

Some New Integral Inequalities for n -Times Differentiable s -Convex and s -Concave Functions in the Second Sense

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Abstract In this article, by using an integral identity together with both the Hölder, Power-Mean integral inequalities and Hermite-Hadamard's inequality, we establish several new inequalities for n -time differentiable s -convex and s -concave functions in the second sense.

Keywords s -Convex Function, s -Concave Function, Hermite-Hadamard's Inequality, Hölder and Power-mean Integral Inequality

1. Introduction

In this paper, by using some classical integral inequalities, Hölder and Power-Mean integral inequality, we establish some new inequalities for functions whose n th derivatives in absolute value are s -convex functions in the second sense. For some inequalities, generalizations and applications concern convexity see [1]-[11]. Recently, in the literature there are so many papers about n -times differentiable functions on several kinds of convexities and s -convex functions. In references [5]-[8], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex and s -convex functions in the second sense see for instance the recent papers [12]-[19] and the references within these papers. There are quite substantial literatures on such problems. Here we mention the results of [1]-[19] and the corresponding references cited therein.

Definition 1.1: A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 1.2: A function $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s -convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s -convex functions is usually denoted by K_s^2 . It can be easily seen that for $s = 1$, s -convexity reduces the ordinary convexity of functions defined on $[0, \infty)$. In the paper [20], some properties of s -convex functions in both senses are considered and various examples and counterexamples are given.

The following Hermite-Hadamard type inequality for s -convex functions in the second sense was demonstrated in [21]. Let $f: [0, \infty) \rightarrow [0, \infty)$ be a s -convex function in the second sense for $s \in (0, 1]$ and let $b > a \geq 0$. If $f \in L[a, b]$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

If f is s -concave, then the above inequalities are reversed.

Throughout this paper we will use the following notations and conventions. Let $J = [0, \infty) \subset \mathbb{R} = (-\infty, +\infty)$, and $a, b \in J$ with $0 < a < b$ and $f' \in L[a, b]$ and

$$A(a, b) = \frac{a+b}{2},$$

$$L_p(a, b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, \quad a \neq b, p \in \mathbb{R}, p \neq -1, 0$$

be the arithmetic, geometric, generalized logarithmic mean for $a, b > 0$ respectively.

We will use the following Lemma [10] for we obtain the main results:

Lemma 1.1: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$, we have the identity

$$\sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx$$

$$= \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx.$$

Especially, we want to note that the results obtained in this article for $s = 1$ coincide with the results in [10].

2. Main Results

Theorem 2.1. For $\forall n \in \mathbb{N}$; let $f: I \subset [0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f^{(n)}|^q \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is s -convex function in the second sense on $[a, b]$, then the following inequality holds:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx \right|$$

$$\leq \frac{1}{n!} (b-a) \left(\frac{2}{s+1} \right)^{\frac{1}{q}} L_{np}^n(a, b) A^{\frac{1}{q}} (|f^{(n)}(a)|^q, |f^{(n)}(b)|^q),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If $|f^{(n)}|^q$ for $q > 1$ is s -convex function in the second sense on $[a, b]$, using Lemma 1.1, the Hölder integral inequality and

$$|f^{(n)}(x)|^q = \left| f^{(n)} \left(\frac{x-a}{b-a} b + \frac{b-x}{b-a} a \right) \right|^q$$

$$\leq \left(\frac{x-a}{b-a} \right)^s |f^{(n)}(b)|^q$$

$$+ \left(\frac{b-x}{b-a} \right)^s |f^{(n)}(a)|^q,$$

we have

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx \right|$$

$$\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx$$

$$\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b \left[\left(\frac{x-a}{b-a} \right)^s |f^{(n)}(b)|^q \right. \right. \right. \\ \left. \left. \left. + \left(\frac{b-x}{b-a} \right)^s |f^{(n)}(a)|^q \right] dx \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\frac{|f^{(n)}(b)|^q}{(b-a)^s} \int_a^b (x-a)^s dx \right. \\ \left. + \frac{|f^{(n)}(a)|^q}{(b-a)^s} \int_a^b (b-x)^s dx \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} \left[\frac{b^{np+1} - a^{np+1}}{np+1} \right]^{\frac{1}{p}} \left[\frac{(b-a)}{s+1} (|f^{(n)}(b)|^q + |f^{(n)}(a)|^q) \right]^{\frac{1}{q}}$$

$$= \frac{1}{n!} (b-a)^{\frac{1}{p}} (b-a)^{\frac{1}{q}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}}$$

$$\times L_{np}^n(a, b) A^{\frac{1}{q}} (|f^{(n)}(a)|^q, |f^{(n)}(b)|^q)$$

$$= \frac{1}{n!} (b-a) \left(\frac{2}{s+1} \right)^{\frac{1}{q}} L_{np}^n(a, b) A^{\frac{1}{q}} (|f^{(n)}(a)|^q, |f^{(n)}(b)|^q)$$

This completes the proof of theorem.

Corollary 2.1. Under the conditions Theorem 2.1 for $n = 1$ we have the following inequality:

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \left(\frac{2}{s+1} \right)^{\frac{1}{q}} L_p(a, b) A^{\frac{1}{q}} (|f'(a)|^q, |f'(b)|^q).$$

Proposition 2.1. Let $a, b \in (0, 1]$ with $a < b$, $q > 1$ and $s \in (0, 1]$, we have the following inequality:

$$L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a, b) \leq \left(\frac{a^s + b^s}{s+1} \right)^{\frac{1}{q}} L_p(a, b)$$

Proof. Under the assumption of the Proposition, let

$$f(t) = \frac{q}{q+s} t^{\frac{s}{q}+1}, t \in [0, 1].$$

Then

$$|f'(t)| = t^{\frac{s}{q}}$$

is s-convex on [0,1] and the result follows directly from Corollary 2.1.

Theorem 2.2. For $\forall n \in \mathbb{N}$; let $f: I \subset [0, \infty) \rightarrow \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f^{(n)}|^q \in L[a, b]$ and $|f^{(n)}|^q$ for $q \geq 1$ is s-convex function on $[a, b]$, then the following inequality holds:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx \right| \leq \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} L_n^{\frac{q-1}{q}} [|f^{(n)}(b)|^q P(n, s, x) + |f^{(n)}(a)|^q Q(n, s, x)]^{\frac{1}{q}}$$

Where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$ and

$$P(n, s, x) = \int_a^b x^n (x-a)^s dx$$

$$Q(n, s, x) = \int_a^b x^n (b-x)^s dx.$$

Proof. From Lemma 1.1 and Power-mean integral inequality, we obtain

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx \right| \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\int_a^b x^n \left[\left(\frac{x-a}{b-a} \right)^s |f^{(n)}(b)|^q + \left(\frac{b-x}{b-a} \right)^s |f^{(n)}(a)|^q \right] dx \right)^{\frac{1}{q}} = \frac{1}{n!} \left(\frac{x^{n+1}}{n+1} \Big|_a^b \right)^{1-\frac{1}{q}} \left(\frac{|f^{(n)}(b)|^q}{(b-a)^s} \int_a^b x^n (x-a)^s dx + \frac{|f^{(n)}(a)|^q}{(b-a)^s} \int_a^b x^n (b-x)^s dx \right)^{\frac{1}{q}} = \frac{1}{n!} \left(\frac{b^{n+1} - a^{n+1}}{n+1} \right)^{1-\frac{1}{q}} \left(\frac{|f^{(n)}(b)|^q}{(b-a)^s} P(n, s, x) + \frac{|f^{(n)}(a)|^q}{(b-a)^s} Q(n, s, x) \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} (b-a)^{\frac{1}{p}} \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{1-\frac{1}{q}} \times \left[\frac{|f^{(n)}(b)|^q}{(b-a)^s} P(n, s, x) + \frac{|f^{(n)}(a)|^q}{(b-a)^s} Q(n, s, x) \right]^{\frac{1}{q}} = \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} L_n^{\frac{q-1}{q}} [|f^{(n)}(b)|^q P(n, s, x) + |f^{(n)}(a)|^q Q(n, s, x)]^{\frac{1}{q}}$$

Theorem 2.3. For $\forall n \in \mathbb{N}$; let $f: I \subset [0, \infty) \rightarrow \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f^{(n)}|^q \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is s-convex function on $[a, b]$, then the following inequalities hold:

$$\left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx \right| \leq \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} \times [|f^{(n)}(b)|^q S(n, s, q, x) + |f^{(n)}(a)|^q T(n, s, q, x)]^{\frac{1}{q}} \left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx \right| \leq \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} L_p(a, b) \times [|f^{(n)}(b)|^q K(n, s, q, x) + |f^{(n)}(a)|^q L(n, s, q, x)]^{\frac{1}{q}} \left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx \right| \leq \frac{(b-a)^{1-\frac{s+1}{q}}}{n!} L_{(n-1)p}^{n-1}(a, b) \times [|f^{(n)}(b)|^q M(s, q, x) + |f^{(n)}(a)|^q N(s, q, x)]^{\frac{1}{q}} \left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx \right| \leq \frac{1}{n!} (b-a) \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \times L_{np}^n(a, b) A^{\frac{1}{q}} (|f^{(n)}(a)|^q, |f^{(n)}(b)|^q)$$

where $S(n, s, q, x) = \int_a^b x^{nq} (x-a)^s dx$

$$T(n, s, q, x) = \int_a^b x^{nq} (b-x)^s dx$$

$$K(n, s, q, x) = \int_a^b x^{(n-1)q} (x-a)^s dx$$

$$L(n, s, q, x) = \int_a^b x^{(n-1)q} (b-x)^s dx$$

$$M(s, q, x) = \int_a^b x^q (x-a)^s dx$$

$$N(s, q, x) = \int_a^b x^q (b-x)^s dx.$$

Proof: If $|f^{(n)}|^q$ for $q > 1$ is s -convex function the second sense on $[a, b]$, using Lemma 1.1 and the Hölder integral inequality, we have the following inequalities respectively:

$$\begin{aligned} & \left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx \right| \\ & \leq \frac{1}{n!} \left(\int_a^b 1^p dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ & \leq \frac{1}{n!} \left(\int_a^b 1 \cdot dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} \left[\left(\frac{x-a}{b-a} \right)^s |f^{(n)}(b)|^q \right. \right. \\ & \quad \left. \left. + \left(\frac{b-x}{b-a} \right)^s |f^{(n)}(a)|^q \right] dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a)^{\frac{1}{p}} \left[\frac{|f^{(n)}(b)|^q}{(b-a)^s} S(n, s, q, x) \right. \\ & \quad \left. + \frac{|f^{(n)}(a)|^q}{(b-a)^s} T(n, s, q, x) \right]^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} \left[|f^{(n)}(b)|^q S(n, s, q, x) \right. \\ & \quad \left. + |f^{(n)}(a)|^q T(n, s, q, x) \right]^{\frac{1}{q}} \\ & \left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx \right| \\ & \leq \frac{1}{n!} \left(\int_a^b x^p dx \right)^{\frac{1}{p}} \left(\frac{|f^{(n)}(b)|^q}{(b-a)^s} \int_a^b x^{(n-1)q} (x-a)^s dx \right. \\ & \quad \left. + \frac{|f^{(n)}(a)|^q}{(b-a)^s} \int_a^b x^{(n-1)q} (b-x)^s dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} \left(\frac{b^{p+1} - a^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|f^{(n)}(b)|^q}{(b-a)^s} K(n, s, q, x) \right. \\ & \quad \left. + \frac{|f^{(n)}(a)|^q}{(b-a)^s} L(n, s, q, x) \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} L_p(a, b) \left[|f^{(n)}(b)|^q K(n, s, q, x) \right. \\ & \quad \left. + |f^{(n)}(a)|^q L(n, s, q, x) \right]^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} & \left| \frac{(-1)^{n+1}}{n!} \int_a^b x^{n-1} x f^{(n)}(x) dx \right| \\ & \leq \frac{1}{n!} \left(\int_a^b x^{(n-1)p} dx \right)^{\frac{1}{p}} \left(\int_a^b x^q |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ & \leq \frac{1}{n!} \left(\int_a^b x^{(n-1)p} dx \right)^{\frac{1}{p}} \left(\int_a^b x^q \left[\left(\frac{x-a}{b-a} \right)^s |f^{(n)}(b)|^q \right. \right. \\ & \quad \left. \left. + \left(\frac{b-x}{b-a} \right)^s |f^{(n)}(a)|^q \right] dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} \left[\frac{b^{(n-1)p+1} - a^{(n-1)p+1}}{(n-1)p+1} \right]^{\frac{1}{p}} \left[\frac{|f^{(n)}(b)|^q}{(b-a)^s} M(s, q, x) \right. \\ & \quad \left. + \frac{|f^{(n)}(a)|^q}{(b-a)^s} N(s, q, x) \right]^{\frac{1}{q}} \\ & = \frac{(b-a)^{1-\frac{s+1}{q}}}{n!} \left[\frac{b^{(n-1)p+1} - a^{(n-1)p+1}}{((n-1)p+1)(b-a)} \right]^{\frac{1}{p}} \\ & \quad \times \left[|f^{(n)}(b)|^q M(s, q, x) + |f^{(n)}(a)|^q N(s, q, x) \right]^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a)^{1-\frac{s+1}{q}} L_{(n-1)p}^{n-1}(a, b) \left[|f^{(n)}(b)|^q M(s, q, x) \right. \\ & \quad \left. + |f^{(n)}(a)|^q N(s, q, x) \right]^{\frac{1}{q}}. \end{aligned}$$

The proof of the last inequality in this Theorem is the same as the proof of Theorem 2.1. This completes the proof of Theorem.

Theorem 2.4. For $\forall n \in \mathbb{N}$; let $f: I \subset [0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f^{(n)}|^q \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is s -concave function in the second sense on $[a, b]$, then the following inequality holds:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx \right| \\ & \leq \frac{1}{n!} (b-a) 2^{\frac{s-1}{q}} L_{np}^n(a, b) \left| f^{(n)} \left(\frac{a+b}{2} \right) \right| \end{aligned}$$

Proof: If $|f^{(n)}|^q$ for $q > 1$ is s -concave function the second sense on $[a, b]$, using Lemma 1.1, the Hermite-Hadamard inequality and the Hölder integral inequality, we have the following inequality:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx \right|$$

$$\begin{aligned}
&\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left((b-a) 2^{s-1} \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \\
&= \frac{1}{n!} (b-a)^{\frac{1}{q}} 2^{\frac{s-1}{q}} \left(\frac{b^{np+1} - a^{np+1}}{np+1} \right)^{\frac{1}{p}} \left| f^{(n)} \left(\frac{a+b}{2} \right) \right| \\
&= \frac{1}{n!} (b-a) 2^{\frac{s-1}{q}} L_{np}^n(a, b) \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|.
\end{aligned}$$

Corollary 2.2. Under the conditions of Theorem 2.4 for $n = 1$, we obtain the inequality

$$\begin{aligned}
&\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq 2^{\frac{s-1}{q}} L_p(a, b) \left| f' \left(\frac{a+b}{2} \right) \right|.
\end{aligned}$$

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