ISSN: 2277-8020, Vol. 5, pp 49-59
doi:10.18052/www.scipress.com/BSMaSS.5.49
2013 SciPress Ltd, Switzerland

# A New Hilbert-type Integral Inequality with the Homogeneous Kernel of Real Degree Form and the Integral in Whole Plane

Xie Zitian[1] K. Raja Rama Gandhi[2] Zheng Zeng[3]

[1]Department of Mathematics, Zhaoqing University, Zhaoqing, Guangdong, China 526061
[2]Resource person in Oxford University,India and BITS-Vizag, INDIA.
[3]Department of Mathematics, Shaoguan University, Shaoguan, Guangdong, China512005

**Keywords:** Hilbert-type integral inequality, weight function, H older's inequality, function.

**Abstract:** In this paper,we build a new Hilbert's inequality with the homogeneous kernel [illegible] real order and the integral in whole plane. The equivalent inequality is considered. [illegible] best con[illegible]ant factor is calculated using Ψ f unction.



## 1 INTRODUCTION

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)\mathrm{d}x < \infty$ $\quad 0 < \int_0^\infty g^2(x)\mathrm{d}[illegible]\infty$; then [illegible]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^{1/2}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/2} \tag{1.1}$$

where the constant factor π is the best possible. Inequa[illegible]ty (1.1) is [illegible]ell-known as Hilbert's integral inequality,which has been extended by Hardy-Riesz as [illegible]

If $p > 1, 1/p + 1/q = 1$ $\; f(x), g(x) \geq 0$; such [illegible]hat $0 < \int_0^{[illegible]}[illegible]\mathrm{d}x < \infty$, and $0 < \int_0^\infty g^q(x)\mathrm{d}x < \infty$;

then we have the following Hardy-Hilbert's inte[illegible]ral [illegible]lity:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\pi/p)}\left(\int [illegible](x)\mathrm{d}x\right)^{1[illegible]}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/q}; \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin[illegible]p)}$ als[illegible] the best possible.

Hilbert's inequality attract[illegible]e attentio[illegible] recent years.Actually, inequalities (1.1)and(1.2) have many generaliza[illegible]ons a[illegible] variations. (1.1) has been strengthened by Yang and others (including double se[illegible] inequali[illegible]) [3,4,6-21].

In 2008, Zitian X[illegible]e and Zheng Zen[illegible] gave a new Hilbert-type Inequality [4] as follows

If $a > 0, b > 0, c > [illegible]$ $p > 1, 1/[illegible] + 1/q = 1$ $\; f(x), g(x) \geq 0$; such that

$0 < \int_0^\infty f[illegible](x)\mathrm{d}x <$ [illegible]nd $0 < \int_0^\infty g^{-1-q/2}(x)\mathrm{d}x < \infty$ then

$$[illegible] < K\left(\int_0 [illegible]^{-p/2}(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^{-1-q/2}(x)\mathrm{d}x\right)^{1/q}; \tag{1.3}$$

where the constant factor $K = \frac{\pi}{(a+b)(a+c)(c+b)}$ is the best possible.

In 2010,Jianhua Xhong and Bicheng Yang gave a new Hilbert-type Inequality [5] as follows :

Assume that

$\lambda, p > 0 (p \neq 1), r > 1,\ 1/p + 1/q = 1, 1/r + 1/s = 1, \phi(x) = x^{p(1-\frac{\lambda}{r})-1}, \varphi(x) = x^{q(1-\frac{\lambda}{s})-1}, x \in (0, \infty),$

$K = \Gamma(\beta+1)\sum_{k=0}^{\infty}(-1)^k \binom{\alpha-\lambda}{k}\left[\frac{1}{(k+\lambda/r)^{\beta+1}} + \frac{1}{(k+\lambda/s)^{\beta+1}}\right]$, and $f, g \geq 0$,

$0<\|f\|_{p,\phi}:=\left\{\int_0^{\infty}x^{p(1-\lambda/r)-1}f^p(x)\,\mathrm{d}x\right\}^{1/p}<\infty,\ 0<\|g\|_{q,\varphi}<\infty$ then

(1) for $p>1$ we have the following equivalent inequalities:

$$\int_0^{\infty}\int_0^{\infty}\frac{|\ln(x/y)|^{\beta}f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^{\alpha}}\mathrm{d}x\mathrm{d}y<K\|f\|_{p,\phi}\|g\|_{q,\varphi}$$

(2)For $0<p<1$ the reverse of (1.5) with the best constant factor K .

$$\int_0^{\infty}\int_0^{\infty}\frac{|\ln(x/y)|^{\beta}f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^{\alpha}}\mathrm{d}x\mathrm{d}y>K\|f\|_{p,\phi}\|g\|_{q,\varphi}$$

The main purpose of this paper is to build a new Hilbert-type inequality with the homogeneous kernel of real order and the integral in whole plane, by estimating the weight function using Ψ function.The equivalent inequality is considered

We knew that (in this paper,γ is the Euler's constant.)

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right),\psi(1)=-\gamma,\psi\left(\frac{1}{2}\right)=-\gamma-2\ln 2.$$

Recent XIE Zitin and ZHOU Qinghua prove that the expression of the Ψ -function admits a finite expression in elementary function for rational number z,and prove that[6]

$$\psi\left(\frac{a}{b}\right)=\Gamma'\left(\frac{a}{b}\right)/\Gamma\left(\frac{a}{b}\right)=-\ln b-\gamma-\ln 2-\frac{\pi}{2}\cot\frac{a\pi}{b}+\sum_{k=1}^{b-1}\cos\frac{2ka\pi}{b}\ln\sin\frac{k\pi}{b}$$

and have

$$\psi\left(\frac{1}{3}\right)=-\gamma-\ln 3-\ln 2-\frac{\pi}{2}\cot\frac{\pi}{3}+\cos\frac{2\pi}{3}\ln\sin\frac{\pi}{3}+\cos\frac{4\pi}{3}\ln\sin\frac{2\pi}{3}$$

$$=-\gamma-\frac{3}{2}\ln 3-\frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{2}{3}\right)=-\gamma-\frac{3}{2}\ln 3+\frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{1}{4}\right)=-\gamma-3\ln 2-\frac{\pi}{2};$$

$$\psi\left(\frac{3}{4}\right)=-\gamma-3\ln 2+\frac{\pi}{2};$$

$$\psi\left(\frac{1}{6}\right)=-\gamma-2\ln [illegible]-\frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{5}{6}\right)=-\gamma [illegible]\ln 2+\frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{1}{5}\right)=-\gamma-\frac{5\ln 5}{4}-\frac{\sqrt{5}}{2}\ln(\sqrt{5}-1)-\frac{\sqrt{5}}{2}\ln 2-\frac{\pi}{40}(5+3\sqrt{5})\sqrt{10-2\sqrt{5}};$$

$$\psi\left(\frac{4}{5}\right)=-\gamma-\frac{5\ln 5}{4}-\frac{\sqrt{5}}{2}\ln(\sqrt{5}-1)-\frac{\sqrt{5}}{2}\ln 2+\frac{\pi}{40}(5+3\sqrt{5})\sqrt{10-2\sqrt{5}}$$

In the following, we always suppose that:

$1/p+1/q=1,p>1,\min\{a\lambda+b\mu,a\mu+b\lambda\}>-1,a\mu+b\lambda\neq 0,a\lambda+b\mu\neq 0,\mu>0,\lambda>0.$

$a+b=1.$

## 2 SOME LEMMAS

We start by introducing some Lemmas.

**Lemma 2.1** If $s>0,r\neq 0,r>-s$, then

$$1)\ \int_0^1 x^{r-1}\ln(1-x^s)dx = -\frac{1}{r}[\gamma+\psi(\frac{r+s}{s})]$$

$$2)\ \int_0^1 x^{r-1}\ln(1+x^s)dx = \frac{1}{r}\ln 2-\frac{1}{2r}[\psi(\frac{r+2s}{2s})-\psi(\frac{r+s}{2s})] \tag{2.1}$$

**Proof.** we obtain,’

$$1)\ -\int_0^1 x^{r-1}\ln(1-x^s)dx = \int_0^1 x^{r-1}\sum_{l=1}^{\infty}\frac{x^{ls}}{l}dx$$

$$=\sum_{l=1}^{\infty}\int_0^1\frac{x^{ls+r-1}}{l}dx=\sum_{l=1}^{\infty}\frac{1}{l(r+ls)}$$

$$=\frac{1}{r}\sum_{l=1}^{\infty}(\frac{1}{l}-\frac{1}{l+r/s})$$

$$=\frac{1}{r}[\gamma+\psi(\frac{r+s}{s})]$$

$$2)\int_0^1 x^{r-1}\ln(1+x^s)dx=\int_0^1 x^{r-1}\sum_{l=1}^{\infty}(-1)^{l-1}\frac{x^{ls}}{l}dx$$

$$=\sum_{l=1}^{\infty}\int_0^1(-1)^{l-1}\frac{x^{ls+r-1}}{l}dx=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l(r+ls)}$$

$$=\lim_{N\to\infty}\left\{\frac{1}{r}\sum_{l=1}^{2N+1}(-1)^{l-1}\frac{1}{l}-\frac{1}{2r}\left[\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+\frac{r+2s}{2s}}\right)-\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+\frac{r+s}{2s}}\right)\right]\right\}$$

$$=\frac{1}{r}\ln 2-\frac{1}{2r}[\psi(\frac{r+2s}{2s})-\psi(\frac{r+s}{2s})]$$

The lemma is proved.

In particular,if $r>-1,\ r\neq 0$, then

$$\int_0^1 u^{r-1}\ln\frac{u^2+u+1}{u^2-u+1}du$$

$$=\int_0^1 u^{r-1}\ln(1-u^3)du-\int_0^1 u^{r-1}\ln(1-u)du-\left[\int_0^1 u^{r-1}\ln(1+u^3)du-\int_0^1 u^{r-1}\ln(1+u)du\right]$$

$$=\frac{1}{r}\left[\psi(r+1)-\psi(\frac{r+3}{3})\right]+\frac{1}{2r}\left[\psi(\frac{r+6}{6})-\psi(\frac{r+3}{6})-\psi(\frac{r+2}{2})+\psi(\frac{r+1}{2})\right]$$

$$(\text{using } \psi(x+1)=\psi(x)+\frac{1}{x})$$

$$=\frac{1}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{2r}\left[\psi(\frac{r}{6})-\psi(\frac{r+3}{6})-\psi(\frac{r}{2})+\psi(\frac{r+1}{2})\right]$$

$$=\frac{1}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{2r}\left[2\psi(\frac{r}{6})-\psi(\frac{r}{6})-\psi(\frac{r+3}{6})-2\psi(\frac{r}{2})+\psi(\frac{r}{2})+\psi(\frac{r+1}{2})\right]$$

$$(\text{using } \psi(x+\frac{1}{2})+\psi(x)=2\psi(2x)-\psi(\frac{r+1}{2})$$

$$=\frac{2}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{r}\left[(\frac{r}{6})-\psi(\frac{r}{2})\right] \tag{2.2}$$

**Lemma 2.2** Define the weight functions as follow:

$$w(x) := \int_{-\infty}^{\infty} \frac{|x|^{a(\mu-\lambda)}}{|y|^{1-b(\mu-\lambda)}} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dy,$$

$$\tilde{w}(y) := \int_{-\infty}^{\infty} \frac{|y|^{b(\mu-\lambda)}}{|x|^{1-a(\mu-\lambda)}} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dx,$$

Then

$$w(x) = \tilde{w}(y)$$
$$= \frac{2}{a\lambda+b\mu}\left[\psi(a\lambda+b\mu) - \psi\left(\frac{a\lambda+b\mu}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\lambda+b\mu}{6}\right) - \psi\left(\frac{a\lambda+b\mu}{2}\right)\right]$$
$$+ \frac{2}{a\mu+b\lambda}\left[\psi(a\mu+b\lambda) - \psi\left(\frac{a\mu+b\lambda}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\mu+b\lambda}{6}\right) - \psi\left(\frac{a\mu+b\lambda}{2}\right)\right].$$
$$:= k \quad (2.3)$$

**Proof** We only prove that $w(x) = k$ for $x \in (-\infty, 0)$

Using lemma 2.1,setting y = ux ,and y=-ux

$$w(x) := \int_{-\infty}^{0} \frac{(-x)^{a(\mu-\lambda)}}{(-y)^{1-b(\mu-\lambda)}} \frac{(\min\{(-x),(-y)\})^{\lambda}}{(\max\{(-x),(-y)\})^{\mu}} \ln \frac{x^2+xy+y^2}{x^2+y^2} dy$$
$$+ \int_{0}^{\infty} \frac{(-x)^{a(\mu-\lambda)}}{y^{1-b(\mu-\lambda)}} \frac{(\min\{(-x),y\})^{\lambda}}{(\max\{(-x),y\})^{\mu}} \ln \frac{x^2+y^2}{x^2+xy+y^2} dy$$
$$:= w_1 + w_2$$

Then

$$w_1 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+u+1}{u^2+1} du + \int_1^{\infty} u^{-1-a\lambda-b\mu} \ln \frac{u^2+u+1}{u^2+1} du$$
$$= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+u+1}{u^2+1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2+u+1}{u^2+1} du$$
$$w_2 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+1}{u^2-u+1} du + \int_1^{\infty} u^{-1-a\lambda-b\mu} \ln \frac{u^2+1}{u^2-u+1} du$$
$$= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+1}{u^2-u+1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2+1}{u^2-u+1} du$$

And

$$w = w_1 + w_2 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+u+1}{u^2-u+1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2+u+1}{u^2-u+1} du$$
$$= \frac{2}{a\lambda+b\mu}\left[\psi(a\lambda+b\mu) - \psi\left(\frac{a\lambda+b\mu}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\lambda+b\mu}{6}\right) - \psi\left(\frac{a\lambda+b\mu}{2}\right)\right]$$
$$+ \frac{2}{a\mu+b\lambda}\left[\psi(a\mu+b\lambda) - \psi\left(\frac{a\mu+b\lambda}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\mu+b\lambda}{6}\right) - \psi\left(\frac{a\mu+b\lambda}{2}\right)\right].$$
$$= k$$

Similarly,setting $x = y/u$ ,and $x = -y/u$

$$\tilde{w}(y) = \int_{-\infty}^{0} \frac{(-y)^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}} \frac{(\min\{(-x),(-y)\})^{\lambda}}{(\max\{(-x),(-y)\})^{\mu}} \ln \frac{x^2+xy+y^2}{x^2+y^2} dx$$
$$+ \int_{0}^{\infty} \frac{y^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}} \frac{(\min\{(-x),y\})^{\lambda}}{(\max\{(-x),y\})^{\mu}} \ln \frac{x^2+y^2}{x^2+xy+y^2} dx$$

$$= \int_{-\infty}^{0} \frac{(-y)^{b(\mu-\lambda)}}{(-y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(-y/u),(-y)\})^{\lambda}}{(\max\{(-y/u),(-y)\})^{\mu}} \ln \frac{(y/u)^2+(y/u)y+y^2}{(y/u)^2+y^2} d(y/u)$$

$$+\int_{0}^{\infty} \frac{y^{b(\mu-\lambda)}}{(y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(y/u),y\})^{\lambda}}{(\max\{(y/u),y\})^{\mu}} \ln \frac{(-y/u)^2+y^2}{(-y/u)^2+(-y/u)y+y^2} d(-y/u)$$

$$= \int_{0}^{\infty} u^{-1+b(\mu-\lambda)} \frac{(\min\{1,u\})^{\lambda}}{(\max\{1,u\})^{\mu}} \ln \frac{u^2+u+1}{u^2+1} du + \int_{0}^{\infty} u^{-1+b(\mu-\lambda)} \frac{(\min\{1,u\})^{\lambda}}{(\max\{1,u\})^{\mu}} \ln \frac{u^2+1}{u^2-u+1} du$$

$$= w_1 + w_2 = k \tag{2.4}$$

and the lemma is proved.

**Lemma 2.3** For ε > 0; and $\min\{a\mu+b\lambda-2\varepsilon/q, a\lambda+b\mu-2\varepsilon/q\} > -1$, define both functions, $\tilde{f},\tilde{g}$ as follow:

$$\tilde{f}(x) = \begin{cases} x^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if} \quad x\in(1,\infty), \\ 0, & \text{if} \quad x\in[-1,1], \\ (-x)^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if} \quad x\in(-\infty,-1), \end{cases}$$

$$\tilde{g}(x) = \begin{cases} x^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if} \quad x\in(1,\infty), \\ 0, & \text{if} \quad x\in[-1,1], \\ (-x)^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if} \quad x\in(-\infty,-1), \end{cases}$$

Then

$$I(\varepsilon) := \varepsilon\left\{\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} \tilde{f}^{p}(x)dx\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} \tilde{g}^{q}(x)dx\right\}^{1/q} = 1; \tag{2.5}$$

$$\tilde{I}(\varepsilon) := \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x)\tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dxdy \to k(\varepsilon\to 0^{+}) \tag{2.6}$$

**Proof** Easily

$$I(\varepsilon) = \varepsilon\left\{2\int_{1}^{\infty} x^{-1}x^{-2\varepsilon}dx\right\}^{1/p} \left\{2\int_{1}^{\infty} y^{-1}y^{-2\varepsilon}dy\right\}^{1/q} = 1$$

Let y-Y, using $\tilde{f}(-x)=\tilde{f}(x)$, $\tilde{g}(-x)=\tilde{g}(x)$ and

$$\tilde{f}(-x)\int_{-\infty}^{\infty} \tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2-xy+y^2}{x^2+y^2}\right| dy$$

$$= \tilde{f}(x)\int_{-\infty}^{\infty} \tilde{g}(Y) \frac{(\min\{|x|,|Y|\})^{\lambda}}{(\max\{|x|,|Y|\})^{\mu}} \left|\ln \frac{x^2+xY+Y^2}{x^2+Y^2}\right| dY$$

we have that $\tilde{f}(x)\int_{-\infty}^{\infty} \tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dy$ is an even function on x, ,then

$$\tilde{I}(\varepsilon) := 2\varepsilon\int_{-\infty}^{\infty} \tilde{f}(x)\left(\int_{-\infty}^{\infty} \tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dy\right)dx$$

$$= 2\varepsilon\left[\int_{1}^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_{-\infty}^{-1} (-y)^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dy\right)dx\right.$$

$$\left. + \int_{1}^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_{1}^{\infty} y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left|\ln \frac{x^2+xy+y^2}{x^2+y^2}\right| dy\right)dx\right]$$

$$:= I_1 + I_2$$

Setting y = tx then

$$I_1 = 2\varepsilon\int_1^\infty x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_1^\infty y^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{x,y\})^\lambda}{(\max\{x,y\})^\mu}\ln\frac{x^2+y^2}{x^2-xy+y^2}dy\right)dx$$

$$=2\varepsilon\int_1^\infty x^{-1-2\varepsilon}\left(\int_{1/x}^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt\right)dx$$

$$=2\varepsilon\int_1^\infty x^{-1-2\varepsilon}\left(\int_1^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt\right)dx$$

$$+2\varepsilon\int_1^\infty x^{-1-2\varepsilon}\left(\int_{1/x}^1 t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt\right)dx$$

$$=\int_1^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$+2\varepsilon\int_0^1 t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}\left(\int_{1/t}^\infty x^{-1-2\varepsilon}dx\right)dt$$

$$=\int_1^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$+\int_0^1 t^{b(\mu-\lambda)-1-2\varepsilon/p}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$=\int_0^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$+\int_0^1 (t^{2\varepsilon/p}-t^{-2\varepsilon/q})t^{b(\mu-\lambda)-1}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$=\int_0^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+1^2}{t^2-t+1^2}dt+\eta_1(\varepsilon)\ (\lim_{\varepsilon\to0^+}\eta_1(\varepsilon)=0)$$

Similarly

$$I_2 = 2\varepsilon\int_1^\infty x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_1^\infty y^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{x,y\})^\lambda}{(\max\{x,y\})^\mu}\ln\frac{x^2+xy+y^2}{x^2+y^2}dy\right)dx$$

$$=\int_0^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu}\ln\frac{t^2+t+1^2}{t^2+1^2}dt+\eta_2(\varepsilon),(\lim_{\varepsilon\to0^+}\eta_2(\varepsilon)=0)$$

by lemma 2.2 we have

$$\tilde{I}(\varepsilon)=I_1+I_2$$

$$=\frac{2}{a\lambda+b\mu-2\varepsilon/q}\left[\psi(a\lambda+b\mu-2\varepsilon/q)-\psi(\frac{a\lambda+b\mu-2\varepsilon/q}{3})\right]$$

$$+\frac{1}{a\lambda+b\mu-2\varepsilon/q}\left[\psi(\frac{a\lambda+b\mu-2\varepsilon/q}{6})-\psi(\frac{a\lambda+b\mu-2\varepsilon/q}{2})\right]$$

$$+\frac{2}{a\mu+b\lambda-2\varepsilon/q}\left[\psi(a\mu+b\lambda-2\varepsilon/q)-\psi(\frac{a\mu+b\lambda-2\varepsilon/q}{3})\right]$$

$$+\frac{1}{a\mu+b\lambda-2\varepsilon/q}\left[\psi(\frac{a\mu+b\lambda-2\varepsilon/q}{6})-\psi(\frac{a\mu+b\lambda-2\varepsilon/q}{2})\right]+\eta_1(\varepsilon)+\eta_2(\varepsilon)$$

we know that $\psi(x)$ is a continuous function,then $\lim_{\varepsilon\to0^+}\tilde{I}(\varepsilon)=k$

The lemma is proved.

**Lemma 2.4** If f(x) is a nonnegative measurable function, and $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$

Then

$$J := \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left( \int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy$$

$$\le k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \qquad (2.7)$$

**Proof** By lemma 2.2,we find

$$\left( \int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p$$

$$= \left( \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \left( \frac{|x|^{[1-a(\mu-\lambda)]/q}}{|y|^{[1-b(\mu-\lambda)]/p}} f(x) \right) \left( \frac{|y|^{[1-b(\mu-\lambda)]/p}}{|x|^{[1-a(\mu-\lambda)]/q}} \right) dx \right)^p$$

$$\le \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx$$

$$\times \left( \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}} dx \right)^{p-1}$$

$$= k^{p-1} |y|^{1-pb(\mu-\lambda)} \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx$$

$$J \le k^{p-1} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx \right] dy$$

$$= k^{p-1} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} dy \right] f^p(x)dx$$

$$= k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx$$

## 3 MAIN RESULTS

**Theorem 3.1** If p > 1; both functions, f(x) and g(x) ,are nonnegative measurable functions, and satisfy and $0 < \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx < \infty$ then $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$

$$I^* := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dxdy$$

$$< k \left( \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \right)^{1/p} \left( \int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx \right)^{1/q} \qquad (3.1)$$

And

$$J = \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left( \int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy$$

$$< k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \qquad (3.2)$$

Inequalities (3.1)and (3.2) are equivalent,and where the constant factors k and kp arethe best possible.

**Proof** If there exist a $y\in(-\infty,0)\cup(0,\infty)$ ,such that (2.7) takes the form of equality, then there exists constants M and N, such that they are not all zero, and

$$M\frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}}f^p(x)=N\frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}}$$ a.e. In $(-\infty,\infty)$

Hence, there exists a constant C, such that

$$M|x|^{p[1-a(\mu-\lambda)]}f^p(x)=N|y|^{q[1-b(\mu-\lambda)]}=C$$ a.e. In $(-\infty,\infty)$

It means that M = 0. In fact, if $M\neq 0$, then

$$|x|^{p[1-a(\mu-\lambda)]-1}f^p(x)=\frac{C}{M|x|}$$ a.e. In $(-\infty,\infty)$

which contradicts the fact that $0<\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^p(x)dx<\infty$ In the same way, we claim that N = 0: This is too a contradiction and hence by (2.7), we have (3.2). By Holder's inequality with weight and (3.2), we have,

$$I^*:=\int_{-\infty}^{\infty}\left[|y|^{b(\mu-\lambda)-\frac{1}{p}}\int_{-\infty}^{\infty}f(x)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dx\right]\left[|y|^{-b(\mu-\lambda)+\frac{1}{p}}g(y)\right]dy$$

$$\le J^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1}g^q(y)dy\right)^{1/q} \tag{3.3}$$

Using (3.2),we have (3.1).

Setting

$$g(y)=|y|^{pb(\mu-\lambda)-1}\left(\int_{-\infty}^{\infty}f(x)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dx\right)^{p-1}$$

Then

$J=\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1}g^q(y)dy$ by (2.?)we have $J<\infty$ if $J=0$ then (3.2) is proved;

If $0<J<\infty$ we obtain

$$0<\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1}g^q(y)dy=J=I^*$$

$$<k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}g^q(x)dx\right)^{1/q}$$

$$\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}g^q(x)dx\right)^{1/p}=J^{1/p}<k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^p(x)dx\right)^{1/p}$$

Inequalities (3.1)and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with h < k),such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$

$$<h\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}g^q(x)dx\right)^{1/q}$$

For $\varepsilon>0$ by (2.5),using lemma 2.3 ,we have

$$k+o(1)<\varepsilon h\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}\tilde{f}^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}\tilde{g}^q(x)dx\right)^{1/q}=h \tag{3.4}$$

Hence we find, k +o(1) < h: For $\varepsilon\to 0^+$ it follows that $k\le h$ which contradicts the fact that h < k .
Hence the constant k in (3.1) is the best possible.
Thus we complete the prove of the theorem.

**Theorem 3.2** If 1 > p > 0; both functions, f(x) and g(x) ,are nonnegative measurable functions, and satisfy

$0<\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx<\infty$ and $0<\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx<\infty$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$>k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx\right)^{1/q} \quad (3.5)$$

$$J>k^p\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \quad (3.6)$$

And

$$L:=\int_{-\infty}^{\infty}|x|^{qa(\mu-\lambda)-1}\left(\int_{-\infty}^{\infty}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}g(y)\right|dy\right)^q dx$$
$$<k^q\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy \quad (3.7)$$

Inequalities (3.5),(3.6)and (3.7) are equivalent,and where the constant factors k, $k^p$ and $k^q$ are the best possible

**Proof** By the reverse Holder's inequality and the same way, we can obtain the reverseforms of (2.7)and (3.3).And then we deduce the (3.5),by the som[illegible] way,we obtain (3.6).

Setting g(y) as the theorem 1,we obtain $J>0$ ,if $J=\infty$ ,[illegible] we have (3.6),if $0<J<\infty$ ,by (3.5)

$$\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy=J=I^*$$
$$>k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx\right)^{1/q}$$

and we have (3.6),and inequalities (3.5)and (3.6) are equivalent.

Setting

$$[g(x)]_n=\begin{cases}\frac{1}{n} & \text{if } g(x)<\frac{1}{n},\\ g(x) & \text{if } \frac{1}{n}\le g(x)\le [illegible]\\ n & [illegible]\ g(x)>n\end{cases}$$

$E_n=\left[[illegible]-\frac{1}{n}\right]\cup\left[\frac{1}{n},n\right]$ then $\exists n_0\in N$, such that $n>n_0$; we have

$$\int_{E_n}|y|^{q[1-b(\mu-[illegible]} g^q(y)dy>0,$$

And

$$[f(x)]_n=|x|^{qa(\mu-\lambda)-1}\left(\int_{E_n}[g(x)]_n\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)^{q-1}$$

$$[L(x)]_n=\int_{E_n}|x|^{qa(\mu-\lambda)-1}\left(\int_{E_n}[g(x)]_n\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)^q dx$$

Then $\exists n_0\in N$, such that $n>n_0$; we have

$$\int_{E_n}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy<\infty,$$

In particular, from (4.1) we get the following particular cases:

1)If $\lambda+\mu=4$; then $k=2\left[\psi(2)-\psi(\frac{2}{3})\right]+\frac{2}{3}\left[\psi(\frac{1}{3})-\psi(1)\right]=2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<\left(2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}\right)\left(\int_{-\infty}^{\infty}|x|^{p(\lambda-1)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(\lambda-1)-1}g^q(x)dx\right)^{1/q} \tag{4.2}$$

2) If $\lambda+\mu=3$; then $k=\frac{8}{3}\left[\psi(\frac{3}{2})-\psi(\frac{1}{2})\right]+\frac{4}{3}\left[\psi(\frac{1}{4})-\psi(\frac{3}{4})\right]=\frac{4}{3}(4-\pi)$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<\left(2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}\right)\left(\int_{-\infty}^{\infty}|x|^{p(\lambda-1)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(\lambda-1)-1}g^q(x)dx\right)^{1/q} \tag{4.3}$$

3) If $\lambda+\mu=2$; then $k=4\left[\psi(1)-\psi(\frac{1}{3})\right]+2\left[\psi(\frac{1}{6})-\psi(\frac{1}{2})\right]=6\ln 3+\frac{2\pi}{\sqrt{3}}-\sqrt{2}\pi$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{2-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<\left(6\ln 3+\frac{2\pi}{\sqrt{3}}-\sqrt{2}\pi\right)\left(\int_{-\infty}^{\infty}|x|^{p\lambda-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q\lambda-1}g^q(x)dx\right)^{1/q} \tag{4.4}$$

B) Let $\lambda=\mu$ in (3.1),then we have a 1ntegral 1nequality with the homogeneous kernel of 0 degree form as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\left(\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}}\right)^{\lambda}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<k(\lambda)\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q-1}g^q(x)dx\right)^{1/q} \tag{4.5}$$

There $k(\lambda)=\frac{4}{\lambda}\left[\psi(\lambda)-\psi(\frac{\lambda}{3})\right]+\frac{2}{\lambda}\left[\psi(\frac{\lambda}{6})-\psi(\frac{\lambda}{2})\right]$


**References**

[1] Hardy G. H., Littlewood J. E. and Polya G., Inequalities,London, Cambridge University Press, Cambridge, 1952.

[2] Hardy G. H., Note on a theorem of Hilbert concerning series of positive terms, Proceedings London Math. Soc., 1925,23(2):45-46.

[3] Zitian Xie and Zheng Zeng, A Hilbert-type integral inequality whose kernel is a homogeneous form of degree-331

[4] Zitian Xie, Zheng Zeng, A Hilbert-type integral inequality with a non- homogeneous form and a best constant factor, Advances and Applications in Mathematical Sciens 2010,3(1),61-71

[5] Jianhua Xhong and Bicheng Yang, On a relation to four basic Hilbert-type integral inequalities, Applied Mathematical Sciences,Vol. 4, 2010,no,19,923-930.

[6] XIE Zitian and ZHOU Qinghua,A note on a Property of psi -function, Wuhan University Journal of Natural Sciences, Wuhan University Journal of Natural Science, 2011,16(2),139-142.

[7] Bicheng Yang, On a Hilber-type integral inequality with a combination kerne and applications, Kyungpook Mathematical Journal, 2010(50),281-288

[8] Zitian Xie , Zheng Zeng, The Hilbert-type integral inequality with the system Kernel of $-\lambda$ degree homogeneous form, Kyungpook Mathematical Journal, 2010(50),297-306

[9] Xie Zitian, A New Hilbert-type integral inequality with the homogeneous kernel of real number-degre, Journal of Jishou University (Natural Science Edition), 2011,32(4)26-30

[10] Zitian Xie and Bicheng Yang,A new Hilbert-type integral inequality with some parameters and its reverse,Kyungpook Mathematical Journal,2008,(48):93-100

[11] Zitian Xie,A new Hilbert-type inequality with the kernel of -30 -homogeneous,Journal of Jilin University (Science edition),2007,45(3):369-373.

[12] Zitian Xie and Juming Murong,A rewerse Hilbert-type inequality with some parameters,Journal of Jilin University (Science edition), 2008,46 (4):66[illegible]

[13] Zitian Xie,A new reverse Hilbert-type inequality with a best constant factor, J.Math.Appl, 2008,(343):1154-1160.

[14] Zheng Zeng and Zi-tian Xie, A Hilbert's inequality with a best constant factor, Journal of Inequalities and Applications, 2009, Article ID 820176, 8 pages doi:10.1155/2009/820176

[15] Xie Zitian ,Zeng Zheng,On generality of Hilbert's inequality with best constant factor, Natural Science Journal of Xiangtan University, 2010, 32(3) : 1-[illegible]

[16] Zheng Zeng and Zitian Xie, On a new Hilbert-type integral inequality with the integral in whole plane, Journal of Inequalities and Applic[illegible] 2010, Article ID 256796, 8 pages, 2010. doi:10.1155/2010/256796

[17] Zitian Xie,Bicheng Yang, Zheng Zeng, A New Hilbert-type integral inequality with the homogeneous kernel of real number-degree ,Journal of Jilin University(Science Edition), 2010,48(6) : 941-945.

[18] Zitian Xie and Fang[illegible] Zhou, A generalization of a Hilbert-type inequality with the best constant factor, Journal of Sichuan Normal University (Natural Science) 2009, 32(5): 626-629

[19] Xie Zitian,Zeng Zheng, A Hilbert-type integral inequality with the integral in whole plane and the kernel of -4 degree homogeneous form, i-manager's Journal on Mathematics, Vol. 1 No. 3, July-september 2012,[illegible]

[20] Zitian Xie,Zheng Zeng,Qinghua Zhou, A new Hilbert-type integral inequality with the homogeneous kernel of real number-degree and its equivalent inequality forms,Journal of Jilin University (Science Edition), 2012,50(4), 693-697.

[21] Xie Zitian,Zeng Zheng, A new half-discrete Hilbert-type inequality with the homogeneous kernel of degree -4μ, Journal of Jishou University(Natural Science Edition), 2012,33 ( 2), 15-19