On smoothness properties of optimal value functions at the boundary of their domain under complete convexity*

Oliver Stein# Nathan Sudermann-Merx‡

June 14, 2013

Abstract

This article studies continuity and directional differentiability properties of optimal value functions, in particular at boundary points of their domain. We extend and complement standard continuity results from [12] for abstract feasible set mappings under complete convexity as well as standard differentiability results from [13] for feasible set mappings in functional form under the Slater condition in the unfolded feasible set.

In particular, we present sufficient conditions for the inner semicontinuity of feasible set mappings and, using techniques from nonsmooth analysis, provide functional descriptions of tangent cones to the domain of the optimal value function. The latter makes the stated directional differentiability results accessible for practical applications.

Keywords: Complete convexity, Slater condition, inner semi-continuity, directional differentiability, nonsmooth linearization cone.

AMS subject classifications: 90C31, 90C25.

---

*This research was partially supported by the DFG (Deutsche Forschungsgemeinschaft) under grant STE 772/13-1

#Institute of Operations Research, Karlsruhe Institute of Technology (KIT), Germany, stein@kit.edu

‡Institute of Operations Research, Karlsruhe Institute of Technology (KIT), Germany, sudermann@kit.edu

1 Introduction

We consider the parametric finite optimization problem

$$P_{f,M}(t): \qquad \min_x f(t,x) \quad \text{s.t.} \quad x \in M(t)$$

with parameter vector $t \in \mathbb{R}^r$, decision vector $x \in \mathbb{R}^n$, a convex function $f : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$ and a non-trivial, outer semi-continuous and graph-convex set-valued mapping $M : \mathbb{R}^r \rightrightarrows \mathbb{R}^n$. The latter assumptions on M mean that its graph,

$$\operatorname{gph} M \;=\; \{(t,x) \in \mathbb{R}^r \times \mathbb{R}^n |\; x \in M(t)\},$$

is a non-empty, closed and convex set, respectively. Following Hogan ([13]), we call this convexity assumption *complete convexity*. Obviously, under complete convexity $P_{f,M}(t)$ is a convex optimization problem for each $t \in \mathbb{R}^r$. For convenience, we will denote the family of optimization problems $P_{f,M}(t)$, $t \in \mathbb{R}^r$, briefly by $P_{f,M}$.

The following will be a blanket assumption throughout this article.

Assumption 1.1 *For each $t \in \mathbb{R}^r$ there exists some level $\alpha(t) \in \mathbb{R}$ such that the set $\{x \in M(t) |\, f(t,x) \leq \alpha(t)\}$ is bounded.*

Note that the set-valued mapping M does not necessarily have to be bounded-valued under Assumption 1.1, that is, $M(t)$ may be unbounded for some $t \in \mathbb{R}^r$.

While the continuity results discussed below will not need a functional description of M, our differentiability analysis shall depend on it, in particular to state the subsequent Assumption 1.2. We will then assume that the graph of M is given by

$$\operatorname{gph} M \;=\; \{(t,x) \in \mathbb{R}^r \times \mathbb{R}^n |\; g_i(t,x) \leq 0,\; i \in I\}$$

with a finite index set $I = \{1, \ldots, p\}$, $p \in \mathbb{N}$, and convex functions $g_i : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$, $i \in I$. Note that M is then outer semi-continuous and graph-convex, and that for a given parameter vector $t \in \mathbb{R}^r$ we obtain

$$M(t) \;=\; \{x \in \mathbb{R}^n |\; g_i(t,x) \leq 0,\; i \in I\}$$

as well as the parametric optimization problem with functional constraints

$$P_{f,g}(t): \qquad \min_x f(t,x) \quad \text{s.t.} \quad g_i(t,x) \leq 0,\; i \in I.$$

In the following, whenever results can be shown without the functional description of M, we will refer to the parametric problem $P_{f,M}$ instead of $P_{f,g}$. The inclusion of finitely many affine equality constraints in the description of $\operatorname{gph} M$ is straightforward, but omitted here for the ease of presentation.

For problem $P_{f,g}$, the non-triviality of M will be strengthened to the following blanket assumption.

Assumption 1.2 *The set* $\operatorname{gph} M$ *satisfies the Slater condition, that is, there exists some* $(t^\star, x^\star) \in \mathbb{R}^r \times \mathbb{R}^n$ *with* $g_i(t^\star, x^\star) < 0$, $i \in I$.

In the subsequent investigation of problem $P_{f,M}$, the (effective) domain of the set-valued mapping M,

$$\operatorname{dom} M \;=\; \{t \in \mathbb{R}^r \mid M(t) \neq \emptyset\},$$

will play a crucial role. Note that $\operatorname{dom} M$ is the orthogonal projection of the non-empty, closed and convex set $\operatorname{gph} M$ to $\mathbb{R}^r$ and, thus, itself at least non-empty and convex.

Due to the continuity of f and Assumption 1.1, the Weierstraß theorem guarantees solvability of $P_{f,M}(t)$ for each $t \in \operatorname{dom} M$. Hence, for $t \in \operatorname{dom} M$ the optimal value of $P_{f,M}(t)$ is the real number

$$f^\star(t) \;=\; \min_{x \in M(t)} f(t,x).$$

As usual, for $t \notin \operatorname{dom} M$ we set $f^\star(t) = +\infty$. Hence, the (effective) domain of $f^\star$,

$$\operatorname{dom} f^\star \;=\; \{t \in \mathbb{R}^r \mid f^\star(t) < +\infty\}$$

and $\operatorname{dom} M$ coincide,

$$\operatorname{dom} f^\star \;=\; \operatorname{dom} M, \tag{1}$$

and thus also $\operatorname{dom} f^\star$ is a non-empty and convex set. Note that Assumption 1.1 rules out the case $f^\star(t) = -\infty$ for any $t \in \mathbb{R}^r$.

We will employ the following example throughout this article to illustrate our results.

Example 1.3 *For* $n = r = 1$ *and* $p = 2$*, the continuously differentiable and convex functions* $f(t,x) = x$, $g_1(t,x) = t^2 + x^2 - 1$ *and* $g_2(t,x) = -t - x$ *define the problem*

$$P_{f,g}(t): \qquad \min_x \; x \quad s.t. \quad t^2 + x^2 \leq 1, \quad -t - x \leq 0$$

with

$$
\begin{aligned}
M(t) &= \begin{cases} \left[\max\left\{-t, -\sqrt{1-t^2}\right\}, \sqrt{1-t^2}\right], & t \in \left[-\frac{1}{\sqrt{2}}, 1\right] \\ \emptyset, & \textit{else,} \end{cases} \\
\operatorname{dom} M &= \left[-\frac{1}{\sqrt{2}}, 1\right] \\
\textit{and} \quad \operatorname{gph} M &= \{(t,x) \in \mathbb{R}^2 |\ t^2 + x^2 \le 1,\ -t - x \le 0\}.
\end{aligned}
$$

It is easy to see that Assumptions 1.1 and 1.2 hold. The optimal value function is

$$f^\star(t) \;=\; \max\left\{-t, -\sqrt{1-t^2}\right\}$$

with $\operatorname{dom} f^\star = \left[-\frac{1}{\sqrt{2}}, 1\right]$.

Note that, throughout this article, the domain $\mathbb{R}^r \times \mathbb{R}^n$ of the functions f, g_i, $i \in I$, could be replaced by any open convex set containing $\operatorname{gph} M$ but, for notational convenience, we will not formulate this explicitly.

The focus of this paper will be on continuity and differentiability properties of $f^\star$ on its whole domain. Our assumptions admit to survey existing and develop new material without veiling the main ideas by too many technicalities. We emphasize that it is not the aim of this paper to prove results under rather weak assumptions, but to provide useful results which still hold under natural assumptions and may easily be implemented in practical applications. Here we think of, in particular, reformulations of generalized Nash equilibrium problems as single optimization problems, where optimal value functions enter the objective function ([4, 10]), and structural properties of generalized semi-infinite optimization problems, where optimal value functions describe the feasible set ([24, 25]).

The article is structured as follows. In Section 2 we study lower and upper semi-continuity of $f^\star$ up to the boundary of $\operatorname{dom} f^\star$, along with the inner semi-continuity of the feasible set mapping M. Given a functional description of M, we also provide a functional description of $\operatorname{dom} f^\star$. Section 3 is devoted to directional differentiability properties of $f^\star$ in directions from the radial cone to $\operatorname{dom} f^\star$, and to functional descriptions of related cones. Some final remarks conclude this article in Section 4.

2 Continuity properties of $f^\star$

There are (at least) two different roads to establish continuity properties of $f^\star$ as, on the one hand, $f^\star$ is an optimal value function and, on the other hand, $f^\star$ is a convex function. We start this section by briefly recalling the latter result.

2.1 Convexity of f^*

The following proposition is well-known even under weaker assumptions on $P_{f,M}$ (cf., e.g., [21, Cor. 3.32]), but we give its short proof for completeness.

Proposition 2.1 *The optimal value function $f^\star$ of $P_{f,M}$ is proper and convex on $\mathbb{R}^r$.*

Proof. The non-triviality and Assumption 1.1 imply that $f^\star$ is proper. The convexity assertion is shown if we can prove that $\text{epi}\, f^\star$, the epigraph of $f^\star$, is a convex set ([19]). In fact, choose (t^1, α^1) and (t^2, α^2) in $\text{epi}\, f^\star$, that is, we have $f^\star(t^i) \le \alpha^i$, $i = 1, 2$. In view of (1), for $i = 1, 2$ the sets $M(t^i)$ are non-empty so that we may choose points $x^i \in M(t^i)$ with $f(t^i, x^i) \le \alpha^i$. The convexity of $\text{gph}\, M$ yields

$$(1-\lambda)x^1 + \lambda x^2 \;\in\; M(\,(1-\lambda)t^1 + \lambda t^2\,)$$

for all $\lambda \in (0, 1)$ so that, together with the convexity of f, we arrive at

$$\begin{aligned}(1-\lambda)\alpha^1 + \lambda\alpha^2 &\ge (1-\lambda) f(t^1, x^1) + \lambda f(t^2, x^2)\\ &\ge f(\,(1-\lambda)t^1 + \lambda t^2, (1-\lambda)x^1 + \lambda x^2\,)\\ &\ge f^\star(\,(1-\lambda)t^1 + \lambda t^2\,),\end{aligned}$$

which means $(1-\lambda)(t^1, \alpha^1) + \lambda(t^2, \alpha^2) \in \text{epi}\, f^\star$ and, thus, convexity of $\text{epi}\, f^\star$. •

As a convex function, $f^\star$ is continuous on the topological interior $\text{int}\,\text{dom}\, f^\star$ of $\text{dom}\, f^\star$ ([19, Th. 10.1]). Example 2.3 below will illustrate that $f^\star$ is not necessarily continuous on all of $\text{dom}\, f^\star$. We will give sufficient conditions to rule out this unsatisfactory situation.

2.2 Lower semi-continuity of f^*

To extend the continuity analysis of $f^\star$ to all of $\operatorname{dom} f^\star$, we consider its lower and upper semi-continuity separately. Note that all continuity properties of $f^\star$ will be meant relative to $\operatorname{dom} f^\star$.

The *lower* semi-continuity of $f^\star$ is shown under different even weaker assumptions in [16, Cor. 2.1] and in [21, Cor. 3.32]. We emphasize that we could give an elementary proof based on Lemma 2.5 below for the case of a bounded-valued feasible set mapping M. In fact, there we will show that under complete convexity the *pointwise* bounded-valuedness of M implies even *local* boundedness of M on its domain, that is, the boundedness holds locally uniformly in the sense that each $\bar{t} \in \operatorname{dom} M$ possesses a neighborhood V such that $\bigcup_{t\in V} M(t)$ is bounded. Without any convexity assumptions a standard result ([12, Th. 5]) then implies lower semi-continuity of $f^\star$ on its domain, as long as f is lower semi-continuous and M is outer semi-continuous.

We do not give this proof explicitly here for two reasons. First, our blanket Assumption 1.1 is more general than bounded-valuedness of M and, second, for convex-valued feasible set mappings M it turns out to be unnecessary to argue via their local boundedness. Instead, we only provide the following result which immediately follows from [16, Cor. 2.1] under our assumptions.

Proposition 2.2 *The optimal value function $f^\star$ of $P_{f,M}$ is lower semi-continuous on* $\operatorname{dom} f^\star$.

2.3 Upper semi-continuity of f^* and inner semi-continuity of M

The following example is a slight modification of [17, Ex. 2] and shows that $f^\star$ is not necessarily continuous on all of $\operatorname{dom} f^\star$.

Example 2.3 *For $n = 1$ and $r = 2$ consider the problem $P_{f,M}$ described by $f(t,x) = -x$, $g_1(t,x) = (t_1 - x)^2 + t_2^2 - (1-x)^2$, $g_2(t,x) = -x$, and $g_3(t,x) = 1 - x$. Then, although g_1 is not convex, the set-valued mapping M is graph-convex as well as non-trivial, outer semi-continuous and bounded-valued. Hence, by Proposition 2.2, the optimal value function $f^\star$ of $P_{f,M}$ is lower semi-continuous on* $\operatorname{dom} f^* = \{t \in \mathbb{R}^2 \mid \|t\|_2 \leq 1\}$.

Take, however, $\bar{t} = (1,0)$ from the boundary of $\operatorname{dom} f^\star$ *and any sequence $t^k \to \bar{t}$ with $\|t^k\|_2 = 1$ for all $k \in \mathbb{N}$. Then we obtain $M(t^k) = \{0\}$ and*

$f^\star(t^k) = 0$ for all $k \in \mathbb{N}$ as well as $M(\bar{t}) = [0,1]$ and $f^\star(\bar{t}) = -1$, so that $f^\star$ is not continuous at $\bar{t}$.

Example 2.3 shows, in particular, that *upper* semi-continuity of $f^\star$ cannot be expected at points from the boundary of $\operatorname{dom} f^\star$ without further assumptions.

While the fact that $f^\star$ is an *optimal value function* was used to prove its lower semi-continuity in Proposition 2.2, there exist well-known sufficient conditions for upper semi-continuity of *convex* functions which we may alternatively apply to $f^\star$ in view of Proposition 2.1. For example, by [5, Th. 2] any convex function is upper semi-continuous at any point at which its domain is locally polyhedral, where the latter means that the domain locally coincides with a polyhedron. In particular, for $r = 1$ the set $\operatorname{dom} f^\star$ is convex in $\mathbb{R}^1$, that is, an interval and, hence, locally polyhedral everywhere. In [11, Prop. 3.1.2] it is also shown directly that any convex function of a single variable is upper semi-continuous on its whole domain. Together with Proposition 2.2 these conditions are easily seen to be sufficient for the continuity of $f^\star$.

In the following, we will complement and improve these results for our special situation in which $f^\star$ is, in fact, also an optimal value function. The key property which is needed to extend upper semi-continuity of an optimal value function $f^\star$ to all of $\operatorname{dom} f^\star$ is the *inner semi-continuity* of the set-valued mapping M on $\operatorname{dom} M$. Note that inner semi-continuity can hold at boundary points of $\operatorname{dom} M$ if it is considered relative to the latter set, that is, for each sequence $t^k \to \bar{t}$ with $t^k \in \operatorname{dom} M$ for all $k \in \mathbb{N}$, and each $\bar{x} \in M(\bar{t})$ there exists a sequence $x^k \to \bar{x}$ with $x^k \in M(t^k)$ for almost all $k \in \mathbb{N}$.

In fact, by [12, Th. 6] inner semi-continuity of M relative to $\operatorname{dom} M$ at some point $\bar{t}$ is *sufficient* for upper semi-continuity of $f^\star$ at $\bar{t}$ relative to $\operatorname{dom} f^\star$. Hence the subsequent discussion of sufficient conditions for inner semi-continuity of $f^\star$ extends known results which show the upper semi-continuity of $f^\star$ directly.

Example 2.4 *In the situation of Example 2.3, the set valued-mapping M is* not *inner semi-continuous relative to* $\operatorname{dom} M$ *at $\bar{t} = (0,1)$, as it is easily seen by choosing $\bar{x} = 1$ and the same type of sequence (t^k) as above.*

A standard sufficient condition for inner semi-continuity of M at $\bar{t}$ is the Slater condition for $M(\bar{t})$ ([12, Th. 12]). This is not helpful in our present analysis for two reasons. First, here we do not assume a functional description of M and, second, even if we did, the Slater condition would necessarily be violated at all boundary points of $\operatorname{dom} f^\star$, as we will discuss in detail in Section 2.4 below. In particular, this cannot help to establish the upper semi-continuity of $f^\star$ outside of the set $\operatorname{int}\operatorname{dom} f^\star$, where it is clear anyway, due to the convexity of $f^\star$. Instead, the following Proposition 2.7 will formulate alternative sufficient conditions for the inner semi-continuity of M.

For the preparation of one of its parts, we first show the announced lemma concerning local boundedness of M.

Lemma 2.5 *In problem $P_{f,M}$ with a graph-convex and outer semi-continuous mapping M, let $M(\bar t)$ be non-empty and bounded. Then M also is locally bounded at $\bar t$.*

Proof. Let $\bar t \in \operatorname{dom} M$ be given and assume that $\bigcup_{t\in V} M(t)$ is unbounded for any neighborhood V of $\bar t$. Then there exist a sequence $t^k \to \bar t$ as well as points $x^k \in M(t^k)$ with $\lim_{k\to\infty} \|x^k\| = +\infty$. For all sufficiently large $k \in \mathbb{N}$ we may define the element $d^k := x^k/\|x^k\|$ of the unit sphere. As the latter is compact, without loss of generality the sequence (d^k) converges, and its limit $\bar d$ satisfies $\|\bar d\| = 1$.

Next, the boundedness of $M(\bar t)$ means that there is some $R > 0$ such that all $x \in M(\bar t)$ satisfy $\|x\| \le R$ and, due to $\bar t \in \operatorname{dom} M$, we may choose some $\bar x \in M(\bar t)$. As the sequence of positive numbers $\lambda^k := 3R/\|x^k\|$, $k \in \mathbb{N}$, converges to zero, for all sufficiently large $k \in \mathbb{N}$ we find $\lambda^k \in [0, 1]$ and may consider the convex combination $(1-\lambda^k)(\bar t, \bar x) + \lambda^k(t^k, x^k)$ of the points $(\bar t, \bar x), (t^k, x^k) \in \operatorname{gph} M$. By the graph-convexity of M, this convex combination is also an element of $\operatorname{gph} M$ and, by the outer semi-continuity of M, also its limit

$$\begin{aligned}\lim_{k\to\infty}(1-\lambda^k)(\bar t,\bar x)+\lambda^k(t^k,x^k) &= \lim_{k\to\infty}\left(1-\frac{3R}{\|x^k\|}\right)(\bar t,\bar x)+\frac{3R}{\|x^k\|}(t^k,x^k)\\ &= (\bar t, \bar x + 3R\bar d)\end{aligned}$$

is. This leads to $\bar x + 3R\bar d \in M(\bar t)$ and, thus, to the contradiction

$$R \;\ge\; \|\bar x + 3R\bar d\| \;\ge\; \big|\, 3R\|\bar d\| - \|\bar x\|\,\big| \;=\; 3R - \|\bar x\| \;\ge\; 2R.$$

•

Remark 2.6 *A quantitative version of Lemma 2.5 is given in [18, Ch. III.1, Lemma 1.1].*

Proposition 2.7 *Let* $\bar{t} \in \operatorname{dom} M$*. Then* M *is inner semi-continuous relative to* $\operatorname{dom} M$ *at* $\bar{t}$ *under any of the following conditions:*

a) $\operatorname{dom} M$ *is locally polyhedral at* $\bar{t}$*,*

b) we have $\bar{t} \in \operatorname{int} \operatorname{dom} M$*,*

c) the parameter dimension is $r = 1$*,*

d) $M(\bar{t})$ *is a singleton.*

Proof. To see the sufficiency of condition a) choose some $r > 0$ and the neighborhood $B_\infty(\bar{t}, r) = \{t \in \mathbb{R}^r | \, \|t - \bar{t}\|_\infty \le r\}$ of $\bar{t}$ such that $\operatorname{dom} M \cap B_\infty(\bar{t}, r)$ is polyhedral and, hence, itself a polytope $\mathcal{P}$. In fact, the local polyhedrality of $\operatorname{dom} M$ at $\bar{t}$ implies that r may be chosen so small that $\mathcal{P} = \operatorname{dom} M \cap B_\infty(\bar{t}, r)$ coincides with $(\bar{t} + \mathcal{T}(\bar{t}, \operatorname{dom} M)) \cap B_\infty(\bar{t}, r)$ where $\mathcal{T}(\bar{t}, \operatorname{dom} M)$ denotes the contingent cone to $\operatorname{dom} M$ at $\bar{t}$.

For any sequence $t^k \to \bar{t}$ with $t^k \in \operatorname{dom} M$, $k \in \mathbb{N}$, we have $t^k \in \mathcal{P}$ for almost all k. For any of the latter k, we put $\tau^k = \|t^k - \bar{t}\|_\infty$ and $\tilde{t}^k = \bar{t} + r s^k$ with $s^k = (t^k - \bar{t})/\tau^k$ if $t^k \ne \bar{t}$, and $s^k = 0$, else. With $\lambda^k = \tau^k / r \in [0, 1]$ we may then write $t^k = (1 - \lambda^k)\bar{t} + \lambda^k \tilde{t}^k$, that is, t^k is a convex combination of $\bar{t}$ and $\tilde{t}^k$. We clearly have $\tilde{t}^k \in B_\infty(\bar{t}, r)$ and, due to $t^k \in \bar{t} + \mathcal{T}(\bar{t}, \operatorname{dom} M)$ and the cone property of $\mathcal{T}(\bar{t}, \operatorname{dom} M)$, also $\tilde{t}^k \in \bar{t} + \mathcal{T}(\bar{t}, \operatorname{dom} M)$, that is, $\tilde{t}^k \in \mathcal{P}$. Hence, if $\{t_j^{\rm ex}, \, j \in J\}$ denotes the extreme point set of $\mathcal{P}$, there are weights $\mu_j^k \ge 0$, $j \in J$, with $\sum_{j \in J} \mu_j^k = 1$ and $\tilde{t}^k = \sum_{j \in J} \mu_j^k \, t_j^{\rm ex}$.

Next, choose any $\bar{x} \in M(\bar{t})$. In view of $\mathcal{P} \subseteq \operatorname{dom} M$ we may also choose points $x_j^{\rm ex} \in M(t_j^{\rm ex})$, $j \in J$, and put

$$x^k \;:=\; (1 - \lambda^k)\bar{x} + \lambda^k \sum_{j \in J} \mu_j^k \, x_j^{\rm ex}.$$

The point (t^k, x^k) then is a convex combination of the points $(\bar{t}, \bar{x})$, $(t_j^{\rm ex}, x_j^{\rm ex})$, $j \in J$, from $\operatorname{gph} M$, so that the graph-convexity of M yields $x^k \in M(t^k)$. Finally, the boundedness of the sequence (μ^k) and $\lim_{k\to\infty} \lambda^k = 0$ imply $\lim_{k\to\infty} x^k = \bar{x}$.

The sufficiency of conditions b) and c) immediately follows from the sufficiency of condition a) since, on the one hand, $\operatorname{dom} M$ is locally polyhedral at any $\bar{t} \in \operatorname{int} \operatorname{dom} M$ and, on the other hand, as a convex set in $\mathbb{R}^1$, $\operatorname{dom} M$ is an interval and, hence, locally polyhedral everywhere.

For the sufficiency proof of condition d), let $\bar{x}$ denote the single element of $M(\bar{t})$. For any sequence $t^k \to \bar{t}$ with $t^k \in \operatorname{dom} M$ for all $k \in \mathbb{N}$, choose an arbitrary point $x^k \in M(t^k)$. By Lemma 2.5 the sequence (x^k) is bounded and, hence, possesses a cluster point. Due to the outer semi-continuity of M, this cluster point must coincide with $\bar{x}$. Moreover, any other cluster point of (x^k) must also coincide with $\bar{x}$ so that, altogether, we find $\lim_{k\to\infty} x^k = \bar{x}$. •

Remark 2.8 *Example 2.4 rules out that the decision variable dimension* $n = 1$ *is a sufficient condition for inner semi-continuity of* M*.*

Remark 2.9 *The sufficiency of condition a) for inner semi-continuity of* M *in Proposition 2.7 is also mentioned in a remark following [12, Cor. 14.1], but without proof. Moreover, recall from our discussion above that, while condition a) is well-known to imply upper semi-continuity of* $f^\star$*, this does not immediately imply the inner semi-continuity of* M*.*

Remark 2.10 *The sufficiency of condition d) for inner semi-continuity of* M *in Proposition 2.7 is also a special case of [15, Th. 1] where, apart from the case of singletons, unbounded images of* M *are considered.*

Remark 2.11 *Condition d) in Proposition 2.7 is mild in the following sense. Assume even that* $\operatorname{gph} M$ *is a bounded set, that is, a so-called convex body in* $\mathbb{R}^{r+n}$*. Consider any point* $\bar{t}$ *from the boundary of* $\operatorname{dom} M$*. Then* $\{\bar{t}\} \times M(\bar{t})$ *is contained in the boundary of* $\operatorname{gph} M$*. Hence, if* $M(\bar{t})$ *is not a singleton, then the boundary of* $\operatorname{gph} M$ *contains a line segment. By [22, Th. 2.3.1], however, the set of unit vectors in* $\mathbb{R}^r \times \mathbb{R}^n$ *that are parallel to a line segment in the boundary of* $\operatorname{gph} M$ *has* $(r+n-1)-$*dimensional Hausdorff measure zero. This means that 'most' small rotations of* $\operatorname{gph} M$ *will result in a situation where* $M(t)$ *is a singleton for all boundary points* t *of* $\operatorname{dom} M$ *or, in other words, the existence of non-singleton sets* $M(t)$ *violating the Slater condition is unstable in this sense.*

Theorem 2.12 *Let* $\bar{t} \in \operatorname{dom} f^\star$*. Then* $f^\star$ *is continuous relative to* $\operatorname{dom} f^\star$ *at* $\bar{t}$ *under any of the conditions a), b), c), or d) from Proposition 2.7.*

Proof. Under any of the four conditions the inner semi-continuity of M relative to $\operatorname{dom} M$ at $\bar{t}$ follows from Proposition 2.7. As discussed above, [12, Th. 6] then yields upper semi-continuity of $f^\star$ relative to $\operatorname{dom} M$ at $\bar{t}$. Thus, together with the identity $\operatorname{dom} M = \operatorname{dom} f^\star$ and Proposition 2.2, the assertion is shown. •

Remark 2.13 *Condition a) in Theorem 2.12 holds, for example, if under a functional description of M in $P_{f,g}$ all functions g_i, $i \in I$, are affine, as orthogonal projections of polyhedral sets are locally polyhedral everywhere. This case is covered in [1, Th. 4.3.7]. However,* $\operatorname{dom} f^\star$ *may also be locally polyhedral everywhere for nonlinear constraints, as Example 1.3 shows. Also condition c) covers continuity of the optimal value function $f^\star$ of $P_{f,M}$ on* all *of* $\operatorname{dom} f^\star$*. On the other hand, condition b) re-establishes continuity of $f^\star$ on the interior of* $\operatorname{dom} f^\star$ *(which is already known by the convexity of $f^\star$), while condition d) covers a large (in the sense of Remark 2.11) set of boundary points of* $\operatorname{dom} f^\star$*, as will become more apparent in the subsequent section.*

Remark 2.14 *Note that the results developed so far do not rely on a functional description of M. This means, in particular, that they remain valid in the semi-infinite case, when M is described by possibly infinitely many inequality constraints (i.e, for $|I| \leq +\infty$ in a functional description).*

2.4 The domain of f^*

In the remainder of this article let us consider a problem $P_{f,g}$ with a feasible set mapping described in functional form with finitely many inequality constraints. On the one hand, this admits a better understanding of the above continuity results and, on the other hand, it also prepares the differentiability analysis in Section 3.

In fact, the functional description of $\operatorname{gph} M$ allows us to state and investigate a functional description of the set $\operatorname{dom} f^\star$ in terms of the non-smooth, albeit convex, function

$$G(t,x) \ := \ \max_{i \in I} \ g_i(t,x).$$

First note that we obviously have $M(t) = \{x \in \mathbb{R}^n |\, G(t,x) \leq 0\}$ for all $t \in \mathbb{R}^r$ as well as $\operatorname{gph} M = \{(t,x) \in \mathbb{R}^r \times \mathbb{R}^n |\, G(t,x) \leq 0\}$. As $\operatorname{dom} f^\star$ coincides with $\operatorname{dom} M$, the orthogonal projection of $\operatorname{gph} M$ to the '$t-$space', we arrive at

$$\operatorname{dom} f^\star \ = \ \{t \in \mathbb{R}^r |\, \exists\, x \in \mathbb{R}^n : \ G(t,x) \leq 0\}. \tag{2}$$

This is the motivation to study the family of unconstrained 'feasibility check' problems

$$F(t): \qquad \min_{x \in \mathbb{R}^n} \ G(t,x)$$

with $t \in \mathbb{R}^r$. *If* the problem $F(t)$ was solvable for any $t \in \mathbb{R}^r$ then, in view of (2), its optimal value function

$$G^\star(t) \ := \ \min_{x \in \mathbb{R}^n} \ G(t,x) \tag{3}$$

would allow the description

$$\operatorname{dom} f^\star \;=\; \{t\in\mathbb{R}^r|\; G^\star(t)\le 0\}.$$

Example 2.15 *In the situation of Example 1.3 the problem $F(t)$ is solvable for any $t\in\mathbb{R}$, and we find*

$$\begin{aligned} G^\star(t) &= \min_{x\in\mathbb{R}} \max\{t^2+x^2-1,\, -t-x\} \\ &= \begin{cases} \frac{1}{2}-\sqrt{\frac{5}{4}-t-t^2}-t\,, & t\in[\frac{-1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}] \\ t^2-1\,, & \textit{else.} \end{cases} \end{aligned}$$

The next lemma makes sure that this is, indeed, the case whenever M is bounded-valued. Note that, by the outer semi-continuity of M, each set $M(t)$ then actually is compact. In fact, we will strengthen Assumption 1.1 to bounded-valuedness of M whenever we use the function $G^\star$, that is, in the present section as well as in Section 3.4 below.

Lemma 2.16 *Let M be bounded-valued. Then for each $t\in\mathbb{R}^r$ the problem $F(t)$ is solvable.*

Proof. Let $\bar t\in\mathbb{R}^r$. The proof is complete if we can show that the objective function $G(\bar t,\cdot)$ of $F(\bar t)$ possesses a non-empty and compact lower level set

$$G^\alpha_\le(\bar t,\cdot) \;=\; \{x\in\mathbb{R}^n|\; G(\bar t,x)\le\alpha\}$$

with some $\alpha\in\mathbb{R}$. For $\bar t\in\operatorname{dom} f^\star$ this is clear with the choice $\alpha=0$, as $G^0_\le(\bar t,\cdot)$ coincides with the non-empty and compact set $M(\bar t)$.

Hence, in the following let $\bar t\notin\operatorname{dom} f^\star$, and let $(t^\star,x^\star)$ be a Slater point of $\operatorname{gph} M$. In view of $\operatorname{dom} f^\star=\operatorname{dom} M$ we have $M(\bar t)=\emptyset$ and, thus $\bar\alpha := G(\bar t,x^\star)>0$. Then the lower level set $G^{\bar\alpha}_\le(\bar t,\cdot)$ is non-empty as it contains $x^\star$, and it is closed as G is continuous as a convex function on the open set $\mathbb{R}^r\times\mathbb{R}^n$.

Assume that $G^{\bar\alpha}_\le(\bar t,\cdot)$ is unbounded. Then there exists a sequence of points $x^k\in G^{\bar\alpha}_\le(\bar t,\cdot)$, $k\in\mathbb{N}$, with $\lim_{k\to\infty}\|x^k\|=+\infty$. The convexity of G yields for any $\lambda\in(0,1)$ and any $k\in\mathbb{N}$

$$\begin{aligned} G(\,(1-\lambda)(t^\star,x^\star)+\lambda(\bar t,x^k)\,) &\le (1-\lambda)\,G(t^\star,x^\star)+\lambda\,G(\bar t,x^k) \\ &\le (1-\lambda)\,G(t^\star,x^\star)+\lambda\,\bar\alpha. \end{aligned}$$

Hence, for some sufficiently small $\bar{\lambda} \in (0,1)$ and all $k \in \mathbb{N}$ we arrive at

$$G(\,(1-\bar{\lambda})(t^\star, x^\star) + \bar{\lambda}(\bar{t}, x^k)\,) \;\leq\; 0$$

and, thus,

$$(1-\lambda)\,x^\star + \lambda\,x^k \;\in\; M(\,(1-\lambda)\,t^\star + \lambda\,\bar{t}\,).$$

Since the sequence of points $(1-\lambda)\,x^\star + \lambda\,x^k$, $k \in \mathbb{N}$, is unbounded, this contradicts the boundedness of the set $M(\,(1-\lambda)\,t^\star + \lambda\,\bar{t}\,)$. •

As explained above, Lemma 2.16 immediately yields part a) of the following result. In part c), $\mathrm{bd}\,\mathrm{dom}\,f^\star$ denotes the topological boundary of $\mathrm{dom}\,f^\star$.

Theorem 2.17 *Let M be bounded-valued. Then the optimal value function $f^\star$ of $P_{f,g}$ satisfies*

a) $\mathrm{dom}\,f^\star \;=\; \{t \in \mathbb{R}^r|\; G^\star(t) \leq 0\}$,

b) $\mathrm{dom}\,f^\star$ *is a closed set, and*

c) $\mathrm{bd}\,\mathrm{dom}\,f^\star \;=\; \{t \in \mathbb{R}^r|\; G^\star(t) = 0\}$.

Proof. First note that $\mathrm{dom}\,G^\star = \mathbb{R}^r$ holds. Along the lines of the proof of Proposition 2.1 one easily sees that the function $G^\star$ is convex and, thus, continuous on $\mathbb{R}^r$. This yields the assertion of part b) as well as the inclusion $\mathrm{bd}\,\mathrm{dom}\,f^\star \subseteq \{t \in \mathbb{R}^r|\; G^\star(t) = 0\}$ in part c).

For the proof of the reverse inclusion choose a Slater point $(t^\star, x^\star)$ of $\mathrm{gph}\,M$. Then we have $G^\star(t^\star) < 0$ so that, in particular, the functional description of $\mathrm{dom}\,f^\star$ from part a) satisfies the Slater condition. Furthermore, choose any $\bar{t} \in \mathbb{R}^r$ with $G^\star(\bar{t}) = 0$ and put $s = \bar{t} - t^\star$. We will show that $\mathrm{dom}\,f^\star$ can be left from $\bar{t}$ along the direction s, so that $\bar{t}$ lies in $\mathrm{bd}\,\mathrm{dom}\,f^\star$. In fact, for any $\tau > 0$ choose $\lambda = 1/(1+\tau)$ which obviously lies in the interval $(0,1)$. Then the convexity of $G^\star$ yields

$$\begin{aligned} 0 \;=\; G^\star(\bar{t}) &\;=\; G^\star(\,(1-\lambda)(\bar{t}-s) + \lambda(\bar{t}+\tau s)\,) \\ &\;\leq\; (1-\lambda)G^\star(t^\star) + \lambda G^\star(\bar{t}+\tau s) \;<\; \lambda G^\star(\bar{t}+\tau s), \end{aligned}$$

so that $\bar{t} + \tau s \notin \mathrm{dom}\,f^\star$ in view of part a). •

It is not hard to see that $G^\star(t) = 0$ holds if and only if $M(t)$ is non-empty and violates the Slater condition. With the definition

$$\mathrm{slater}\,M \;=\; \{t \in \mathbb{R}^r|\; M(t) \;\text{ satisfies the Slater condition}\}$$

the next corollary is, thus, basically a reformulation of Theorem 2.17.

Corollary 2.18 *Let M be bounded-valued. Then the optimal value function $f^\star$ of $P_{f,g}$ satisfies*

a) $\operatorname{int}\operatorname{dom} f^\star \;=\; \operatorname{slater} M$ *and*

b) $\operatorname{bd}\operatorname{dom} f^\star \;=\; \operatorname{dom} M \setminus \operatorname{slater} M$.

Example 2.19 *Exactly at the two boundary points of*

$$\operatorname{dom} f^\star \;=\; \left[-\frac{1}{\sqrt{2}}, 1\right]$$

the set $M(t)$ violates the Slater condition, where $M(-\frac{1}{\sqrt{2}}) = \{\frac{1}{\sqrt{2}}\}$ and $M(1) = \{0\}$.

Remark 2.20 *In view of Proposition 2.7, Theorem 2.12, and Corollary 2.18, the inner semi-continuity of (a bounded-valued) M and the continuity of $f^\star$ are not only clear on the set* $\operatorname{slater} M$, *but they also hold at each $\bar t$ with $M(\bar t)$ violating the Slater condition, as long as either the parameter dimension is $r = 1$, or* $\operatorname{dom} M$ *is locally polyhedral at $\bar t$, or $M(\bar t)$ is a singleton. Recall that the latter condition is weak in the sense of Remark 2.11.*

3 Differentiability properties of $f^\star$

Throughout this section we will consider problems of the form $P_{f,g}$ and assume that all functions f, g_i, $i \in I$, are continuously differentiable on $\mathbb{R}^r \times \mathbb{R}^n$. We will require Assumption 1.1, but the bounded-valuedness of M will only be needed in Section 3.4.

3.1 Standard results

As the optimal value function $f^\star$ of $P_{f,g}$ is convex by Proposition 2.1, its directional differentiability in the extended-valued sense at every $t \in \operatorname{dom} f^\star$ is clear without further assumptions from [19, Th. 23.1], as $f^\star$ takes finite values on its domain. Here, the (weak) directional derivative of $f^\star$ at t in direction $s \in \mathbb{R}^r$ is defined as

$$(f^\star)'(t,s) \;=\; \lim_{\tau \searrow 0} \frac{f^\star(t+\tau s) - f^\star(t)}{\tau},$$

and for given $t \in \operatorname{dom} f^\star$ the function $(f^\star)'(t,\cdot)$ is a convex function with effective domain

$$\mathcal{C}(t, \operatorname{dom} f^\star) \;=\; \{s \in \mathbb{R}^r |\; \exists\, \tau > 0 : t + \tau s \in \operatorname{dom} f^\star\},$$

that is, $(f^\star)'(t,s) < +\infty$ holds if and only if s lies in the so-called radial cone $\mathcal{C}(t, \operatorname{dom} f^\star)$ to $\operatorname{dom} f^\star$ at t ([19]). Note, however, that $(f^\star)'(t,s) = -\infty$ may occur for $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$, as Example 1.3 shows for $t = 1$ and $s = -1$. Also note that $(f^\star)'(t,0) = 0$ trivially holds for any $t \in \operatorname{dom} f^\star$.

Our focus is on formulas for $(f^\star)'(t,s)$ which use the information that $f^\star$ is not only a convex function but also an optimal value function. Such formulas are standard in the case $t \in \operatorname{int}\operatorname{dom} f^\star$, and we briefly recall them in the sequel. For their statement, let

$$M^\star(t) \;=\; \{x \in M(t) |\; f(t,x) \le f^\star(t)\}$$

denote the set of optimal points of $P_{f,g}(t)$, let

$$L(t,x,\lambda) \;=\; f(t,x) + \langle \lambda, g(t,x)\rangle$$

denote the Lagrange function of $P_{f,g}(t)$, and let

$$KKT(t) \;=\; \{\lambda \in \mathbb{R}^p |\; \nabla_x L(t,x,\lambda) = 0,\; \lambda \ge 0,\; \langle \lambda, g(t,x)\rangle = 0\}$$

be the set of Karush-Kuhn-Tucker multipliers for $x \in M^\star(t)$. Note that $KKT(t)$ does not depend on x as $P_{f,g}(t)$ is a convex problem ([7, 13]), and that for each $t \in \mathbb{R}^r$ the set $KKT(t)$ is a polyhedron, but not necessarily a polytope, that is, non-empty and bounded.

Lemma 3.1 *We have* $\operatorname{slater} M = \{t \in \mathbb{R}^r |\; KKT(t)$ *is a polytope*$\}$.

Proof. By a result from [6], $KKT(t)$ is a polytope if and only if MFCQ holds at some $x \in M^\star(t)$. The latter is equivalent to the Slater condition holding in $M(t)$, which shows the assertion. •

Note that the theory of perturbation functions also allows to interpret the set $KKT(t)$ as the subdifferential of a certain convex function (e.g., [19, Th. 29.1]) which paves the way for an alternative proof of Lemma 3.1 without using the differentiability structure via MFCQ. In order to avoid technicalities, however, we continue without discussing its details.

Theorem 3.2 (e.g. [7, 13, 20]) *$f^\star$ is directionally differentiable at each $t \in \operatorname{slater} M$ with*

$$(f^\star)'(t,s) \;=\; \min_{x\in M^\star(t)} \max_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle \tag{4}$$

for all $s \in \mathbb{R}^r$.

In fact, under complete convexity it is shown in [13] that the right hand side of (4) does not depend on the actual choice of $x \in M^\star(t)$. The combination of Theorem 3.2 with Corollary 2.18a) hence yields the following result.

Corollary 3.3 *$f^\star$ is directionally differentiable at each $t \in \operatorname{int}\operatorname{dom} f^\star$ with*

$$(f^\star)'(t,s) \;=\; \max_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle \tag{5}$$

for all $s \in \mathbb{R}^r$, where $x \in M^\star(t)$ may be chosen arbitrarily.

Example 3.4 *In Example 1.3, the only nondifferentiability point of $f^\star$ in $\operatorname{int}\operatorname{dom} f^\star$ is $t = 1/\sqrt{2}$. To determine the directional derivative $(f^\star)'(1/\sqrt{2}, s)$ for $s \in \mathbb{R}$, note that we have $M^\star(1/\sqrt{2}) = \{-1/\sqrt{2}\}$, so that both inequality constraints are active at $x \in M^\star(1/\sqrt{2})$. With $L(t,x,\lambda) = x + \lambda_1(t^2 + x^2 - 1) + \lambda_2(-t - x)$ this leads to*

$$KKT\left(\frac{1}{\sqrt{2}}\right) \;=\; \left\{ \begin{pmatrix} \theta \\ 1-\sqrt{2}\theta \end{pmatrix} \,\Big|\, \theta \in \left[0, \frac{1}{\sqrt{2}}\right] \right\}$$

and, thus, for $t = 1/\sqrt{2}$ and any $s \in \mathbb{R}$

$$\max_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle \;=\; \max_{\theta\in[0,1/\sqrt{2}]} (2\sqrt{2}\theta - 1)s \;=\; |s|.$$

Due to Corollary 3.3, this results in $(f^\star)'(1/\sqrt{2}, s) = |s|$ for all $s \in \mathbb{R}$.

3.2 The optimal value condition

In the sequel we will develop the proof of a more general result than Corollary 3.3, which streamlines the ideas from [13] under our slightly stronger assumptions. In fact, we will prove an analogue of formula (5) also for $t \in \operatorname{dom} f^\star \cap \operatorname{bd}\operatorname{dom} f^\star$ and appropriate choices of s. We note that for nonconvex problems explicit estimates for upper and lower directional derivatives of $f^\star$ at boundary points of $\operatorname{dom} f^\star$ are given in [14] under the constant rank constraint qualification and under the notions of upper and lower stability, respectively, which are, however, not easily verified. A related result is presented in [9], but under a uniform Mangasarian-Fromovitz constraint qualification which is, again, not easily checked. Directional differentiability results at boundary points of $\operatorname{dom} f^\star$ are also given in [8], but without explicit formulas. For a discussion of our improvements of the results from [13] see Remark 3.24 below.

As a first step, for given $t \in \operatorname{dom} f^\star$ and $x \in M^\star(t)$ consider the radial cone

$$\mathcal{C}((t,x), \operatorname{gph} M) \;=\; \{(s,d) \in \mathbb{R}^r \times \mathbb{R}^n |\; \exists\, \tau > 0 : (t,x) + \tau(s,d) \in \operatorname{gph} M\}$$

to $\operatorname{gph} M$ at (t,x). Note that, by convexity of $\operatorname{gph} M$, for $(t,x) \in \operatorname{gph} M$ the pair (s,d) is in $\mathcal{C}((t,x), \operatorname{gph} M)$ if and only if $(t,x) + \tau(s,d) \in \operatorname{gph} M$ holds for all $\tau \in (0, \bar\tau)$ with some $\bar\tau > 0$. In the following, pr_t shall denote the orthogonal projection into the '$t-$space' $\mathbb{R}^r$.

Lemma 3.5 *For any $t \in \operatorname{dom} f^\star$ we have*

$$\mathcal{C}(t, \operatorname{dom} f^\star) \;=\; \operatorname{pr}_t \mathcal{C}((t,x), \operatorname{gph} M),$$

where $x \in M^\star(t)$ may be chosen arbitrarily.

Proof. For any $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$ there exists some $\tau > 0$ with $t + \tau s \in \operatorname{dom} f^\star = \operatorname{dom} M$. Hence there exists some $x(\tau) \in M(t + s\tau)$, and we may put $d(\tau) := (x(\tau) - x)/\tau$ for which $(s, d(\tau)) \in \mathcal{C}((t,x), \operatorname{gph} M)$ is easily seen. This shows $\mathcal{C}(t, \operatorname{dom} f^\star) \subseteq \operatorname{pr}_t \mathcal{C}((t,x), \operatorname{gph} M)$. The reverse inclusion is trivial. •

By Lemma 3.5 a direction s lies in $\mathcal{C}(t, \operatorname{dom} f^\star)$ if and only if the fiber

$$C_{(t,x)}(s) \;=\; \{d \in \mathbb{R}^n |\; (s,d) \in \mathcal{C}((t,x), \operatorname{gph} M)\}$$

is non-empty. In other words, we have $\mathcal{C}(t, \operatorname{dom} f^\star) = \operatorname{dom} C_{(t,x)}$, which yields $\operatorname{dom} (f^\star)'(t, \cdot) = \operatorname{dom} C_{(t,x)}$ for arbitrary $x \in M^\star(t)$.

Next, for $t \in \operatorname{dom} f^\star$, $x \in M^\star(t)$ and $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$ consider the optimization problem

$$LP^{\mathcal{C}}_{(t,x)}(s): \quad \min_{d \in \mathbb{R}^n} \langle \nabla f(t,x), (s,d) \rangle \quad \text{s.t.} \quad d \in C_{(t,x)}(s)$$

with optimal value function $v^{\mathcal{C}}_{(t,x)}(s) = \inf_{d \in C_{(t,x)}(s)} \langle \nabla f(t,x), (s,d) \rangle$. Note that, by the above discussion, $LP^{\mathcal{C}}_{(t,x)}(s)$ is consistent for $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$, so that $v^{\mathcal{C}}_{(t,x)}(s) < +\infty$ is guaranteed.

Proposition 3.6 *$f^\star$ is directionally differentiable at each $t \in \operatorname{dom} f^\star$ in each direction $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$ with*

$$(f^\star)'(t,s) \;=\; v^{\mathcal{C}}_{(t,x)}(s),$$

where $x \in M^\star(t)$ may be chosen arbitrarily.

Proof. Choose any $t \in \operatorname{dom} f^\star$ and $x \in M^\star(t)$. As $f^\star$ is convex, its directional differentiability at t is clear, and from $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$ we also know that $(f^\star)'(t,s) < \infty$ holds.

Next, in view of $C_{(t,x)}(s) \neq \emptyset$ we may choose some $d \in C_{(t,x)}(s)$, that is, we have $(s,d) \in \mathcal{C}((t,x), \operatorname{gph} M)$ and, hence, $x + \tau d \in M(t + \tau s)$ for all $\tau \in (0, \bar\tau)$ with some $\bar\tau > 0$. For these τ the optimal value function $f^\star$ satisfies

$$f^\star(t+\tau s) \;\leq\; f(t+\tau s, x+\tau d),$$

and due to $f^\star(t) = f(t,x)$ also

$$\frac{f^\star(t+\tau s) - f^\star(t)}{\tau} \;\leq\; \frac{f(t+\tau s, x+\tau d) - f(t,x)}{\tau}.$$

Taking the limit $\tau \searrow 0$ leads to

$$(f^\star)'(t,s) \;\leq\; \langle \nabla f(t,x), (s,d)\rangle$$

and, as $d \in C_{(t,x)}(s)$ was arbitrary, also $(f^\star)'(t,s) \leq v^{\mathcal{C}}_{(t,x)}(s)$. Note that this also covers the cases $(f^\star)'(t,s) = -\infty$ and $v^{\mathcal{C}}_{(t,x)}(s) = -\infty$.

To see the reverse inequality, choose any $\varepsilon > 0$. Then, due to $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$, for all sufficiently small $\tau > 0$ the set $M(t+\tau s)$ is non-empty, and there is some point $x(\varepsilon,\tau) \in M(t+\tau s)$ with

$$f^\star(t+\tau s) + \tau\varepsilon \;\geq\; f(t+\tau s, x(\varepsilon,\tau)).$$

With the same construction as in the proof of Lemma 3.5 we obtain a direction $d(\varepsilon,\tau) = (x(\varepsilon,\tau) - x)/\tau \in C_{(t,x)}(s)$. Together with the convexity of f this yields

$$\begin{aligned}
\frac{f^\star(t+\tau s) - f^\star(t)}{\tau} + \varepsilon \;&\geq\; \frac{f(\,(t,x) + \tau(s, d(\varepsilon,\tau))\,) - f(t,x)}{\tau} \\
&\geq\; \langle \nabla f(t,x), (s, d(\varepsilon,\tau))\rangle \;\geq\; v^{\mathcal{C}}_{(t,x)}(s).
\end{aligned}$$

After taking the limits $\tau \searrow 0$ and $\varepsilon \searrow 0$ we arrive at $(f^\star)'(t,s) \geq v^{\mathcal{C}}_{(t,x)}(s)$. Again, it is not hard to see that these arguments cover the cases $(f^\star)'(t,s) = -\infty$ and $v^{\mathcal{C}}_{(t,x)}(s) = -\infty$. This concludes the proof. •

To derive the statement of Corollary 3.3 from Proposition 3.6, we need an explicit expression for the optimal value function of $LP^{\mathcal{C}}_{(t,x)}(s)$. This expression is readily available if we consider a slightly different optimization problem, where the fiber $C_{(t,x)}(s)$ of the radial cone $\mathcal{C}((t,x), \operatorname{gph} M)$ is replaced by the fiber

$$T_{(t,x)}(s) \;=\; \{d \in \mathbb{R}^n |\; (s,d) \in \mathcal{T}((t,x), \operatorname{gph} M)\}$$

of the contingent cone $\mathcal{T}((t,x), \operatorname{gph} M)$. This results in the optimal value function $v^{\mathcal{T}}_{(t,x)}$ of the problem

$$LP^{\mathcal{T}}_{(t,x)}(s): \quad \min_{d\in\mathbb{R}^n} \langle \nabla f(t,x), (s,d)\rangle \quad \text{s.t.} \quad d \in T_{(t,x)}(s).$$

Lemma 3.7

a) For any $t \in \operatorname{dom} f^\star$ with $KKT(t) \neq \emptyset$, $x \in M^\star(t)$ and $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$ we have

$$v^{\mathcal{T}}_{(t,x)}(s) \;=\; \max_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle.$$

b) For any $t \in \operatorname{dom} f^\star$ with $KKT(t) = \emptyset$, $x \in M^\star(t)$ and $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$ we have

$$v^{\mathcal{T}}_{(t,x)}(s) \;=\; -\infty.$$

Proof. As $\operatorname{gph} M$ satisfies the Slater condition, the Abadie constraint qualification is satisfied at (t,x) in $\operatorname{gph} M$, that is, the contingent cone coincides with the outer linearization cone

$$\mathcal{L}((t,x), \operatorname{gph} M) \;=\; \{(s,d) \in \mathbb{R}^r \times \mathbb{R}^n |\; \langle \nabla g_i(t,x), (s,d)\rangle \le 0,\; i \in I_0(t,x)\},$$

where $I_0(t,x) = \{i \in I |\, g_i(t,x) = 0\}$ denotes the active index set at (t,x). The problem $LP^{\mathcal{T}}_{(t,x)}(s) = LP^{\mathcal{L}}_{(t,x)}(s)$ is, thus, a standard linear programming problem with dual problem

$$D_{(t,x)}(s): \quad \max_{\lambda\in\mathbb{R}^p} \langle \nabla_t L(t,x,\lambda), s\rangle \quad \text{s.t.} \quad \lambda \in KKT(t).$$

Moreover, the primal problem $LP^{\mathcal{L}}_{(t,x)}(s)$ is consistent since, due to $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$, there exists some $d \in \mathbb{R}^n$ with $(s,d) \in \mathcal{C}((t,x), \operatorname{gph} M) \subseteq \mathcal{T}((t,x), \operatorname{gph} M) = \mathcal{L}((t,x), \operatorname{gph} M)$. The strong duality theorem of linear programming now immediately yields the assertions of parts a) and b). •

Note that, by the usual convention $\sup_\emptyset = -\infty$, the assertion of Lemma 3.7 may as well be written as $v^{\mathcal{T}}_{(t,x)}(s) = \sup_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle$ for any $t \in \operatorname{dom} f^\star$, $x \in M^\star(t)$ and $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$.

Moreover, recall from Lemma 3.1 that $KKT(t) \neq \emptyset$ certainly holds for all $t \in \operatorname{int} \operatorname{dom} f^\star = \operatorname{slater} M$. On the other hand, for $t \in \operatorname{dom} f^\star \cap \operatorname{bd} \operatorname{dom} f^\star$, the set $KKT(t)$ either is empty or unbounded. We emphasize that, for $t \in \operatorname{dom} f^\star \cap \operatorname{bd} \operatorname{dom} f^\star$ with *unbounded* set $KKT(t)$ and $s \in \mathcal{C}(t, \operatorname{dom} f^\star)$, Lemma 3.7 states that $v^{\mathcal{T}}_{(t,x)}(s)$ is *finite*.

We sum up our discussion so far in the following result.

Proposition 3.8 *$f^\star$ is directionally differentiable at each $t \in \operatorname{dom} f^\star$ in each direction $s \in \mathcal{C}(t, \operatorname{dom} f^\star) \setminus \{0\}$ with*

$$(f^\star)'(t, s) \;=\; \sup_{\lambda \in KKT(t)} \langle \nabla_t L(t, x, \lambda), s\rangle \;<\; +\infty \tag{6}$$

whenever $x \in M^\star(t)$ may be chosen such that the optimal value condition $v^{\mathcal{C}}_{(t,x)}(s) = v^{\mathcal{T}}_{(t,x)}(s)$ holds.

Note that, for $s \notin \mathcal{C}(t, \operatorname{dom} f^\star)$, the formal validity of (6) cannot be expected as then $(f^\star)'(t, s) = +\infty$ holds while the right-hand side of (6) may be strictly smaller than $+\infty$, e.g. due to $KKT(t) = \emptyset$. For a similar reason, the trivial direction $s = 0$ is excluded in the assertion of Proposition 3.8.

3.3 The fiber condition

It remains to be discussed under which conditions in Proposition 3.8 $x \in M^\star(t)$ may be chosen such that the optimal value condition $v^{\mathcal{C}}_{(t,x)}(s) = v^{\mathcal{T}}_{(t,x)}(s)$ is satisfied. Note that the underlying optimization problems $LP^{\mathcal{C}}_{(t,x)}(s)$ and $LP^{\mathcal{T}}_{(t,x)}(s)$ only differ in their feasible sets $C_{(t,x)}(s)$ and $T_{(t,x)}(s)$, where the latter set is closed as a fiber of the closed set $\mathcal{T}((t, x), \operatorname{gph} M)$. Sufficient for the optimal value condition hence is the identity $\operatorname{cl} C_{(t,x)}(s) = T_{(t,x)}(s)$, where cl denotes the topological closure. In the following, we refer to the latter identity as the *fiber condition*. Recall that we have $\operatorname{dom} (f^\star)'(t, \cdot) = \operatorname{dom} C_{(t,x)}$, so that no void sets are involved in the fiber condition for the directions s of interest. Also note that, for unbounded problems $LP^{\mathcal{T}}_{(t,x)}(s)$, the fiber condition yields $v^{\mathcal{C}}_{(t,x)}(s) = v^{\mathcal{T}}_{(t,x)}(s) = -\infty$. This proves the following reformulation of Proposition 3.8.

Proposition 3.9 *$f^\star$ is directionally differentiable at each $t \in \operatorname{dom} f^\star$ in each direction $s \in \mathcal{C}(t, \operatorname{dom} f^\star) \setminus \{0\}$ with $(f^\star)'(t, s)$ given by (6) whenever $x \in M^\star(t)$ may be chosen such that the fiber condition $\operatorname{cl} C_{(t,x)}(s) = T_{(t,x)}(s)$ holds.*

It is not hard to see that the inclusion $\mathrm{cl}\, C_{(t,x)}(s) \subseteq T_{(t,x)}(s)$ always holds, while simple examples show that the reverse inclusion might fail. In the following we shall, hence, identify situations in which $T_{(t,x)}(s) \subseteq \mathrm{cl}\, C_{(t,x)}(s)$ holds.

First note that at least $\mathrm{cl}\, \mathcal{C}((t,x), \mathrm{gph}\, M)$ is known to coincide with $\mathcal{T}((t,x), \mathrm{gph}\, M)$ ([2]), so that the closedness of $\mathcal{C}((t,x), \mathrm{gph}\, M)$ implies that the fibers $C_{(t,x)}(s)$ and $T_{(t,x)}(s)$ coincide and, thus, the optimal value condition holds. Moreover, the closedness of $\mathcal{C}((t,x), \mathrm{gph}\, M)$ is known to follow from the local polyhedrality of $\mathrm{gph}\, M$ at (t,x) (this can readily be seen using the arguments from the proof of Proposition 2.7c)). The cone $\mathcal{C}((t,x), \mathrm{gph}\, M)$ then is polyhedral, so that its orthogonal projection $\mathcal{C}(t, \mathrm{dom}\, f^\star)$ also is polyhedral and, in particular, closed. This proves the following result.

Proposition 3.10 *$f^\star$ is directionally differentiable at each $t \in \mathrm{dom}\, f^\star$ in each direction $s \in \mathrm{cl}\, \mathcal{C}(t, \mathrm{dom}\, f^\star) \setminus \{0\}$ with $(f^\star)'(t,s)$ given by* (6) *whenever $x \in M^\star(t)$ may be chosen such that* $\mathrm{gph}\, M$ *is locally polyhedral at* (t,x).

The next proposition will provide a different sufficient condition for the fiber condition. We prepare it with the following lemma which is of independent interest. Here,

$$\mathcal{L}^<((t,x), \mathrm{gph}\, M) \;=\; \{(s,d) \in \mathbb{R}^r \times \mathbb{R}^n |\ \langle \nabla g_i(t,x), (s,d)\rangle < 0,\ i \in I_0(t,x)\}$$

denotes the inner linearization cone to $\mathrm{gph}\, M$ at (t,x).

Lemma 3.11 *At each $t \in \mathrm{dom}\, f^\star$ we have*

$$\mathrm{int}\, \mathcal{C}(t, \mathrm{dom}\, f^\star) \;=\; \mathrm{pr}_t \mathcal{L}^<((t,x), \mathrm{gph}\, M),$$

where $x \in M^\star(t)$ may be chosen arbitrarily.

Proof. Choose some $t \in \mathrm{dom}\, f^\star$ and any $x \in M^\star(t)$. By Taylor expansion one easily verifies the relation $\mathcal{L}^<((t,x), \mathrm{gph}\, M) \subseteq \mathcal{C}((t,x), \mathrm{gph}\, M)$ which implies $\mathrm{pr}_t \mathcal{L}^<((t,x), \mathrm{gph}\, M) \subseteq \mathcal{C}(t, \mathrm{dom}\, f^\star)$ in view of Lemma 3.5. Since $\mathrm{pr}_t \mathcal{L}^<((t,x), \mathrm{gph}\, M)$ is open as the orthogonal projection of an open set, we arrive at the inclusion $\mathrm{pr}_t \mathcal{L}^<((t,x), \mathrm{gph}\, M) \subseteq \mathrm{int}\, \mathcal{C}((t,x), \mathrm{gph}\, M)$.

We split the proof of the reverse inclusion into two steps by showing the chain of inclusions $\mathrm{int}\, \mathcal{C}(t, \mathrm{dom}\, f^\star) \subseteq \mathcal{C}(t, \mathrm{int}\, \mathrm{dom}\, f^\star) \subseteq \mathrm{pr}_t \mathcal{L}^<((t,x), \mathrm{gph}\, M)$. In fact, let $s \in \mathrm{int}\, \mathcal{C}(t, \mathrm{dom}\, f^\star)$. Due to $s \in \mathcal{C}(t, \mathrm{dom}\, f^\star)$, we have $t +$

$\tau_0 s \in \operatorname{dom} f^\star$ for some $\tau_0 > 0$. Assume that τ_0 cannot be chosen such that even $t + \tau_0 s \in \operatorname{int}\operatorname{dom} f^\star$ holds. Then there exists some $\bar\tau > 0$ with $t + \tau s \in \operatorname{dom} f^\star \cap \operatorname{bd}\operatorname{dom} f^\star$ for all $\tau \in (0, 2\bar\tau]$, in particular for $\tau = \bar\tau$. Thus, by [19, Cor. 11.6.1] there exists some nontrivial normal direction $\eta \in \mathbb{R}^r$ to $\operatorname{dom} f^\star$ at $t + \bar\tau s$, so that we obtain $t + \bar\tau s + \sigma\eta \notin \operatorname{dom} f^\star$ for all $\sigma > 0$. With $s(\sigma) := s + (\sigma/\bar\tau)\eta$ we may then write $t + \bar\tau s + \sigma\eta = t + \bar\tau s(\sigma)$ for any $\sigma > 0$. Next, in view of $s \in \operatorname{int}\mathcal{C}(t, \operatorname{dom} f^\star)$ there exists some $\bar\sigma > 0$ with $s(\bar\sigma) \in \mathcal{C}(t, \operatorname{dom} f^\star)$, so that there exists some $\tau(\bar\sigma) \in (0, \bar\tau)$ with $t + \tau(\bar\sigma)s(\bar\sigma) \in \operatorname{dom} f^\star$. For $\lambda := \bar\tau/(2\bar\tau - \tau(\bar\sigma))$ this yields $\lambda \in (0, 1)$ so that, as the convex combination of the points $t + 2\bar\tau s, t + \tau(\bar\sigma)s(\bar\sigma) \in \operatorname{dom} f^\star$, the point

$$(1-\lambda)(t + 2\bar\tau s) + \lambda(t + \tau(\bar\sigma)s(\bar\sigma)) \;=\; t + \bar\tau s + \frac{\tau(\bar\sigma)\bar\sigma}{2\bar\tau - \tau(\bar\sigma)}\eta$$

lies in $\operatorname{dom} f^\star$. At the same time, with the choice $\sigma := (\tau(\bar\sigma)\bar\sigma)/(2\bar\tau - \tau(\bar\sigma)) > 0$ it does not lie in $\operatorname{dom} f^\star$, a contradiction. This shows $s \in \mathcal{C}(t, \operatorname{int}\operatorname{dom} f^\star)$.

In the second step we will show that s lies in $\operatorname{pr}_t \mathcal{L}^<((t, x), \operatorname{gph} M)$. In fact, as seen in the first step there exists some $\tau_0 > 0$ with $t + \tau_0 s \in \operatorname{int}\operatorname{dom} f^\star = \operatorname{slater} M$, so that we may choose some $x^0 \in \mathbb{R}^n$ with $g_i(t + \tau_0 s, x^0) < 0$ for all $i \in I$. Next, with any $x \in M^\star(t)$ we put $d^0 = (x^0 - x)/\tau_0$ which yields $g_i((t, x) + \tau_0(s, d^0)) < 0$ for all $i \in I$. For any $i \in I_0(t, x)$ the convexity of g_i now implies $\langle \nabla g_i(t, x), (s, d^0)\rangle < 0$, that is, $(s, d^0) \in \mathcal{L}^<((t, x), \operatorname{gph} M)$ and, thus, $s \in \operatorname{pr}_t \mathcal{L}^<((t, x), \operatorname{gph} M)$. •

Lemma 3.12 *For all $t \in \operatorname{dom} f^\star$ and $s \in \operatorname{int}\mathcal{C}(t, \operatorname{dom} f^\star)$ we have*

$$T_{(t,x)}(s) \;\subseteq\; \operatorname{cl} C_{(t,x)}(s).$$

Proof. In view of Lemma 3.11 we may first choose some d^0 with $(s, d^0) \in \mathcal{L}^<((t, x), \operatorname{gph} M)$. Furthermore, choose any $d^1 \in T_{(t,x)}(s)$. Since the Slater condition holds in $\operatorname{gph} M$, the latter fiber coincides with the fiber $L_{(t,x)}(s)$ of the outer linearization cone $\mathcal{L}((t, x), \operatorname{gph} M)$ to $\operatorname{gph} M$ at (t, x).

Hence, for any $\theta \in (0, 1)$ the vector $d^\theta = (1-\theta)d^0 + \theta d^1$ satisfies

$$\langle \nabla g_i(t, x), (s, d^\theta)\rangle \;=\; (1-\theta)\langle \nabla g_i(t, x), (s, d^0)\rangle + \theta\langle \nabla g_i(t, x), (s, d^1)\rangle \;<\; 0$$

for all $i \in I_0(t, x)$, that is, also (s, d^θ) is an element of $\mathcal{L}^<((t, x), \operatorname{gph} M)$. By Taylor expansion one can now see that $g_i((t, x) + \tau(s, d^\theta)) < 0$ holds for all $i \in I_0(t, x)$ and sufficiently small $\tau > 0$, so that $(s, d^\theta) \in \mathcal{C}((t, x), \operatorname{gph} M)$ and, thus, $d^\theta \in C_{(t,x)}(s)$ hold. Taking the limit $\theta \to 1$ shows $d^1 \in \operatorname{cl} C_{(t,x)}(s)$. •

We, thus, arrive at the following result.

Proposition 3.13 *$f^\star$ is directionally differentiable at each $t \in \operatorname{dom} f^\star$ in each direction $s \in \operatorname{int} \mathcal{C}(t, \operatorname{dom} f^\star) \setminus \{0\}$ with $(f^\star)'(t,s)$ given by* (6)*, where $x \in M^\star(t)$ may be chosen arbitrarily.*

Remark 3.14 *Corollary 3.3 is a direct consequence of Proposition 3.13 as for $t \in \operatorname{int} \operatorname{dom} f^\star$ we have $\mathcal{C}(t, \operatorname{dom} f^\star) = \mathbb{R}^r$.*

Remark 3.15 *For $t \in \operatorname{dom} f^\star \cap \operatorname{bd} \operatorname{dom} f^\star$ the condition $s \in \operatorname{pr}_t \mathcal{L}^<((t,x), \operatorname{gph} M)$ is also known as Gollan's condition ([8]) or, in more general settings, as directional regularity ([3]).*

3.4 Functional descriptions of tangent cones in the parameter space

In this section we derive functional descriptions of the sets $\operatorname{cl} \mathcal{C}(t, \operatorname{dom} f^\star)$ and $\operatorname{int} \mathcal{C}(t, \operatorname{dom} f^\star)$ from Propositions 3.10 and 3.13, respectively. To this end, recall from Section 2.4 that for bounded-valued feasible set mappings M we know the functional description of the domain of $f^\star$,

$$\operatorname{dom} f^\star \;=\; \{t \in \mathbb{R}^r \mid\, G^\star(t) \le 0\},$$

with the function $G^\star$ from (3). Thus, we will use the bounded-valuedness of M as a blanket assumption throughout Section 3.4.

Recall that $G^\star$ is convex and, from Lemma 2.16, that $\operatorname{dom} G^\star = \mathbb{R}^r$ holds. Hence, $G^\star$ is directionally differentiable on $\mathbb{R}^r$ and, in particular, at each $t \in \operatorname{bd} \operatorname{dom} f^\star$. As in [23], we may thus define the inner and outer linearization cones to $\operatorname{dom} f^\star$ at $t \in \operatorname{bd} \operatorname{dom} f^\star$, that is,

$$\mathcal{L}^<(t, \operatorname{dom} f^\star) \;=\; \{s \in \mathbb{R}^r \mid\, (G^\star)'(t,s) < 0\}$$

and

$$\mathcal{L}(t, \operatorname{dom} f^\star) \;=\; \{s \in \mathbb{R}^r \mid\, (G^\star)'(t,s) \le 0\},$$

respectively. For $t \in \operatorname{int} \operatorname{dom} f^\star$ we put $\mathcal{L}^<(t, \operatorname{dom} f^\star) = \mathcal{L}(t, \operatorname{dom} f^\star) = \mathbb{R}^r$.

To derive a formula for the directional derivatives of $G^\star$, observe that by the epigraph reformulation $G^\star(t)$ coincides with $\alpha^\star(t)$ for all $t \in \mathbb{R}^r$, where $\alpha^\star(t)$ is the optimal value of the (solvable) problem

$$F_{\text{epi}}(t): \qquad \min_{(x,\alpha)} \alpha \quad \text{s.t.} \quad g_i(t,x) \le \alpha, \quad i \in I.$$

With its Lagrangian

$$\ell(t,x,\alpha,\mu) \;:=\; \alpha(1-\langle\mu,e\rangle)+\langle\mu,g(t,x)\rangle,$$

where e denotes the all ones vector in $\mathbb{R}^p$, it is not hard to see that its set of Karush-Kuhn-Tucker multipliers at an optimal point (x,α) is

$$kkt(t) \;:=\; \{\mu\in\Sigma|\ \nabla_x g(t,x)\,\mu=0,\ \langle\mu,g(t,x)\rangle=0\}$$

where

$$\Sigma \;=\; \{\mu\in\mathbb{R}^p|\ \mu\geq 0,\ \langle\mu,e\rangle=1\}$$

denotes the standard simplex in $\mathbb{R}^p$. Here we used that for $t\in\operatorname{bd}\operatorname{dom} f^\star$ any optimal point (x,α) of $F_{\rm epi}(t)$ satisfies $\alpha=G^\star(t)=0$ by Theorem 2.17c). Furthermore, it is $\nabla_t\ell(t,x,\alpha,\mu)=\nabla_t g(t,x)\,\mu$. Thus, by Corollary 3.3, at any $t\in\operatorname{int}\operatorname{dom}\alpha^\star=\operatorname{int}\operatorname{dom}G^\star=\mathbb{R}^r$ the function $\alpha^\star$ is directionally differentiable with

$$(\alpha^\star)'(t,s) \;=\; \max_{\mu\in kkt(t)}\langle\nabla_t g(t,x)\mu,s\rangle$$

for all $s\in\mathbb{R}^r$, where (x,α) is any optimal point of $F_{\rm epi}(t)$. As the functions $G^\star$ and $\alpha^\star$ coincide, the same formula holds for the directional derivatives of $G^\star$ with any optimal point x of $F(t)$. The inner and outer linearization cones to $\operatorname{dom} f^\star$ at $t\in\operatorname{bd}\operatorname{dom} f^\star$ thus have the description

$$\mathcal{L}^<(t,\operatorname{dom} f^\star) \;=\; \{s\in\mathbb{R}^r|\ \max_{\mu\in kkt(t)}\langle\nabla_t g(t,x)\mu,s\rangle<0\}$$

and

$$\mathcal{L}(t,\operatorname{dom} f^\star) \;=\; \{s\in\mathbb{R}^r|\ \max_{\mu\in kkt(t)}\langle\nabla_t g(t,x)\mu,s\rangle\leq 0\},$$

respectively, where x is any optimal point of $F(t)$.

Example 3.16 *In the situation of Example 1.3 we can compute the inner and outer linearization cones at the boundary points of* $\operatorname{dom} f^\star=[-1/\sqrt{2},1]$ *as follows. At* $t=-1/\sqrt{2}$ *the unique optimal point of* $F(t)$ *is* $x=1/\sqrt{2}$ *with active index set* $I_0(t,x)=\{1,2\}$*. This results in*

$$kkt\left(-\frac{1}{\sqrt{2}}\right) \;=\; \left\{\left(\frac{1}{1+\sqrt{2}},\frac{\sqrt{2}}{1+\sqrt{2}}\right)^\intercal\right\}$$

and

$$\max_{\mu\in kkt(t)}\langle\nabla_t g(t,x)\mu,s\rangle \;=\; -\frac{2\sqrt{2}}{1+\sqrt{2}}\,s$$

for all $s \in \mathbb{R}$. *We arrive at* $L^<(-1/\sqrt{2}, \operatorname{dom} f^\star) = \{s \in \mathbb{R} |\, s > 0\}$ *and* $L(-1/\sqrt{2}, \operatorname{dom} f^\star) = \{s \in \mathbb{R} |\, s \geq 0\}$.

For $t = 1$, *with the unique optimal point* $x = 0$ *of* $F(t)$ *and* $kkt(1) = \{(1, 0)^\intercal\}$ *one analogously sees* $L^<(1, \operatorname{dom} f^\star) = \{s \in \mathbb{R} |\, s < 0\}$ *and* $L(1, \operatorname{dom} f^\star) = \{s \in \mathbb{R} |\, s \leq 0\}$.

In analogy to the projection result for the radial cone in Lemma 3.5, the inner and outer linearization cones are orthogonal projections of the corresponding cones in the product space $\mathbb{R}^r \times \mathbb{R}^n$.

Lemma 3.17 *For any* $t \in \operatorname{dom} f^\star$ *we have*

$$\mathcal{L}^<(t, \operatorname{dom} f^\star) \;=\; \operatorname{pr}_t \mathcal{L}^<(\,(t, x), \operatorname{gph} M)$$

as well as

$$\mathcal{L}(t, \operatorname{dom} f^\star) \;=\; \operatorname{pr}_t \mathcal{L}(\,(t, x), \operatorname{gph} M),$$

where $x \in M^\star(t)$ *may be chosen arbitrarily.*

Proof. First consider the case $t \in \operatorname{int} \operatorname{dom} f^\star$. Then we have $\mathcal{L}^<(t, \operatorname{dom} f^\star) = \mathcal{L}(t, \operatorname{dom} f^\star) = \mathbb{R}^r$ so that, in view of $\operatorname{pr}_t \mathcal{L}^<(\,(t, x), \operatorname{gph} M) \subseteq \operatorname{pr}_t \mathcal{L}(\,(t, x), \operatorname{gph} M)$ the assertions follow if we can show the relation $\mathbb{R}^r \subseteq \operatorname{pr}_t \mathcal{L}^<(\,(t, x), \operatorname{gph} M)$. In fact, for any $s \in \mathbb{R}^r$ there exists some $\tau_0 > 0$ with $t + \tau_0 s \in \operatorname{int} \operatorname{dom} f^\star$. As in the second step of the proof of Lemma 3.11 one can construct some $d^0 \in \mathbb{R}^n$ with $(s, d^0) \in \mathcal{L}^<(\,(t, x), \operatorname{gph} M)$ which shows $s \in \operatorname{pr}_t \mathcal{L}^<(\,(t, x), \operatorname{gph} M)$ and, thus, $\mathbb{R}^r \subseteq \operatorname{pr}_t \mathcal{L}^<(\,(t, x), \operatorname{gph} M)$.

To see the assertions in the case $t \in \operatorname{bd} \operatorname{dom} f^\star$, note that by linear programming duality the identity

$$\max_{\mu \in kkt(t)} \langle \nabla_t g(t, x) \mu, s \rangle \;=\; \min_{d \in \mathbb{R}^n} \max_{i \in I_0(t,x)} \langle \nabla g_i(t, x), (s, d) \rangle$$

holds. •

Next, we clarify the relation of the linearization cones $\mathcal{L}^<(t, \operatorname{dom} f^\star)$ and $\mathcal{L}(t, \operatorname{dom} f^\star)$ to the radial cone $\mathcal{C}(t, \operatorname{dom} f^\star)$.

Proposition 3.18 *At each* $t \in \operatorname{dom} f^\star$ *we have*

$$\operatorname{int} \mathcal{C}(t, \operatorname{dom} f^\star) \;=\; \mathcal{L}^<(t, \operatorname{dom} f^\star).$$

Proof. The assertion immediately follows from the combination of Lemma 3.11 with Lemma 3.17. •

Proposition 3.19 *At each $t \in \operatorname{dom} f^\star$ we have*

$$\operatorname{cl} \mathcal{C}(t, \operatorname{dom} f^\star) \; = \; \mathcal{L}(t, \operatorname{dom} f^\star).$$

Proof. For $t \in \operatorname{dom} f^\star$ choose any $x \in M^\star(t)$. Then the relations

$$\mathcal{L}^<((t,x), \operatorname{gph} M) \; \subseteq \; \mathcal{C}((t,x), \operatorname{gph} M) \; \subseteq \; \mathcal{L}((t,x), \operatorname{gph} M)$$

are easily verified. They imply the relations

$$\operatorname{pr}_t \mathcal{L}^<((t,x), \operatorname{gph} M) \; \subseteq \; \operatorname{pr}_t \mathcal{C}((t,x), \operatorname{gph} M) \; \subseteq \; \operatorname{pr}_t \mathcal{L}((t,x), \operatorname{gph} M)$$

which, in view of Lemma 3.5 and Lemma 3.17, lead to

$$\mathcal{L}^<(t, \operatorname{dom} f^\star) \; \subseteq \; \mathcal{C}(t, \operatorname{dom} f^\star) \; \subseteq \; \mathcal{L}(t, \operatorname{dom} f^\star). \tag{7}$$

After taking the closures of the sets in (7), and using the closedness of $\mathcal{L}(t, \operatorname{dom} f^\star)$, the assertion follows if we can show the inclusion $\mathcal{L}(t, \operatorname{dom} f^\star) \subseteq \operatorname{cl} \mathcal{L}^<(t, \operatorname{dom} f^\star)$.

Again, first consider the case $t \in \operatorname{int} \operatorname{dom} f^\star$. Then we have $\mathcal{L}^<(t, \operatorname{dom} f^\star) = \mathcal{L}(t, \operatorname{dom} f^\star) = \mathbb{R}^r$, so that the assertion trivially follows.

For $t \in \operatorname{bd} \operatorname{dom} f^\star$, first note that $\mathcal{L}^<(t, \operatorname{dom} f^\star) \neq \emptyset$ holds due to the existence of a Slater point in $\operatorname{dom} f^\star$ and [23, Th. 3.1]. Choose any $s^0 \in \mathcal{L}^<(t, \operatorname{dom} f^\star)$ and $s^1 \in \mathcal{L}(t, \operatorname{dom} f^\star)$. With a similar construction as in the proof of Lemma 3.12 we put $s^\theta = (1-\theta)s^0 + \theta s^1$ for any $\theta \in (0,1)$. The sub-additivity of the maximum operator then implies

$$\begin{aligned}\max_{\mu \in kkt(t)} \langle \nabla_t g(t,x)\mu, s^\theta \rangle \; &\leq \; (1-\theta) \max_{\mu \in kkt(t)} \langle \nabla_t g(t,x)\mu, s^0 \rangle \\ &\quad + \theta \max_{\mu \in kkt(t)} \langle \nabla_t g(t,x)\mu, s^1 \rangle \; < \; 0,\end{aligned}$$

so that $s^\theta \in \mathcal{L}^<(t, \operatorname{dom} f^\star)$ holds for all $\theta \in (0,1)$. Taking the limit $\theta \to 1$ yields $s^1 \in \operatorname{cl} \mathcal{L}^<(t, \operatorname{dom} f^\star)$, which completes the proof. •

Remark 3.20 *From, e.g., [2] it is known that $\operatorname{cl} \mathcal{C}(t, \operatorname{dom} f^\star)$ coincides with the contingent cone $\mathcal{T}(t, \operatorname{dom} f^\star)$, so that Proposition 3.19 also shows that the Abadie condition $\mathcal{T}(t, \operatorname{dom} f^\star) = \mathcal{L}(t, \operatorname{dom} f^\star)$ holds any $t \in \operatorname{dom} f^\star$. This can as well be shown directly by using [23, Prop. 3.1].*

The combination of Propositions 3.18 and 3.19 with Propositions 3.13 and 3.10, respectively, yield our main results.

Theorem 3.21 *$f^\star$ is directionally differentiable at each $t \in \operatorname{bd}\operatorname{dom} f^\star$ in each direction s with $\max_{\mu\in kkt(t)} \langle \nabla_t g(t,\hat{x})\mu, s\rangle < 0$ ($\hat{x}$ being any optimal point of $F(t)$), with*

$$(f^\star)'(t,s) \;=\; \sup_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle \;<\; +\infty$$

where $x \in M^\star(t)$ may be chosen arbitrarily.

Theorem 3.22 *$f^\star$ is directionally differentiable at each $t \in \operatorname{bd}\operatorname{dom} f^\star$ in each direction $s \neq 0$ with $\max_{\mu\in kkt(t)} \langle \nabla_t g(t,\hat{x})\mu, s\rangle \leq 0$ ($\hat{x}$ being any optimal point of $F(t)$), with*

$$(f^\star)'(t,s) \;=\; \sup_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle \;<\; +\infty$$

whenever $x \in M^\star(t)$ may be chosen such that $\operatorname{gph} M$ is locally polyhedral at (t,x).

Example 3.23 *In the situation of Example 1.3, consider the parameter $t = -1/\sqrt{2} \in \operatorname{bd}\operatorname{dom} f^\star$. As $M(t)$ does not contain a Slater point, $KKT(t)$ cannot be a polytope. In fact, with $M^\star(t) = 1/\sqrt{2}$ one computes the unbounded Karush-Kuhn-Tucker set*

$$KKT\left(-\frac{1}{\sqrt{2}}\right) \;=\; \left\{ \begin{pmatrix} \theta \\ 1+\sqrt{2}\theta \end{pmatrix} \;\middle|\; \theta \geq 0 \right\}.$$

In view of Theorem 3.21 and Example 3.16, this results in

$$(f^\star)'(t,s) \;=\; \sup_{\lambda\in KKT(t)} \langle \nabla_t L(t,x,\lambda), s\rangle \;=\; \sup_{\theta\geq 0}\,(-2\sqrt{2}\theta - 1)s \;=\; -s$$

for all $s > 0$.

For $t = 1$, the other boundary point of $\operatorname{dom} f^\star$, it is easily seen that $KKT(t)$ is void, so that Theorem 3.21 yields $(f^\star)'(t,s) = -\infty$ for all $s < 0$.

Remark 3.24 *Our improvement of the results from [13] is threefold. First, in [13, Eq. (17a)] the explicit formula for the directional derivative is given* *only in primal form, but not in the dual form using Karush-Kuhn-Tucker multipliers. Second, in [13, Cor. 3.2(a)] the directional differentiability result in the case* $t \in \operatorname{bd} \operatorname{dom} f^\star$ *is stated for* $s \in \mathcal{C}(t, \operatorname{int} \operatorname{dom} f^\star)$*, while we state it for the more natural choice* $s \in \operatorname{int} \mathcal{C}(t, \operatorname{dom} f^\star)$*, as well as for* $s \in \operatorname{cl} \mathcal{C}(t, \operatorname{dom} f^\star)$ *under a polyhedrality assumption. Most importantly, however, using techniques from nonsmooth analysis, we give functional descriptions of the cones* $\operatorname{int} \mathcal{C}(t, \operatorname{dom} f^\star)$ *and* $\operatorname{cl} \mathcal{C}(t, \operatorname{dom} f^\star)$ *which makes the directional differentiability results more tractable for applications.*

4 Final remarks

In the nonpolyhedral case, the radial cone $\mathcal{C}(t, \operatorname{dom} f^\star)$ at some boundary point t of $\operatorname{dom} f^\star$ may still be closed (as, e.g., the Lorentz cone), or it may not be closed while still some directions $s \in \mathcal{C}(t, \operatorname{dom} f^\star) \cap \operatorname{bd} \mathcal{C}(t, \operatorname{dom} f^\star)$ exist. Directional derivatives in such boundary directions cannot be calculated via Theorems 3.21 or 3.22, but the according assertion is still true under the fiber condition in view of Proposition 3.9. The identification of computable sufficient conditions for the fiber condition in this situation is subject of future research.

As the optimal value function $f^\star$ is convex under the complete convexity assumption, one may also wish to investigate its convex subdifferential. In view of the intimate relationship of the latter with directional derivatives, explicit representations of convex subdifferentials of $f^\star$ are easily obtained using the results of the present article. Moreover, in polyhedral settings one may invoke [19, Th. 23.10] for results on the polyhedrality of the subdifferential.

Finally, we mention that Example 2.3 is slightly unsatisfactory for the illustration of the lack of inner semi-continuity of M under our assumptions. In fact, while it covers the setting of abstract feasible set mappings, we were not able to construct a description of a completely convex feasible set mapping by convex functions whose graph satisfies the Slater condition, but inner semi-continuity fails at some point in the (boundary of) the domain. We leave the quest for such an example or (more interestingly) the proof that it cannot exist as an open question for future research.

Acknowledgments

We are grateful to Christian Kanzow and Diethard Klatte for fruitful discussions on the subject of this paper, and to Georg Still for pointing out the reference [22] to us.

References

[1] B. Bank, J. Guddat, D. Klatte, B. Kummer, K. Tammer, *Non-Linear Parametric Optimization,* Akademie-Verlag, Berlin, 1982.

[2] J.M. Borwein, A.S. Lewis, *Convex Analysis and Nonlinear Optimization,* Springer, 2006.

[3] J.F. Bonnans, A. Shapiro, *Optimization problems with perturbations: a guided tour,* SIAM Review, Vol. 40 (1998), 228-264.

[4] A. Dreves, C. Kanzow, O. Stein, *Nonsmooth optimization reformulations of player convex generalized Nash equilibrium problems,* Journal of Global Optimization, Vol. 53 (2012), 587-614.

[5] D. Gale, V. Klee, R.T. Rockafellar, *Convex functions on convex polytopes,* Proceedings of the American Mathematical Society, Vol. 19 (1968), 867-873.

[6] J. Gauvin, *A necessary and sufficient regularity condition to have bounded multipliers in nonconvex optimization,* Mathematical Programming, Vol. 12 (1977), 136-138.

[7] E.G. Gol'stein, *Theory of Convex Programming,* Translations of Mathematical Monographs, Vol. 36, American Mathematical Society, Providence, Rhode Island, 1972.

[8] B. Gollan, *On the marginal function in nonlinear programming,* Mathematics of Operations Research, Vol. 9 (1984), 208-221.

[9] M. Gugat, *Parametric convex optimization: One-sided derivatives of the value function in singular parameters,* in: Butzer, P. L. (ed.) et al.: *Karl der Große und sein Nachwirken. 1200 Jahre Kultur und Wissenschaft in Europa. Band 2: Mathematisches Wissen*, Brepols, 1998, 471-483.

[10] N. Harms, C. Kanzow, O. Stein, *On differentiability properties of player convex generalized Nash equilibrium problems,* Optimization, iFirst, DOI: 10.1080/02331934.2012.752822.

[11] J.-B. Hiriart-Urruty, C. Lemaréchal, *Convex Analysis and Minimization Algorithms I,* Springer, 1996.

[12] W.W. Hogan, *Point-to-set maps in mathematical programming,* SIAM Review, Vol. 15 (1973), 591-603.

[13] W.W. Hogan, *Directional derivatives for extremal-value functions with applications to the completely convex case,* Operations Research, Vol. 21 (1973), 188-209.

[14] R. Janin, *Directional derivative of the marginal function in nonlinear programming,* Mathematical Programming Study, Vol. 21 (1984), 110-126.

[15] D. Klatte, *A sufficient condition for lower semicontinuity of solutions sets of systems of convex inequalities,* Mathematical Programming Study, Vol. 21 (1984), 139-149.

[16] D. Klatte, *Lower semicontinuity of the minimum in parametric convex programs,* Journal of Optimization Theory and Applications, Vol. 94 (1997), 511-517.

[17] P. Maćkowiak, *Some remarks on lower hemicontinuity of convex multivalued mapping,* Economic Theory, Vol. 28 (2006), 227-233.

[18] B.N. Psenichny, *Convex Analysis and Extremal Problems (in Russian)*, Nauka, Moskow, 1980.

[19] R.T. Rockafellar, *Convex Analysis,* Princeton University Press, Princeton, New Jersey, 1970.

[20] R.T. Rockafellar, *Directional differentiability of the optimal value function in a nonlinear programming problem,* Mathematical Programming Study, Vol. 21 (1984), 213-226.

[21] R.T. Rockafellar, R.J.B. Wets, *Variational Analysis,* Springer, Berlin, 1998.

[22] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory,* Cambridge University Press, Cambridge, 1993.

[23] O. Stein, *On constraint qualifications in nonsmooth optimization,* Journal of Optimization Theory and Applications, Vol. 121 (2004), 647-671.

[24] O. Stein, *Bi-level Strategies in Semi-infinite Programming,* Kluwer, Boston, 2003.

[25] O. Stein, *How to solve a semi-infinite optimization problem,* European Journal of Operational Research, Vol. 223 (2012), 312-320.