Amplitude Independent Frequency Synchroniser for a Cubic Planar Polynomial System

Islam Boussaada
Laboratoire des Signaux et Systèmes (L2S),
Supélec-CNRS-Université Paris,
Sud 3 rue Joliot-Curie,
91192 Gif-sur-Yvette cedex (France);
Laboratoire de Modélisation et
Calcul Scientifique (LMCS),
Institut Polytechnique des Sciences,
Avancées 7 rue Maurice Grandcoing,
94200 Ivry-sur-Seine (France)
e-mail: islam.boussaada@gmail.com

The problem of local linearizability of the planar linear center perturbed by cubic nonlinearities in all generalities on the system parameters (14 parameters) is far from being solved. The synchronization problem (as noted in Pikovsky, A., Rosenblum, M., and Kurths, J., 2003, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge Nonlinear Science Series, Cambridge University Press, UK, and Blekhman, I. I., 1988, Synchronisation in Science and Technology, ASME Press Translations, New York) consists in bringing appropriate modifications on a given system to obtain a desired dynamic. The desired phase portrait along this paper contains a compact region around a singular point at the origin in which lie periodic orbits with the same period (independently from the chosen initial conditions). In this paper, starting from a five parameters non isochronous Chouikha cubic system (Chouikha, A. R., 2007, "Isochronous Centers of Lienard Type Equations and Applications," J. Math. Anal. Appl., ***331****, pp. 358–376) we identify all possible monomial perturbations of degree $d \in \{2, 3\}$ insuring local linearizability of the perturbed system. The necessary conditions are obtained by the Normal Forms method. These conditions are real algebraic equations (multivariate polynomials) in the parameters of the studied ordinary differential system. The efficient algorithm FGb (J. C. Faugère, "FGb Salsa Software," http://fgbrs.lip6.fr) for computing the Gröbner basis is used. For the family studied in this paper, an exhaustive list of possible parameters values insuring local linearizability is established. All the found cases are already known in the literature but the contexts are different since our object is the synchronisation rather than the classification. This paper can be seen as a direct continuation of several new works concerned with the hinting of cubic isochronous centers, (in particular Bardet, M., and Boussaada, I., 2011, "Compexity Reduction of C-algorithm," App. Math. Comp., in press; Boussaada, I., Chouikha, A. R., and Strelcyn, J.-M., 2011, "Isochronicity Conditions for some Planar Polynomial Systems," Bull. Sci. Math,* ***135****(1), pp. 89–112; Bardet, M., Boussaada, I., Chouikha, A. R., and Strelcyn, J.-M., 2011, "Isochronicity Conditions for some Planar Polynomial Systems," Bull. Sci. Math,* ***135****(2), pp. 230–249; and furthermore, it can be considered as an adaptation of a qualitative theory method to a synchronization problem.* [DOI: 10.1115/1.4005322]

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1 Introduction

We consider the planar dynamical system,

$$\frac{dx}{dt} = \dot{x} = X(x,y), \quad \frac{dy}{dt} = \dot{y} = Y(x,y) \tag{1.1}$$

where (x, y) belongs to an open connected subset $U \subset \mathbb{R}^2$, $X, Y \in C^k(U, \mathbb{R})$, and $k \geq 1$. Due to Poincaré: an isolated singular point $p \in U$ of Eq. (1.1) is a center if and only if there exists a punctured neighborhood $V \subset U$ of p such that every orbit in V is a cycle surrounding p. A center is said to be isochronous if all the orbits surrounding it have the same period. An overview of Chavarriga and Sabatini [1] present the methods and basic results concerning the problem of the isochronicity, see also Refs. [2–6].

The synchronization problem consists of bringing appropriate modifications on a given system to obtain a desired dynamic, see Refs. [7,8]. Along this paper, the desired phase portrait contains a compact region around a singular point at the origin in which lies periodic orbits with the same period (independently from the chosen initial conditions which is not always the case). More concretely, in this paper we consider the following problem: starting from a non-isochronous polynomial planar system, we seek to discover if there is any polynomial perturbation which insures the local linearizability of the perturbed system. In this paper, we adopt the normal forms method often used in qualitative theory investigations: the center-focus problem, bifurcation problem and local linearizability problem. The problem of local linearizability conditions of the planar linear center perturbed by cubic nonlinearities (in all generalities on the system parameters 14 parameters) is far from being solved.

In this paper, starting from a five-parameters non-isochronous Chouikha cubic system [3], we identify all possible monomial perturbations of degree $d \in \{2, 3\}$ insuring local linearizability of the perturbed system. Investigations are based on the normal forms Theory.

In the following system as well as in all other considered systems, all parameters are reals.

Consider the real Lienard Type equation:

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \tag{1.2}$$

or, equivalently, its associated two dimensional (planar) system

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$$\left.\begin{aligned}\dot{x}&=y\\ \dot{y}&=-g(x)-f(x)y^2\end{aligned}\right\}\qquad(1.3)$$

The study of isochronicity of Eq. (1.2) was established first in Sabatini's paper [9]. The sufficient conditions of the isochronicity of the origin O for system (Eq. (1.3)) with f and g of class C^1 are given. In the analytic case, the necessary and sufficient conditions for isochronicity are given by Chouikha in Ref. [3]. In the same paper, the author implemented a new algorithmic method for computing isochronicity conditions for a system (Eq. (1.3)) called C-algorithm. As an application of this algorithm, the author studied the following cubic system:

$$\left.\begin{aligned}\dot{x}&=-y+\tilde{a}_{1,2,1}x^2y\\ \dot{y}&=x+\tilde{a}_{2,2,0}x^2+\tilde{a}_{2,0,2}y^2+\tilde{a}_{2,3,0}x^3+\tilde{a}_{2,1,2}xy^2\end{aligned}\right\}\qquad(1.4)$$

where all the parameters values for which system Eq. (1.4) has an isochronous center at the origin are established in the following theorem.

We note that the coefficient $a_{i,j,k}$ denotes the parameter of the monomial perturbation of the ith equation of the linear isochronous center ($\dot{x}=-y$, $\dot{y}=x$) of degree j in x and of degree k in y.

Theorem 1.1. *According to Chouikha [3], the system represented in Eq. (1.4) has an isochronous center at 0 if and only if its parameters satisfy one of the following conditions:*

1. $\tilde{a}_{2,3,0}=-(2/3)\tilde{a}_{1,2,1},\tilde{a}_{2,1,2}=3\tilde{a}_{1,2,1},\tilde{a}_{2,2,0}=\tilde{a}_{2,0,2}=0$
2. $\tilde{a}_{2,1,2}=\tilde{a}_{1,2,1},\ \tilde{a}_{2,2,0}=\tilde{a}_{2,3,0}=\tilde{a}_{2,0,2}=0$
3. $\tilde{a}_{2,3,0}=(1/14)\tilde{a}^2_{2,0,2},\tilde{a}_{2,1,2}=(3/7)\tilde{a}^2_{2,0,2},$ $\tilde{a}_{1,2,1}=(1/7)\tilde{a}^2_{2,0,2},\tilde{a}_{2,2,0}=-(1/2)\tilde{a}_{2,0,2}$
4. $\tilde{a}_{2,1,2}=\tilde{a}^2_{2,0,2},\tilde{a}_{2,3,0}=0,\quad \tilde{a}_{1,2,1}=(1/2)\tilde{a}^2_{2,0,2},$ $\tilde{a}_{2,2,0}=-(1/2)\tilde{a}_{2,0,2}$

A one-parameter perturbation of the system represented in Eq. (1.4) is studied in Ref. [10]. Namely, the following system

$$\left.\begin{aligned}\dot{x}&=-y+\underline{a_{1,1,1}xy}+a_{1,2,1}x^2y\\ \dot{y}&=x+a_{2,2,0}x^2+a_{2,0,2}y^2+a_{2,3,0}x^3+a_{2,1,2}xy^2\end{aligned}\right\}\qquad(1.5)$$

which is system Eq. (1.4) perturbed by the underlined term. All the values of the parameters in the above system, Eq. (1.5), have an isochronous center at the origin that was found.

Note that the above system is still reducible to the Liénard type equation for which the C-algorithm is applicable, see Ref. [5].

Section 2 is devoted to recall the headlines of the methodology of the Normal Forms algorithm, called in the sequel NF algorithm, which will allow us to obtain isochronicity necessary conditions.

The last section is concerned with the main result, which is an application of the NF method. Indeed, we consider an unknown one-directional monomial perturbation of degree two or three of the system Eq. (1.4), namely,

$$\left.\begin{aligned}\dot{x}&=-y+a_{1,2,1}x^2y+\underline{\Psi_1(x,y)}\\ \dot{y}&=x+a_{2,2,0}x^2+a_{2,0,2}y^2+a_{2,3,0}x^3+a_{2,1,2}xy^2+\underline{\Psi_2(x,y)}\end{aligned}\right\}\qquad(1.6)$$

in which only one of the monomials Ψ_1 or Ψ_2 is non-zero monomial ($\Psi_1\Psi_2=0$) of degree $d\in\{2,3\}$.

The problem turns to studying eight polynomial cubic systems which are not reducible by the transformations described in Ref. [5] to the Liénard type equation. For each system, we identify the values of the parameters for which the singular point at the origin is an isochronous center. Hence it is done for Eq. (1.6).

2 The Normal Forms Method

The normal form theory, which is due essentially to Poincaré, presents a basic tool in understanding the qualitative behavior of orbit structures of a vector field near equilibria [11]. It was used for the study of center conditions and the nature of bifurcation of a given vector field. We also recall a pioneer work in this field established by Pleshkan (see Ref. [12]), in which the author presented an investigation method of isochronicity in the case of a linear center perturbed by homogeneous cubic nonlinearity. The principle of Pleshkan's algorithm is very close to the one presented in Algaba et.al's paper [13], where the normal form theory is used in the analysis of isochronicity and gave a recursive method for the isochronicity investigation. In the last cited paper, the authors studied a cubic Liénard equation and obtained a characterization of the reversible Liénard equation having an isochronous center at the origin.

Let $x=(x_1,\ x_2)\in\mathbb{R}^2$ and $f(x_1,\ x_2)\in\mathbb{R}[x_1,\ x_2]\times\mathbb{R}[x_1,\ x_2]$ and consider the general planar system

$$\dot{x}=Lx+f(x)=Lx+f_2(x)+f_3(x)+\cdots+f_k(x)+\cdots\qquad(2.1)$$

where Lx represents the linear part, L the Jacobian matrix associated to system Eq. (2.1) and $f_k(x)$ denotes the kth order vector homogeneous polynomials of x. We assume that the system admits an equilibrium at the origin O. The essential idea of the Normal Form theory is to find a near identity transformation

$$x=y+h(y)=y+h_2(y)+h_3(y)+\cdots+h_k(y)+\cdots\qquad(2.2)$$

by which the resulting system

$$\dot{y}=Ly+g(x)=Ly+g_2(y)+g_3(y)+\cdots+g_k(y)+\cdots\qquad(2.3)$$

becomes as simple as possible. In this sense, the terms that are not essential in the local dynamical behavior are removed from the analytical expression of the vector field. Let us denote by $h_k(y)$ and $g_k(y)$ the kth order vectors homogeneous polynomials of y. According to Takens normal form theory, we define an operator as follows:

$$L_k:H_k\rightarrow H_k,\quad U_k\in H_k\mapsto L_k(U_k)=\left[U_k,u_1\right]\in H_k\qquad(2.4)$$

where $u_1=Ly$ is the linear part of the vector field and H_k denotes a linear vector space containing the kth degree homogeneous vector polynomials of $y=(y_1,\ y_2)$. The operator [.,.] is called the Lie Bracket, defined by

$$\left[U_k,u_1\right]=LU_k-D(U_k)u_1$$

where D denotes the frechet derivative.

Next, we define the spaces R_k and K_k as the range of L_k and the complementary space of R_k respectively. Thus, $H_k=R_k+K_k$ and one can then choose bases for K_k and R_k. The normal form theorem determines how it is possible to reduce the analytic expression of the vector field (see Gukenheimer-Holmes [11]). The authors explicitly provide an analysis for the quadratic and the cubic cases. Consequently, a vector homogeneous polynomial $f_k\in H_k$ can be split into two parts, such that one of them can be spanned in K_k and the remaining part in R_k.

Normal form theory shows that the part belonging to R_k can be eliminated and the remaining part can be retained in the normal form. By the Eqs. (2.1), (2.2), and (2.3), we can obtain algebraic equations one order after another.

Theorem 2.1. *According to the work of Yu et al., [14], the recursive formula for computing the normal form coefficients and the nonlinear transformation are given by:*

$$g_k = f_k + [h_k, Ly] + \sum_{i=2}^{k-1} (Df_i h_{k-i+1} - Dh_{k-i+1} g_i)$$
$$+\sum_{i=2}^{[k/2]} \frac{1}{i!} \sum_{j=i}^{k-i} D^i f_j \sum_{l_1+l_2+\cdots+l_i=k-(j-i)2\le l_1,l_2,\ldots,l_i\le k+2-(i+j)} h_{l_1} h_l 2 \ldots h_{l_i}$$

for $k = 2, 3, \ldots$

see also Refs. [15,16].

System Eq. (2.3) can be transformed to the polar coordinate system with $y_1 = rcos(\theta)$, $y_2 = rsin(\theta)$ so that

$$\dot{r} = \sum_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O(r^{2N+3}),$$
$$\dot{\theta} = 1 + \sum_{j=1}^{N} \beta_{2j+1} r^{2j} + O(r^{2N+2}) \quad (2.5)$$

Recall that a necessary condition to have a center at the origin is that all the focal values α_{2j+1} vanish. By the Hilbert's basis theorem, the set of focal values has a finite basis in the ring of polynomials in the coefficients of the initial system Eq. (2.1). Since the non-vanishing of one of the angular component coefficient implies dependency of an associated period constant, a necessary condition for which this center is isochronous is that β_{2j+1} vanish.

Recall that our study is motivated by the interest of describing a synchronizer for a desired dynamic but also to underline the key role that classification of centers and isochronous centers of polynomial systems can have in applications such that synchronization problems.

3 Main Results: Applications of the NF Algorithm to Cubic Systems

In our study, we use Maple. To compute the Gröbner basis of the obtained polynomial equations in the ring of characteristic 0, we employ the *Salsa Software* [17]. More precisely, we use the *FGb* algorithm which is the most efficient algorithm in computing Gröbner basis [18], at least for the polynomial systems studied in this paper. We note also that we have used DRL ordering in all computations established in this paper.

Since our approach in the investigation of isochronicity conditions is based on an algorithmic method, we can guess that every simplification is beneficial in the goal of speeding the computations and reducing the necessary memory size. Solving multivariate algebraic equations (real polynomials) can be a very hard task if we try to manipulate the polynomial equations without tricks. Interested readers can find in the website of *Salsa Software* [17], more precisely the page of J. C. Faugère, some important rules in solving polynomial systems and about Gröbner basis.

Let us consider the more general cubic perturbation of linear center:

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,0}x^2 + a_{1,0,2}y^2 + a_{1,1,1}xy + a_{1,3,0}x^3 \\ &\quad + a_{1,2,1}x^2y + a_{1,1,2}xy^2 + a_{1,0,3}y^3 \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,1,1}xy \\ &\quad + a_{2,30}x^3 + a_{2,2,1}x^2y + a_{2,1,2}xy^2 + a_{2,0,3}y^3 \end{aligned}\right\} \quad (3.1)$$

or, equivalently, the following one:

$$\left.\begin{aligned} \dot{x} &= -y + \tilde{a}_{1,2,0}x^2 + \tilde{a}_{1,0,2}y^2 + \tilde{a}_{1,1,1}xy + \tilde{a}_{1,3,0}x^3 \\ &\quad + \tilde{a}_{1,2,1}x^2y + \tilde{a}_{1,1,2}xy^2 + \tilde{a}_{1,0,3}y^3 \\ \dot{y} &= x + \tilde{a}_{2,2,0}x^2 + \tilde{a}_{2,0,2}y^2 + \tilde{a}_{2,1,1}xy \\ &\quad + \tilde{a}_{2,30}x^3 + \tilde{a}_{2,2,1}x^2y + \tilde{a}_{2,1,2}xy^2 + \tilde{a}_{2,0,3}y^3 \end{aligned}\right\}$$

Observe that we can easily reconstruct the coefficient $\tilde{a}_{i,j,k}$ from the ones in Eq. (3.1) by the change of coordinates $(x, y) \mapsto (-x, y)$.

The classification of all the isochronous centers of the above system is a very hard task. By any recursive method from those quoted in Ref. [1], solving the isochronicity problem for system Eq. (3.1) is very difficult in the sense of solving multivariate polynomials. Here, the variables are the 14 parameters of the polynomial differential system Eq. (3.1). Hence it needs very important computation supports.

With a realistic point of view, several authors have chosen some particular cases of the above system for investigation like the homogeneous cubic perturbations of the linear center [1,12] and time reversible cubic systems [19,20].

In our case, we focus on an unknown one-directional one-parameter perturbation of the system Eq. (1.4) which is system Eq. (1.6). Therefore, we ramify our study to all possible cases.

In the sequel, every subsection will be concerned with a possible one-parameter perturbation of system Eq. (1.4)

Recall that $a_{i,j,k}$ denotes the parameter of the monomial perturbation of the ith equation in system Eq. (1.4) of degree j in x and of degree k in y.

3.1 Perturbation $a_{1,1,1}$.

As a continuation of the result of Ref. [3] on system Eq. (1.4), Chouikha, Romanovski and Chen [10] investigated a one-parameter perturbation of Eq. (1.4) which is Eq. (1.5).

Theorem 3.1. *According to Chouikha et al. [10], system Eq. (1.5) has an isochronous center at the origin O if and only if* $a_{1,2,1} = \frac{-a_{1,1,1}+a_{1,1,1}a_{2,2,0}-10a_{2,2,0}^2+5a_{1,1,1}a_{2,0,2}-10a_{2,0,2}a_{2,0,2}-4a_{2,0,2}^2+9a_{2,3,0}+3a_{2,1,2}}{3}$ *and its parameters satisfy one of the following sets of conditions:*

1. $2a_{2,2,0} + a_{2,0,2} - a_{1,1,1} = 0$
   $a_{2,0,2}^2 - 3a_{2,0,2}a_{1,1,1} + 2a_{1,1,1}^2 - 6a_{2,3,0} - \frac{4a_{2,1,2}}{3} = 0$
   $a_{2,0,2}a_{2,3,0} - 3a_{2,3,0}a_{1,1,1} + \frac{a_{2,1,2}a_{1,1,1}}{6} - \frac{a_{2,0,2}a_{2,1,2}}{6} = 0$
   $a_{2,3,0}^2 a_{1,1,1}^2 - 3a_{2,3,0}^2 + \frac{a_{1,1,1}^2 a_{2,3,0}a_{2,1,2}}{6} + \frac{a_{2,3,0}^2 a_{2,1,2}}{3} + \frac{5a_{2,1,2}^2 a_{2,3,0}}{36} - \frac{a_{2,1,2}^3}{5} = 0$
   $a_{2,1,2}a_{1,1,1}a_{2,0,2} + 6a_{2,3,0}a_{1,1,1}^2 - a_{2,1,2}a_{1,1,1}^2 - 18a_{2,3,0}^2 - a_{2,3,0}a_{2,1,2} + \frac{2a_{2,1,2}^2}{3} = 0$
2. $a_{2,1,2} = 0$ *and*
   (a) $a_{2,3,0} = a_{2,0,2} - 1/4a_{1,1,1} = a_{2,2,0} = 0$
   (b) $4a_{2,3,0} - 3a_{1,1,1} = a_{2,2,0} + a_{1,1,1} = a_{1,1,1}^2 - 3a_{2,3,0} = 0$
   (c) $a_{2,0,2} - 2a_{1,1,1} = a_{2,3,0} - a_{1,1,1}^2 = a_{2,2,0} + 2a_{1,1,1} = 0$
   (d) $a_{2,0,2} - 1/3a_{1,1,1} = a_{2,2,0} + 2/3a_{1,1,1} = 9/2a_{2,3,0} - a_{1,1,1}^2 = 0$
3. $a_{2,3,0} = 0$ *and*
   (a) $a_{2,2,0} + 1/2a_{2,0,2} - 1/2a_{1,1,1} = a_{2,0,2}^2 - 3a_{2,0,2}a_{1,1,1} + 2a_{1,1,1}^2 - a_{2,1,2} = 0$
   (b) $2a_{2,0,2} - a_{1,1,1} = 2a_{2,2,0} + a_{1,1,1} = 0$
   (c) $a_{2,0,2} - a_{1,1,1} = a_{2,2,0} = 0$
   (d) $a_{2,0,2} = a_{1,1,1}^2 - 9a_{1,2,1} = a_{2,2,0} = 0$

We contribute by classifying the isochronous centers of all the remaining one-parameter perturbations of system Eq. (1.4). Eight systems are studied to do this.

For these perturbations, first, we check if the center (at the origin O) conditions are satisfied and after we give necessary and sufficient isochronicity conditions depending only on the six real parameters.

3.2 Perturbation $\mathbf{a_{1,0,3}}$.

We are concerned by the following system

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y + \underline{a_{1,0,3}y^3} \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 \end{aligned}\right\} \quad (3.2)$$

Lemma 3.2. *The investigation of isochronicity criteria of system Eq. (3.2) reduces to the investigation in the following three cases:*

1. $a_{1,0,3} = 0$
2. $a_{1,0,3} = 1$
3. $a_{1,0,3} = -1$

Proof. For the case $a_{1,0,3} \neq 0$, two cases are to be analyzed

First we assume that $a_{1,0,3} > 0$. We use on Eq. (3.2) the change of coordinates:

$$(x, y) \mapsto a_{1,0,3}^{1/2}(x, y) \tag{3.3}$$

to obtain

$$\left.\begin{aligned} \dot{x} &= y + \frac{a_{1,2,1}}{a_{1,0,3}}x^2y + y^3 \\ \dot{y} &= -x + \frac{a_{2,2,0}}{a_{1,0,3}^{1/2}}x^2 + \frac{a_{2,0,2}}{a_{1,0,3}^{1/2}}y^2 + \frac{a_{2,3,0}}{a_{1,0,3}}x^2 + \frac{a_{2,1,2}}{a_{1,0,3}}xy^2 \end{aligned}\right\} \tag{3.4}$$

when the solutions of the isochronicity problem of system Eq. (3.4) are established, we can easily reconstruct those of the original system Eq. (3.2) by the transformation:

$$(x, y) \mapsto (1/a_{1,0,3}^{1/2})(x, y) \tag{3.5}$$

If $a_{1,0,3} < 0$, then Eq. (3.2) can be written as

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y - \tilde{a}_{1,0,3}y^3 \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 \end{aligned}\right\}$$

with $-a_{1,0,3} = \tilde{a}_{1,0,3} > 0$. Applying the following change of coordinates

$$(x, y) \mapsto \tilde{a}_{1,0,3}^{1/2}(x, y) \tag{3.6}$$

yields

$$\left.\begin{aligned} \dot{x} &= y + \frac{a_{1,2,1}}{\tilde{a}_{1,0,3}}x^2y - y^3 \\ \dot{y} &= -x + \frac{a_{2,2,0}}{\tilde{a}_{1,0,3}^{1/2}}x^2 + \frac{a_{2,0,2}}{\tilde{a}_{1,0,3}^{1/2}}y^2 + \frac{a_{2,3,0}}{\tilde{a}_{1,0,3}}x^2 + \frac{a_{2,1,2}}{\tilde{a}_{1,0,3}}xy^2 \end{aligned}\right\} \tag{3.7}$$

Furthermore, the reconstruction of the solutions of Eq. (3.2) can be obtained from those of Eq. (3.7) by the change of coordinates

$$(x, y) \mapsto (1/\tilde{a}_{1,0,3}^{1/2})(x, y)$$

□

Remark 3.3: if one is concerned with quadratic perturbations of system Eq. (1.4) with the parameter $a_{i,j,2-j}$,

we can consider the two cases namely,

1. $a_{i,j,2-j} = 0$
2. $a_{i,j,2-j} = 1$

Indeed, when $a_{i,j,2-j} \neq 0$, the change of coordinates

$$(x, y) \mapsto a_{i,j,2-j}(x, y)$$

reduces the problem to the case $a_{i,j,2-j} = 1$.

Lastly, thanks to the transformation,

$$(x, y) \mapsto 1/a_{i,j,2-j}(x, y)$$

we can easily reconstruct the solutions of the problem when $a_{i,j,2-j} \neq 0$.

Theorem 3.4. *System Eq. (3.2) has an isochronous center at O if and only if its parameters satisfy one of the following conditions:*

1. $a_{1,0,3} = 0$
   (a) $a_{1,2,1} = a_{2,1,2}, a_{2,2,0} = a_{2,0,2} = a_{2,3,0} = 0$
   (b) $a_{1,2,1} = -3/2\ a_{2,3,0}, a_{2,1,2} = -9/2\ a_{2,3,0}, a_{2,2,0} = a_{2,0,2} = 0$
   (c) $a_{1,2,1} = a_{2,2,0} = -1/2, a_{2,0,2} = 1, a_{2,1,2} = -1, a_{2,3,0} = 0$
   (d) $a_{1,2,1} = -1/7,\ a_{2,2,0} = -1/2,\ a_{2,0,2} = 1,\ a_{2,3,0} = -1/14, a_{2,1,2} = -3/7$
2. $a_{1,0,3} = 1$
   (a) $a_{1,2,1} = -9/2, a_{2,1,2} = -3/2, a_{2,2,0} = a_{2,0,2} = a_{2,3,0} = 0$
   (b) $a_{2,3,0} = 1, a_{1,2,1} = a_{2,1,2} = -3, a_{2,2,0} = a_{2,0,2} = 0$
   (c) $a_{2,0,2} = -3/2, a_{2,3,0} = 0, a_{2,1,2} = a_{1,2,1} = a_{2,2,0} = 0$
   (d) $a_{2,0,2} = 3/2, a_{2,3,0} = 0, a_{2,1,2} = a_{1,2,1} = a_{2,2,0} = 0$
   (e) $a_{1,2,1} = -1, a_{2,2,0} = a_{2,0,2} = \pm\frac{\sqrt{2}}{2}, a_{2,1,2} = -2, a_{2,3,0} = 0$
3. $a_{1,0,3} = -1$
   (a) $a_{2,3,0} = -1, a_{1,2,1} = a_{2,1,2} = 3, a_{2,2,0} = a_{2,0,2} = 0$
   (b) $a_{1,2,1} = 9/2, a_{2,1,2} = 3/2, a_{2,3,0} = a_{2,2,0} = a_{2,0,2} = 0$

In this proof, we do not present the algorithm generated polynomials because they are too long.

Proof. we use the strategy given in lemma 3.2. We note also that we investigate only the real values of the parameters for which system Eq. (3.2) has an isochronous center at the origin.

1. Assume $a_{1,0,3} = 0$ and then solve the isochronicity problem for system Eq. (3.2) under this assumption. We use the following change of coordinates $(x,\ y) \mapsto (-x, y)$ to obtain system Eq. (1.4) studied in Ref. [3]. The investigation following the two cases:
   (a) $a_{2,0,2} = 0$
   (b) $a_{2,0,2} = 1$
   which covers (with respect to a linear change of coordinates) all the values of the parameters for which the center at the origin of system Eq. (1.4), see Remark 3.3. In Ref. [3], the author used the C- algorithm which characterizes isochronicity by establishing an associated Urabe function.
2. Consider the case $a_{1,0,3} = 1$. Computations of normal forms of the initial system in polar form give Eq. (2.5):

$$\dot{r} = \sum_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O(r^{2N+3}),$$

$$\dot{\theta} = 1 + \sum_{j=1}^{N} \beta_{2j+1} r^{2j} + O(r^{2N+2})$$

We obtain in the radial component $\alpha_{2j\ +1} = 0$ until order $N = 6$. So that the first six nec essary conditions to have a center are satisfied. Analyzing isochronicity involves the angular component. Using *FGb* for computing the Gröbner basis of the obtained first six quantities in the angular component, we obtain a Gröbner basis of 27 polynomials denoted G_1 such that it is first element is

$$-a_{2,3,0}a_{2,2,0}\left(-a_{2,1,2}^2 + 9a_{2,3,0}\right)$$

Then we analyze the isochronicity problem in the following three cases, which are given by the vanishing of one factor of the above expression.

(a) $a_{2,3,0} = 0$, we substitute this condition into G_1 and we compute again the associated Gröbner basis; we obtain a basis of 14 polynomials. When we solve it, we obtain the following four real solutions to the problem:
   (i) $a_{1,0,3} = 1,\quad a_{2,1,2} = -3/2,\quad a_{1,2,1} = -9/2,\quad a_{2,3,0} = a_{2,2,0} = a_{2,0,2} = 0$
   (ii) $a_{1,0,3} = 1,\quad a_{2,0,2} = 3/2,\quad a_{2,3,0} = a_{1,2,1} = a_{2,2,0} = a_{2,1,2} = 0$
   (iii) $a_{1,0,3} = 1,\quad a_{2,0,2} = -3/2,\quad a_{2,3,0} = a_{1,2,1} = a_{2,2,0} = a_{2,1,2} = 0$
   (iv) $a_{1,0,3} = 1,\quad a_{1,2,1} = -1,\quad a_{2,2,0} = a_{2,0,2} = \pm\frac{\sqrt{2}}{2}, a_{2,1,2} = -2$

(b) For $a_{2,2,0} = 0$, substituting this assumption in G_1 and computing it is associated Gröbner basis which contains seven polynomials, after solving it we obtain the solutions:

(i) $a_{1,0,3}=1, \quad a_{2,0,2}=3/2, \quad a_{2,2,0}=a_{1,2,1}=a_{2,3,0}=a_{2,1,2}=0$

(ii) $a_{1,0,3}=1, \quad a_{2,0,2}=-3/2, \quad a_{2,2,0}=a_{1,2,1}=a_{2,3,0}=a_{2,1,2}=0$

(iii) $a_{1,0,3}=1, \quad a_{2,1,2}=-3/2, \quad a_{1,2,1}=-9/2, \quad a_{2,2,0}=a_{2,3,0}=a_{2,0,2}=0$

(iv) $a_{1,0,3}=a_{2,3,0}=1, \quad a_{2,1,2}=a_{1,2,1}=-3, \quad a_{2,2,0}=a_{2,0,2}=0$

(c) Similarly for $a_{2,3,0}=(1/9)a_{2,1,2}^2$, we obtain the solutions:

(i) $a_{1,0,3}=1, \quad a_{2,0,2}=-3/2, \quad a_{2,2,0}=a_{1,2,1}=a_{2,3,0}=a_{2,1,2}=0$

(ii) $a_{1,0,3}=1, \quad a_{2,0,2}=3/2, \quad a_{2,2,0}=a_{1,2,1}=a_{2,3,0}=a_{2,1,2}=0$

(iii) $a_{1,0,3}=a_{2,3,0}=1, \quad a_{2,1,2}=a_{1,2,1}=-3, \quad a_{2,2,0}=a_{2,0,2}=0$

It is easy to see that several solutions are repeated above. For example, cases (2(a)ii $\equiv$ 2(b)i) and (2(a)iii $\equiv$ 2(b)ii). We claim that there are only five solutions to the problem when $a_{1,0,3}=1$ which are given in the theorem.

Analysis of the Theorem Cases with $a_{1,0,3}=1$.

- 2(a) In this case system Eq. (3.2) reduces to

$$\left.\begin{aligned}\dot{x} &= y-9/2yx^2+y^3\\ \dot{y} &= -x-3/2xy^2\end{aligned}\right\}$$

which is a cubic homogeneous perturbation of a linear center with an isochronous center at the origin. Indeed, we use the change of coordinates $(x,y)\mapsto\left(\frac{y}{\sqrt{2}},\frac{x}{\sqrt{2}}\right)$ and we obtain system $\widetilde{S}^*_3$ given in Refs. [1,12]. A first integral, a linearizing change of coordinate, and a transversal commuting system are established for homogeneous perturbations (see Refs. [1,21]).

- 2(b) System Eq. (3.2) reduces to

$$\left.\begin{aligned}\dot{x} &= y-3x^2y+y^3\\ \dot{y} &= -x+x^3-3xy^2\end{aligned}\right\}$$

which is an homogeneous perturbation of linear center, by the change of coordinates $(x,y)\mapsto(y,x)$ reduces to system S^*_1 of [1,12,21].

- For cases 2(c) and 2(d), system Eq. (3.2) reduces to

$$\left.\begin{aligned}\dot{x} &= y+y^3\\ \dot{y} &= -x\pm 3/2y^2\end{aligned}\right\}$$

We see that by the change of coordinate $(x,y)\mapsto(y,x)$, we have a Liénard systems $x''+f(x)x'+g(x)=0$ satisfying

$$g(x)=g'(0)x+\frac{1}{x^3}\left(\int_0^x sf(s)ds\right)^2 \tag{3.8}$$

then the origin is an isochronous center. See Refs. [13,22,23,24] for more details about characterization of isochronicity for the Liénard equation.

- In 2(e), the case system Eq. (3.2) is a time- reversible system with an isochronous center at O. Indeed, in polar coordinates it reduces to

$$\left.\begin{aligned}\dot{r} &= -1/2\sin(\theta)r^2\sqrt{2}+1/2r^3\sin(2\theta)\\ \dot{\theta} &= 1+(-1/2\cos(2\theta)+1/2)r^2-1/2r\cos(\theta)\sqrt{2}\end{aligned}\right\}$$

which belongs to the family (ii) (with $R_1=r_1=-\frac{\sqrt{2}}{2}$ if $a_{2,2,0}=a_{2,0,2}=\frac{\sqrt{2}}{2}$) and ($R_1=r_1=\frac{\sqrt{2}}{2}$ if $a_{2,2,0}=a_{2,0,2}=-\frac{\sqrt{2}}{2}$) of theorem 8.11 in the Garcia thesis, see also system C R4 of Ref. [1].

3. $a_{1,0,3}=-1$. When executing the normal form Maple code, there is no change from the case $a_{1,0,3}=1$; the coefficients of radial component of system Eq. (2.5) are such that $\alpha_{2j+1}=0$ until order $N=6$. The first six necessary conditions to have a center are satisfied. Similarly to the case $a_{1,0,3}=1$, for analyzing isochronicity, we are concerned by the angular component. We compute the Gröbner basis, denoted G_{-1}, of the obtained first six quantities in the angular component. We obtain an ideal of 27 polynomials such that its first element is

$$a_{2,2,0}a_{2,3,0}\left(a_{2,1,2}^2+9a_{2,3,0}\right)$$

Then we analyze the isochronicity problem in the following three cases, which are the vanishing of each of the factors of the above expression.

(a) For $a_{2,3,0}=0$, we substitute this assumption in G_{-1}. We compute again the Gröbner basis associated to this case. We obtain a basis of 14 polynomials. When we solve it, we obtain the unique real solution to the problem:

(i) $a_{1,0,3}=-1, \quad a_{1,2,1}=9/2, \quad a_{2,1,2}=3/2, \quad a_{2,3,0}=a_{2,2,0}=a_{2,0,2}=0$

(b) For $a_{2,2,0}=0$ we do the same as for the last case and obtain the two real solutions:

(i) $a_{1,0,3}=a_{2,3,0}=-1, \quad a_{1,2,1}=a_{2,1,2}=3, \quad a_{2,2,0}=a_{2,0,2}=0$

(ii) $a_{1,0,3}=-1, \quad a_{1,2,1}=9/2, \quad a_{2,1,2}=3/2, \quad a_{2,3,0}=a_{2,2,0}=a_{2,0,2}=0$

(c) $a_{2,3,0}=-(1/9)a_{2,1,2}^2$ in the same way as for the first and second cases we obtain

(i) $a_{1,0,3}=a_{2,3,0}=-1, \quad a_{1,2,1}=a_{2,1,2}=3, \quad a_{2,2,0}=a_{2,0,2}=0$

We conclude that in the case $a_{1,0,3}=-1$, we have only two solutions to the isochronicity problem.

Analysis of the Theorem Cases with $a_{1,0,3}=-1$. These two solutions are cubic homogeneous perturbations of the linear center, which can be found in Ref. [1].

- 3(a) In this case, system Eq. (3.2) can be written as

$$\left.\begin{aligned}\dot{x} &= y+3x^2y-y^3\\ \dot{y} &= -x-x^3+3xy^2\end{aligned}\right\}$$

by the change of coordinates $(x,y)\mapsto(-x,-y)$, it reduces to system S^*_1 of Refs. [1,12,21], which have an isochronous center at the origin.

- 3(b) System Eq. (3.2) can be written as

$$\left.\begin{aligned}\dot{x} &= y+9/2yx^2-y^3\\ \dot{y} &= -x+3/2xy^2\end{aligned}\right\}$$

we use the change of coordinates $(x,y)\mapsto\left(\frac{y}{\sqrt{2}},\frac{x}{\sqrt{2}}\right)$ we obtain system S^*_3 with isochronous center at the origin O given in Refs. [1,12].

This concludes the proof. □

3.3 Perturbation $a_{1,3,0}$. Consider the system

$$\left.\begin{aligned}\dot{x} &= y+a_{1,2,1}x^2y+\underline{a_{1,3,0}x^3}\\ \dot{y} &= -x+a_{2,2,0}x^2+a_{2,0,2}y^2+a_{2,3,0}x^3+a_{2,1,2}xy^2\end{aligned}\right\} \tag{3.9}$$

Theorem 3.5. *System Eq. (3.9) with $a_{1,3,0} \neq 0$ has no center at the origin. Moreover, system Eq. (3.9) has an isochronous center at the origin O if and only if it reduces to system Eq. (1.4) and its parameters satisfy one of isochronicity cases from those of Theorem (1.1).*

Proof. Analogously to lemma 3.2, we have to analyze $a_{1,3,0} = \pm 1$.

Executing the *Maple* code, which gives the normal form of Eq. (3.9); in its polar form Eq. (2.5):

$$\dot{r} = \sum_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O(r^{2N+3}),$$
$$\dot{\theta} = 1 + \sum_{j=1}^{N} \beta_{2j+1} r^{2j} + O(r^{2N+2})$$

we obtain in the radial component such that $\alpha_3 = \pm \frac{3}{8}$; in this case, the singular point cannot be a center. □

3.4 Perturbation $a_{2,0,3}$.

This perturbation represents system Eq. (1.4) perturbed by the additional monomial with the parameter $a_{2,0,3}$:

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 + \underline{a_{2,0,3}y^3} \end{aligned}\right\} \tag{3.10}$$

Theorem 3.6. *System Eq. (3.10) with $a_{2,0,3} \neq 0$ has no center at the origin. Moreover, system Eq. (3.10) has an isochronous center at O if and only if it reduces to system Eq. (1.4) and its parameters satisfy one case of isochronicity conditions given in Theorem (1.1).*

Proof. Analogously to lemma 3.2,

consider $a_{2,0,3} = \pm 1$.

Executing the Maple code, which gives the normal form of Eq. (3.10); in its polar form Eq. (2.5):

$$\dot{r} = \sum_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O(r^{2N+3}), \dot{\theta} = 1 + \sum_{j=1}^{N} \beta_{2j+1} r^{2j} + O(r^{2N+2})$$

we obtain in the radial component such that $\alpha_3 = \pm \frac{3}{8}$; in this case, the singular point cannot be a center. □

3.5 Perturbation $a_{1,1,2}$.

This case represents system Eq. (1.4) perturbed by the monomial with the parameter $a_{1,1,2}$:

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y + \underline{a_{1,1,2}xy^2} \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 \end{aligned}\right\} \tag{3.11}$$

Theorem 3.7. *System Eq. (3.11) with $a_{1,1,2} \neq 0$ has no center at the origin. Moreover, system Eq. (3.11) has an isochronous center at O if and only if it reduces to one case of isochronicity from those of system Eq. (1.4).*

Proof. When $a_{1,1,2} \neq 0$, we obtain in the radial component $\alpha_3 = \pm \frac{1}{8}$; in this case, the singular point at the origin cannot be a center. □

3.6 Perturbation $a_{2,2,1}$.

This case represents system Eq. (1.4) perturbed by the monomial with the parameter $a_{2,2,1}$:

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 + \underline{a_{2,2,1}x^2y} \end{aligned}\right\} \tag{3.12}$$

Theorem 3.8. *System Eq. (3.12) with $a_{2,2,1} \neq 0$ has no center at the origin. Moreover, system Eq. (3.12) has an isochronous center at O if and only if it reduces to one case of isochronicity from those of system Eq. (1.4).*

Proof. By the same reason from the one of the last case, we substitute $a_{2,2,1} = \pm 1$ in the system, then we obtain in the radial component $\alpha_3 = \pm \frac{1}{8}$; in this case, the singular point at the origin cannot be a center. □

3.7 Perturbation $a_{1,0,2}$.

This perturbation represents system Eq. (1.4) with an additional monomial $a_{1,0,2}y^2$ in the first equation:

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y + \underline{a_{1,0,2}y^2} \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 \end{aligned}\right\} \tag{3.13}$$

Theorem 3.9. *System Eq. (3.13) with $a_{1,0,2} \neq 0$ has no isochronous center at the origin. Moreover, system Eq. (3.13) has an isochronous center at O if and only if $a_{1,0,2} = 0$ and it reduces (modulo a linear change of coordinates) to system Eq. (1.4) such as its parameters satisfy one of the four cases given in Theorem 1.1.*

Proof. Since we have a quadratic perturbation of system Eq. (1.4), thanks to Remark 1 we study the cases $a_{1,0,2} \in \{0,1\}$. If $a_{1,0,2} = 0$, system Eq. (3.13) admits an isochronous center at the origin if and only if it reduces to Eq. (1.4) and its parameters satisfy one of the isochronicity conditions given by Theorem 1.1. We assume $a_{1,0,2} = 1$, and we first compute the radial component of the normal form in polar coordinates. The first radial component $\alpha_3 = -a_{2,0,2}/4$, then we substitute the assumption $a_{2,0,2} = 0$ in the remaining five α_{2j+1} and when we compute the associated Gröbner base, we find $\alpha_5 = a_{2,2,0}\,(a_{2,1,2} + a_{1,2,1})$. We continue the analysis in these two cases:

1. $a_{2,2,0} = 0$
2. $(a_{2,1,2} + a_{1,2,1}) = 0$

Unfortunately, we computed in each case the Gröbner base and there are no common roots between the multivariate polynomials β_{2j+1} of the angular component. □

3.8 Perturbation $a_{1,2,0}$.

Consider the system:

$$\left.\begin{aligned} \dot{x} &= y + a_{1,2,1}x^2y + \underline{a_{1,2,0}x^2} \\ \dot{y} &= -x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 \end{aligned}\right\} \tag{3.14}$$

Theorem 3.10. *System Eq. (3.14) with $a_{1,2,0} \neq 0$ has an isochronous center at O if and only if, modulo a linear change of coordinates, its parameters satisfy*

$$a_{2,3,0} = -4/9, \quad a_{2,1,2} = 0, \quad a_{1,2,1} = 0, \quad a_{2,2,0} = 0,$$
$$a_{2,0,2} = 0, \quad a_{1,2,0} = 1$$

Proof. Consider the case $a_{1,2,0} = 1$. We compute first, under this assumption, the radial component of the normal form of system Eq. (3.14). We obtain $\alpha_3 = a_{2,2,0}/4$. Then the first necessary condition to have a center at the origin is the vanishing of $a_{2,2,0}$. We substitute this additional assumption in the remaining coefficients of the radial component ($\alpha_5 \ldots \alpha_{13}$). A common factor appears which is $a_{2,0,2}$.

Hence we obtain two cases to analyze center conditions $a_{2,0,2} = 0$ and $a_{2,0,2} \neq 0$.

For the case $a_{2,0,2} \neq 0$, we divide all the expressions of the coefficients of the radial component by $a_{2,0,2}$. We compute the associated Gröbner basis which is generated by eight polynomials and gives three cases for each one and the first six necessary conditions for the center are satisfied. Note the following:

$$\{a_{2,1,2}=0,\quad a_{1,2,1}=0,\quad a_{2,2,0}=0,\quad a_{2,0,2}=0,\quad a_{1,2,0}=1,\quad a_{2,3,0}=-3/4\}$$

$$\{a_{2,2,0}=0,\ a_{1,2,0}=1,\ a_{2,3,0}=0,\ a_{2,0,2}=-1,\ a_{2,1,2}=-a_{1,2,1}\}$$

$$\{a_{2,2,0}=0,\ a_{1,2,0}=1,\ a_{2,3,0}=0,\ a_{2,1,2}=-a_{1,2,1},\ a_{2,0,2}=1\}$$

the first solution is rejected because we have assumed that $a_{2,0,2}\neq 0$.

For the second and the third solution of the center condition investigation, we substitute each of those into the angular component coefficients expressions. We compute the Gröbner bases of the obtained multivariate polynomial systems. Unfortunately, in the two cases it gives Gröbner basis $\equiv 1$ which means that there are not common roots.

We return to the remaining case $a_{1,2,0}=1,\ a_{2,2,0}=a_{2,0,2}=0$ which ensures the first six necessary conditions of the singular point at the origin to be a center. Substituting this assumption in the angular component coefficients and computing its associated Gröbner basis which is generated by three polynomials. This gives the unique solution to the problem of isochronicity

$$a_{2,3,0}=-4/9, a_{1,2,0}=1, a_{2,1,2}=a_{1,2,1}=a_{2,2,0}=a_{2,0,2}=0$$

Then system Eq. (3.14) reduces to

$$\left.\begin{array}{l}\dot{x}=y+x^2\\ \dot{y}=-x-4/9x^3\end{array}\right\}$$

Which is a Liénard isochronous system with $f(x)=-2x$, $g(x)=x+4/9\,x^3$ satisfying

$$g(x)=g'(0)x+\frac{1}{x^3}\left(\int_0^x sf(s)ds\right)^2$$

□

3.9 Perturbation $a_{2,1,1}$.

$$\left.\begin{array}{l}\dot{x}=y+a_{1,2,1}x^2y\\ \dot{y}=-x+a_{2,2,0}x^2+a_{2,0,2}y^2+\underline{a_{2,1,1}xy}+a_{2,3,0}x^3+a_{2,1,2}xy^2\end{array}\right\} \quad (3.15)$$

Theorem 3.11. *System Eq. (3.15) with $a_{2,1,1}\neq 0$ has an isochronous center at O if and only if its parameters satisfy one of the following two cases:*

1. $a_{2,1,1}=1, a_{2,3,0}=-1/9, a_{2,1,2}=a_{1,2,1}=a_{2,2,0}=a_{2,0,2}=0$
2. $a_{2,1,1}=1, a_{2,1,2}=-2/9, a_{1,2,1}=1/9, a_{2,2,0}=a_{2,0,2}=a_{2,3,0}=0$

Proof. Consider that $a_{2,1,1}=1$;

We compute first, under this assumption, the radial component of the normal form of sys tem Eq. (3.15). We obtain $\alpha_3=(a_{2,0,2}+a_{2,2,0})/8$, then the first necessary condition to have a center at the origin is the vanishing of α_3.

We substitute this additional assumption $a_{2,0,2}=-a_{2,2,0}$ in the following coefficient of the radial component $\alpha_5=-1/48\ a_{2,0,2}(a_{1,2,1}-a_{2,1,2}+a_{2,3,0})$, then it appears as two cases to analyze: $\{a_{2,0,2}=a_{2,2,0}=0\}$ and $\{a_{2,0,2}=-a_{2,2,0}, a_{1,2,1}-a_{2,1,2}+a_{2,3,0}=0\}$.

1. $a_{2,0,2}=a_{2,2,0}=0$; we substitute this additional assumption in the remaining coefficients of the radial component $\alpha_7,\ldots,\alpha_{13}$ which gives $\alpha_3=\alpha_5\ldots\alpha=\alpha_{13}=0$. Hence, the first six center necessary conditions are satisfied.
   Substituting the assumptions into the angular component coefficients expressions, com puting associated Gröbner basis gives the two real solutions. The first one is:

$$\{a_{2,1,1}=1,\quad a_{2,3,0}=-1/9,\quad a_{2,1,2}=a_{2,2,0}=a_{2,0,2}=a_{1,2,1}=0\}$$

under which system Eq. (3.15) reduces to

$$\left.\begin{array}{l}\dot{x}=y\\ \dot{y}=-x+xy-1/9x^3\end{array}\right\}$$

in addition, we have an isochronous Liénard systems with

$$f(x)=-x \quad \text{and} \quad g(x)=x+1/9x^3$$

which satisfies

$$g(x)=g'(0)x+\frac{1}{x^3}\left(\int_0^x sf(s)ds\right)^2$$

The second one is:

$$\{a_{2,2,0}=0,\quad a_{2,0,2}=0,\quad a_{2,1,1}=1,\quad a_{2,3,0}=0,\quad a_{2,1,2}=-2/9,\quad a_{1,2,1}=1/9\}$$

under which system Eq. (3.15) reduces to

$$\left.\begin{array}{l}\dot{x}=y+1/9x^2y\\ \dot{y}=-x+xy-2/9xy^2\end{array}\right\}$$

By the change of coordinates $(x, y)\mapsto(y, x)$ we obtain

$$\left.\begin{array}{l}\dot{x}=-y+xy-2/9x^2y\\ \dot{y}=-x+1/9xy^2\end{array}\right\}$$

which belongs to the Liénard Type equation, Eq. (1.2), with $f(x)=-3\ (2x-3)^{-1}$ and $g(x)=1/9x\ (2x-3)\ (x-3)$. The isochronicity of this last system is proved since it belongs to the case 9 of Theorem 3 of Chouikha et al. in Ref. [10].

2. $a_{2,0,2}=-a_{2,2,0}$ and $a_{1,2,1}-a_{2,1,2}+a_{2,3,0}=0$

Unfortunately, in this case, after computing Gröbner basis, no real solutions are found. □

Theorem 3.12. *System Eq. (1.4) has an amplitude independent frequency synchronizer at O if and only if its parameters satisfy one of the cases of Theorem 3 through Theorem 11.*

To summarize, for the eight monomial perturbations studied in this paper, we have identified all possible amplitude independent frequency synchronizers. Moreover, we claim that all isochronous cases are known in a fragmented literature; however, our study insures that for the studied family there are no other (necessary conditions). Each of them was found in a classification of a specific family of planar differential systems which are different from the context of the ones studied in this paper (synchronization). For isochronous centers of Liénard systems, the reader can see for instance Refs. [1,23]. Refs. [3,10] are concerned with the planar differential Liénard Type equations. For cubic time reversible systems see Refs. [1,19,20] and for cubic homogeneous perturbations of linear center see Refs. [1,20,21].

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