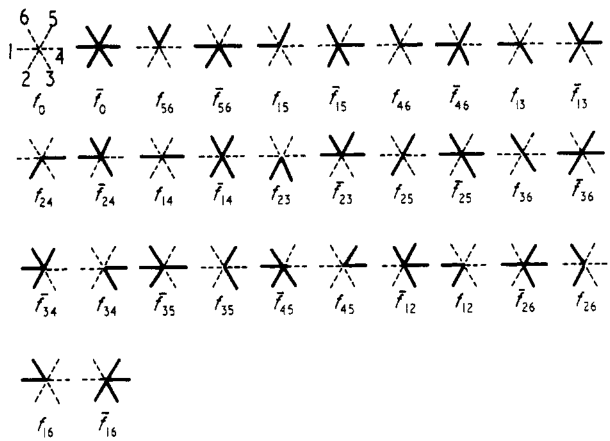

Figure 1. The 32 vertex configurations and the associated weights. The first 20 configurations satisfy the ice rule.

vertex having bonds incident along edges $i, \ldots, j$, is assigned a weight $f_{i\ldots j}$ (f_0 for vertices with no bonds). The generating partition function is then defined by

$$Z = \sum \prod f_{i\ldots j}^{n(i\ldots j)} \tag{1}$$

where the summation is over all bond graphs and $n(i\ldots j)$ is the number of sites having bonds on edges $i, \ldots, j$. Our problem is to evaluate the 'free energy'

$$\psi = \lim_{N\to\infty} \frac{1}{N} \ln Z. \tag{2}$$

In the following we shall also use the 'complementary' notations

$$\bar{f}_0 = f_{123456}, \qquad \bar{f}_{12} = f_{3456} \tag{3}$$

etc, whenever it is convenient.

In a ferroelectric model, the vertex weights are the Boltzmann factors

$$f_{i\ldots j} = \mathrm{e}^{-\beta\epsilon(i\ldots j)} \tag{4}$$

where $\beta = 1/kT$ and $\epsilon(i\ldots j)$ is the vertex energy. Then ψ is a function of T and the model exhibits a phase transition at T_c where ψ fails to be analytic.

Some exact results of this 32-vertex model are readily derived. If all $f_{i\ldots j} = 1$ then ψ is the entropy S and it can be easily derived that

$$S = \ln 4. \tag{5}$$

This is to be compared with the entropy of the triangular ice (Baxter 1969)

$$S_{\Delta\,\mathrm{ice}} = \ln(\tfrac{3}{2}\sqrt{3}). \tag{6}$$

The 32-vertex model is reduced to the 8-vertex model on a square lattice if $f_{i\ldots j} = 0$ whenever any of the indices is among the set $\{1, 4\}$, $\{2, 5\}$ or $\{3, 6\}$. Certainly in this case the critical exponents depend on the vertex weights. We shall see by explicitly solving some special cases in §§ 3, 4 that this is also the case even though none of the vertex weights vanishes.

3. An exactly soluble case

If the vertex weights are

$$\begin{aligned} f_0 &= \bar{f}_0 = \omega_0 \\ f_{ij} &= \bar{f}_{ij} = \omega_1 < \omega_0 \end{aligned} \tag{7}$$

the model can be solved exactly. At low temperatures the system will certainly favour an ordered state. To see that there is a phase transition, we rewrite the partition function as

$$Z(\omega_0, \omega_1) = \left(\frac{\omega_1}{\cosh 2K}\right)^N Z(\cosh 6K, \cosh 2K) \tag{8}$$

where

$$\cosh 2K = \tfrac{1}{2}(\omega_0/\omega_1 + 3)^{1/2} > 1. \tag{9}$$

The partition function on the right-hand side of (8) can be related to that of an Ising model with pure three-spin interactions on a triangular lattice (Baxter and Wu 1973, 1974, Baxter 1974). The Ising lattice, shown in figure 2, has the Hamiltonian

$$\mathscr{H} = -\beta^{-1}K \sum \sigma_i\sigma_j\sigma_k \tag{10}$$

where the summation is over all $6N$ triangular faces. First, we take a partial trace over the N spins belonging to one of the three (triangular) sublattices. For one such spin (the open circle in figure 2) surrounded by spins $\sigma_1, \ldots, \sigma_6$, this partial trace gives rise to a factor

$$2\cosh[K(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_5 + \sigma_5\sigma_6 + \sigma_6\sigma_1)] \tag{11}$$

which is either $2\cosh 6K$ or $2\cosh 2K$. Then in the usual way (Wu 1971) we construct a 32-vertex model on the triangular sublattice by drawing bonds between two σ's of opposite signs. The vertex weights are therefore given by (11) and we obtain

$$Z(2\cosh 6K, 2\cosh 2K) = \tfrac{1}{2}Z_{\text{Ising}}(K) \tag{12}$$

where the factor $\frac{1}{2}$ is due to the two-to-one mapping of the spin and the bond configurations. Substituting (12) into (8) and making use of the result on the three-spin model, we find for the 32-vertex model (7)

$$\psi = \ln \omega_1 - \tfrac{1}{2}\ln[4(t^2+1)] + \tfrac{3}{2}\ln(6yt) \tag{13}$$

where $t = \frac{1}{2}(\omega_0/\omega_1 - 1)^{1/2}$, and $1 \leqslant y < \infty$ is the solution of the quartic equation

$$(y-1)^3(1+3y)/y^3 = 2(1-t)^4/t(1+t^2). \tag{14}$$

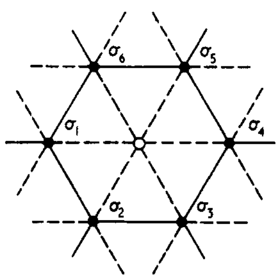

Figure 2. Partial trace is taken over the spin denoted by the open circle.

The free energy is now non-analytic at $t = 1$ or

$$\omega_0 = 5\omega_1 \tag{15}$$

and the specific heat diverges at (15) with the exponents $\alpha = \alpha' = \frac{2}{3}$.

4. Free-fermion solution

Another soluble case of the 32-vertex model is the free-fermion model. Generally, a vertex model is soluble if the vertex weights satisfy certain conditions so that the partition function is reducible to the S matrix of a many-fermion system (Hurst 1966). In the present problem these constraints relate $\bar{f}_0$ and $\bar{f}_{ij}$ to f_0 and f_{ij}, so there are 16 independent conditions. Following Hurst (1966) it is not difficult to see that the conditions are

$$f_0 f_{ijkl} = f_{ij}f_{kl} - f_{ik}f_{jl} + f_{il}f_{jk} \tag{16}$$

and

$$f_0\bar{f}_0 = f_{12}\bar{f}_{12} - f_{13}\bar{f}_{13} + f_{14}\bar{f}_{14} - f_{15}\bar{f}_{15} + f_{16}\bar{f}_{16}. \tag{17}$$

For a meaningful physical model we require these 'free-fermion' conditions to hold at all temperatures. One such model, which we refer to as the 'modified KDP' model (Wu 1967), is a special case of the ice-rule model

$$f_{12} = \bar{f}_{12} = f_{16} = \bar{f}_{16} = f_{26} = \bar{f}_{26} = f_{34} = \bar{f}_{34} = f_{35} = \bar{f}_{35} = f_{45} = \bar{f}_{45} = 0 \tag{18}$$

with

$$\begin{aligned} &f_0 = \bar{f}_{13} = \bar{f}_{23} = \bar{f}_{36} = \bar{f}_{46} = \bar{f}_{56} = 0 \\ &\bar{f}_{14} = \bar{f}_{15} = \bar{f}_{24} = \bar{f}_{25} = x \\ &f_{13} = f_{14} = f_{15} = f_{23} = f_{24} = f_{25} = f_{36} = f_{46} = f_{56} = y. \end{aligned} \tag{19}$$

It is also possible to satisfy (16) and (17) with nonzero weights. One example is given in § 5.

It is straightforward to carry out the analysis of Hurst (1966) for the 32-vertex free-fermion model. In fact, using equation (38) of Hurst (1966), we can immediately write down the free energy:

$$\psi = \frac{1}{8\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln(f_0^2 D(\theta, \varphi)). \tag{20}$$

$D(\theta, \varphi) =$

$$\begin{vmatrix} 0 & -h_{12} & -h_{13} & e^{i\theta} - h_{14} & -h_{15} & -h_{16} \\ h_{12} & 0 & -h_{23} & -h_{24} & e^{i(\theta+\varphi)} - h_{25} & -h_{26} \\ h_{13} & h_{23} & 0 & -h_{34} & -h_{35} & e^{i\varphi} - h_{36} \\ -e^{-i\theta} + h_{14} & h_{24} & h_{34} & 0 & -h_{45} & -h_{46} \\ h_{15} & -e^{-i(\theta+\varphi)} + h_{25} & h_{35} & h_{45} & 0 & -h_{56} \\ h_{16} & h_{26} & -e^{-i\varphi} + h_{36} & h_{14} & h_{56} & 0 \end{vmatrix}$$

where $h_{ij} = f_{ij}/f_0$.

After some algebra and making use of (16) and (17), we find

$$\begin{aligned} f_0^2 D(\theta, \varphi) = {} & \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2 - 2(\Omega_1\Omega_3 - \Omega_2\Omega_4)\cos\theta - 2(\Omega_1\Omega_4 - \Omega_2\Omega_3)\cos\varphi \\ & + 2(\Omega_3\Omega_4 - \Omega_5\Omega_6)\cos(\theta+\varphi) + 2(\Omega_5\Omega_6 - \Omega_1\Omega_2)\cos(\theta-\varphi) \\ & - 4a\sin\varphi\sin(\theta+\varphi) - 4b\sin\theta\sin(\theta+\varphi) - 2c\sin^2(\theta+\varphi) \\ & - 2d\sin^2\theta - 2e\sin^2\varphi \end{aligned} \tag{21}$$

where

$$\begin{aligned} \Omega_1 &= f_0 + \bar{f}_0, \qquad \Omega_2 = f_{25} + \bar{f}_{25} \\ \Omega_3 &= f_{14} + \bar{f}_{14}, \qquad \Omega_4 = f_{36} + \bar{f}_{36} \\ \Omega_5\Omega_6 &= f_{15}f_{24} + \bar{f}_{15}\bar{f}_{24} + f_{14}\bar{f}_{25} + f_{25}\bar{f}_{14} \\ a &= f_{12}f_{45} + \bar{f}_{12}\bar{f}_{45} - f_0\bar{f}_{36} - \bar{f}_0 f_{36} \\ b &= f_{23}f_{56} + \bar{f}_{23}\bar{f}_{56} - f_0\bar{f}_{14} - \bar{f}_0 f_{14} \\ c &= f_0\bar{f}_0 + f_{13}\bar{f}_{13} - f_{12}\bar{f}_{12} - f_{23}\bar{f}_{23} \\ d &= f_0\bar{f}_0 + f_{26}\bar{f}_{26} - f_{16}\bar{f}_{16} - f_{12}\bar{f}_{12} \\ e &= f_0\bar{f}_0 + f_{15}\bar{f}_{15} - f_{56}\bar{f}_{56} - f_{16}\bar{f}_{16}. \end{aligned} \tag{22}$$

As a function of T, ψ, given by a double integral of the form (20), is analytic unless we can make

$$D(\theta, \varphi) = 0 \tag{23}$$

at some T, θ and φ. From (21) it is seen that this can be achieved at $\{\theta, \varphi\} = 0$ or π with the critical temperature T_c determined by

$$\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 = 2\max(\Omega_1, \Omega_2, \Omega_3, \Omega_4). \tag{24}$$

In the free-fermion vertex models that have been previously solved (Fan and Wu 1970, Hsue *et al* 1975, Wu and Lin 1975) the actual critical temperature is always given by (24). We shall suppose that this is also the case here, although we have not established it in the general case.

To investigate the non-analytic behaviour of ψ, we may proceed by carrying out one of the two integrations in (20). However, neither ψ nor its derivatives can be expressed in terms of standard functions. Therefore, we shall be content to discuss its critical behaviour. Consider, for definiteness, the non-analyticity of ψ at $\Omega_1 = \Omega_2 + \Omega_3 + \Omega_4$ which occurs at $\theta = \varphi = 0$. To extract the singular behaviour of ψ we expand $D(\theta, \varphi)$ about $\theta = \varphi = 0$ and obtain

$$\psi_{\text{singular}} \sim \int_0 \mathrm{d}\theta \int_0 \mathrm{d}\varphi \ln[(\Omega_1 - \Omega_2 - \Omega_3 - \Omega_4)^2 + \alpha\theta^2 + \beta\theta\varphi + \gamma\varphi^2] \tag{25}$$

where only the lower integration limits are needed.

The integration in (25) can be performed by first diagonalizing the quadratic form in θ and φ. One finds (Hsue *et al* 1975)

$$\psi_{\text{singular}} \sim |T - T_c|^2 \ln|T - T_c|, \tag{26}$$

which leads to the Ising behaviour with logarithmically divergent specific heat. The argument breaks down if

$$\beta^2 = 4\alpha\gamma \tag{27}$$

at T_c (Hsue *et al* 1975). For the cases of interest considered below, this is not the case. In the modified KDP model (19), for example, we find the critical temperature

$$\bar{f}_0 = 2x + 3y \tag{28}$$

at which (27) does not hold. In contrast, the modified KDP model on a square lattice satisfies (27) and exhibits a different critical behaviour.

5. Multiple phase transitions

In this section we show explicitly that the free-fermion 32-vertex model on the triangular lattice may exhibit multiple phase transitions. This follows from the fact that each Ω_i in (24) is the sum of two Boltzmann factors. If the two largest Boltzmann factors do not belong to the same Ω_i, then there is the possibility that (24) may have three solutions.

To visualize what happens, it is convenient to consider an Ising representation of the 32-vertex model. Following Wu (1971), we place a spin σ in each *face* of the triangular lattice and let the six spins surrounding a vertex interact with two-, four- and six-spin interactions in all possible combinations. There are then $C_0^6 + C_2^6 + C_4^6 + C_6^6 = 32$ Ising parameters. As before, there is a two-to-one mapping between the spin and the vertex configurations and hence a unique transformation between the spin and the vertex models.

As a matter of fact, the free-fermion 32-vertex model with *nonzero* weights always transforms into an Ising model with non-crossing two-spin interactions. It is then convenient to think in terms of the Ising language. To illustrate our result, consider one such model, the Ising model on a Union Jack lattice whose unit cell is shown in figure 3. Note that this is a generalization of the model ($J_2 = J$) considered by Vaks *et al* (1966). Our result now permits a discussion for $J_2 \neq J$.

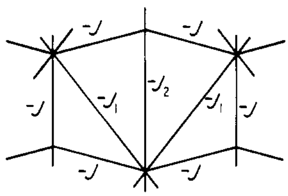

Figure 3. Unit cell of an Ising model on a Union Jack lattice.

Let $K = \beta J$, $K_1 = \beta J_1$, $K_2 = \beta J_2$. We find

$$\begin{aligned}
&f_0 = \exp(3K + 2K_1 + K_2), \qquad \bar{f}_0 = \exp(-3K + 2K_1 - K_2)\\
&f_{12} = f_{34} = \exp(K + 2K_1 + K_2)\\
&f_{13} = f_{14} = f_{23} = f_{24} = \exp(K - 2K_1 - K_2)\\
&f_{15} = f_{25} = f_{36} = f_{46} = \exp(K - K_2)\\
&f_{16} = f_{26} = f_{35} = f_{45} = \exp(K + K_2)\\
&f_{56} = \exp(K + 2K_1 - K_2),
\end{aligned} \tag{29}$$

all other vertex weights being given by (16). From (22) and (29) we find

$$\begin{aligned} \Omega_1 &= 2e^{2K_1}\cosh(3K+K_2) \\ \Omega_3 &= 2e^{-2K_1}\cosh(K-K_2) \\ \Omega_2 &= \Omega_4 = 2\cosh(K-K_2). \end{aligned} \tag{30}$$

The critical condition (24) now has two possible realizations:

$$\begin{aligned} \Omega_1 &= \Omega_2+\Omega_3+\Omega_4 \\ \Omega_3 &= \Omega_1+\Omega_2+\Omega_4. \end{aligned} \tag{31}$$

Let $J_1 = -a|J|$, $J_2 = bJ$, we find that the first equation in (31) has at most one solution while in a limited region of a and b the second equation has two solutions. Thus the system may exhibit up to three phase transitions. Results are shown in figure 4

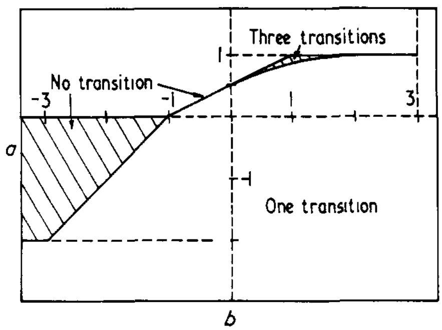

Figure 4. Regions of one, three and no transitions of the Union Jack Ising lattice.

where regions of three, one and no transitions are given. A region of three transitions exists in $0 < b < 3$. Typical boundaries of this region are

$$\begin{aligned} &0.7468 < a < 0.75 \qquad &&\text{for } b = 0.5 \\ &0.9068 < a < 1 \qquad &&\text{for } b = 1 \\ &0.9847 < a < 1 \qquad &&\text{for } b = 1.5. \end{aligned} \tag{32}$$

For $b = 1$ this is the result of Vaks *et al* (1966) (in their paper the lower bound of a is erroneously given as 0·94). We also find no transition in the region

$$\begin{aligned} &-2 \leqslant a \leqslant 0 \qquad &&\text{for } b \leqslant -3 \\ &b+1 \leqslant a \leqslant 0 \qquad &&\text{for } -3 \leqslant b \leqslant -1 \end{aligned} \tag{33}$$

and

$$a = \tfrac{1}{2}(b+1) \qquad \text{for } -1 \leqslant b \leqslant 0.$$

What the system does at the three transition points is most easily seen by considering the Ising representation (Vaks *et al* 1966). Three phase transitions occur in a limited range of ferromagnetic $-J$, $-J_2$ and antiferromagnetic $-J_1$. At low temperatures the system is ferromagnetic which makes the vertex without bond (weight f_0 in figure 1) favoured. As the temperature rises, the system first goes into a disordered state, then an antiferromagnetic state dominated by the second-neighbour interactions $-J_1$. It may be seen that this antiferromagnetic state implies the existence of chains of bonds running across the lattice from left to right in the vertex model. Above the third transition point the system is again disordered.

Finally we remark that a 32-vertex model may also exhibit two phase transitions. Such is the case if the equivalent Ising model breaks into two overlapping *non-identical* nearest-neighbour Ising lattices. Because of the existence of 'crossing' interactions, the free-fermion conditions (16) and (17) do not hold in this case. It is possible that a weak-graph transformation similar to that used for the 8-vertex model (Wu 1969) will bring it to a free-fermion model which explicitly reveals the decomposition.

6. Conclusions

We have studied some soluble cases of the 32-vertex model on a triangular lattice. The specific heat is found to diverge with the critical exponents $\alpha = \alpha' = \frac{2}{3}$ in one case, while in the other case the system exhibits the usual Ising behaviour. Our results also indicate that the 32-vertex model would generally exhibit multiple phase transitions.

References

Baxter R J 1969 *J. Math. Phys.* **10** 1211–6
—— 1972 *Ann. Phys., NY* **70** 193–228
—— 1974 *Aust. J. Phys.* **27** 369–81
Baxter R J and Wu F Y 1973 *Phys. Rev. Lett.* **31** 1294–7
—— 1974 *Aust. J. Phys.* **27** 357–67
Fan C and Wu F Y 1970 *Phys. Rev.* B **2** 723–33
Hsue C S, Lin K Y and Wu F Y 1975 *Phys. Rev.* B **12** 429–37
Hurst C A 1966 *J. Math. Phys.* **7** 305–10
Kelland S B 1974a *J. Phys.* A: *Math., Nucl. Gen.* **7** 1907–12
—— 1974b *Aust. J. Phys.* **27** 813–9
Vaks C G, Larkin A I and Urchinnikov Y N 1966 *Sov. Phys.–JETP* **22** 820–6
Wu F Y 1967 *Phys. Rev. Lett.* **18** 605–7
—— 1969 *Phys. Rev.* **183** 604–7
—— 1971 *Phys. Rev.* B **4** 2312–4
—— 1974 *J. Math. Phys.* **15** 687–91
Wu F Y and Lin K Y 1975 *Phys. Rev.* B **12** 419–28