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Research Article

Wave Breaking for the Modified Two-Component Camassa-Holm System

Wujun Lv and Weiyi Zhu

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Correspondence should be addressed to Weiyi Zhu; zwy@zjnu.cn

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Some new sufficient conditions to guarantee wave breaking for the modified two-component Camassa-Holm system are established.

1. Introduction

This paper concerns the following modified two-component Camassa-Holm system (MCH2, for simplicity):

$$\begin{aligned} & u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = -g\rho\overline{\rho}_x, \\ & \qquad t > 0, \quad x \in \mathbb{R}, \\ & \rho_t + (\rho u)_x = 0, \quad t > 0, \ x \in \mathbb{R}, \\ & u(x, t = 0) = u_0(x), \quad x \in \mathbb{R}, \\ & \rho(x, t = 0) = \rho_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where $\rho(x,t) = (1-\partial_x^2)(\overline{\rho} - \overline{\rho}_0)(x,t)$, $u(x,t)$ expresses the velocity field, and g is the downward constant acceleration of gravity in applications to shallow water waves. In this paper, we let $g = 1$.

Let $\Lambda = (1-\partial_x^2)^{(1/2)}$; then the operator Λ^{-2} can be denoted by its associated Green's function $G = (1/2)e^{-|x|}$ as

$$\left(\Lambda^{-2} f\right)(x) = (G * f)(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} f(y)\, dy. \tag{2}$$

Let $\gamma(x,t) = (\overline{\rho} - \overline{\rho}_0)(x,t)$ and $(G * \rho)(x,t) = \gamma(x,t)$. So system (1) is equivalent to the following one:

$$\begin{aligned} & u_t + uu_x + \partial_x G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right) = 0, \\ & \qquad t > 0, \quad x \in \mathbb{R}, \\ & \gamma_t + u\gamma_x + G * \left(\left(u_x\gamma_x\right)_x + u_x\gamma\right) = 0, \quad t > 0, \ x \in \mathbb{R}, \\ & u(x, t = 0) = u_0(x), \quad x \in \mathbb{R}, \\ & \gamma(x, t = 0) = \gamma_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3}$$

The MCH2 system admits peaked solutions in the velocity and average density and we refer it to reference [1]. The local posedness, precise blow-up scenarios, and the existence of strong solutions which blow up in finite time can be found in [2–5]. Note that the MCH2 system is a modified version of the 2-component Camassa-Holm (CH2, for simplicity) system to allow a dependence on the average density $\overline{\rho}$ (or depth, in the shallow water interpretation) as well as the pointwise density ρ. Meanwhile, the MCH2 may not be integrable unlike the CH2 system. The characteristic is that it will amount to strengthening the norm for $\overline{\rho}$ from L^2 to H^1 in the potential energy term [5]. Also, the MCH2 admits the following conserved quantity:

$$E_1 = \int_{\mathbb{R}} \left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right) dx. \tag{4}$$

This paper mainly studies wave breaking phenomenon, and we aim at improving previous results which were proved in [3, 6]. Our method is partially motivated by [7]. The remaining of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we establish a new blow-up criterion for the MCH2. Finally, we establish a similar criterion for the CH2 system in Section 4.

2. Preliminaries

In this section, we recall some results without the proofs for conciseness. The first one is concerning local well-posedness and blow-up scenario.

Lemma 1 (see [2]). *Given $X_0 = (u_0, \gamma_0)^T \in H^s \times H^s$ to system* (3), *$s \geq 3/2$, there exists a maximal $T = T(\|X_0\|_{H^s\times H^s}) > 0$, and a unique solution $X = (u, \gamma)^T \in H^s \times H^s$ to system* (3). *Then the corresponding solutions blow up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}} \{u_x(x,t)\} = -\infty \quad or \quad \liminf_{t\to T}\inf_{x\in\mathbb{R}} \{\gamma_x(x,t)\} = -\infty. \tag{5}$$

We also need to introduce the standard particle trajectory [8]. Let $q(x,t)$ be the particle line evolved by the solution; that is, it satisfies

$$\begin{gathered} q_t = u(q,t), \quad 0 < t < T,\ x \in \mathbb{R}, \\ q(x,0) = x, \quad x \in \mathbb{R}. \end{gathered} \tag{6}$$

Taking the derivative with respect to x, we get

$$\frac{dq_t}{dx} = q_{xt} = u_x(q,t)\,q_x, \quad t \in (0,T). \tag{7}$$

Hence

$$q_x(x,t) = \exp\left\{\int_0^t u_x(q,s)\,ds\right\}, \qquad q_x(x,0) = 1. \tag{8}$$

Thus, the map $q(\cdot,t)$ is a diffeomorphism of the real line.

3. Blowup for the MCH2 System

In this section, we establish a new sufficient condition to guarantee blowup for system (3), which is an improvement of that in [3].

Theorem 2. *Suppose $X_0 = (u_0, \gamma_0)^T \in H^s \times H^s$ to system* (3), *$s > 3/2$ and $\rho_0(x_0) = 0$. And the initial data satisfies the following two conditions:*

$$\begin{aligned} (i)\quad & \rho_0(x_0) \geq 0 \quad on\ (-\infty, x_0), \\ & \rho_0(x_0) \leq 0 \quad on\ (x_0, \infty), \end{aligned} \tag{9}$$

$$(ii)\quad u_0'(x_0) < -|u_0(x_0)|, \tag{10}$$

for some point $x_0 \in \mathbb{R}$. Then the solution $X = (u, \gamma)^T$ to our system (3) *with initial value X_0 blows up in finite time.*

Remark 3. In [17] conditions $\int_{-\infty}^{x_0} e^{\xi} y_0(\xi)d\xi \geq 0$ and $\int_{x_0}^{\infty} e^{-\xi} y_0(\xi)d\xi \leq 0$ are needed to guarantee blowup, which implies condition (10). In addition, $y_0(x_0) = 0$ is required. So obviously Theorem 2 is an improvement of that in [3]. On the other hand, our condition is a local version and is easy to check. For nonlocal conditions, we refer to [5, 9].

Now we give a proof for Theorem 2.

Proof. Let us first consider the case $X_0 = (u_0, \gamma_0)^T \in H^2 \times H^2$. As in [10], we will look for $(d/dt)u_x(q(x,t),t)$. Applying $\partial_x^2(G * f) = G * f - f$ to differentiate (3) with respect to x yields

$$\begin{aligned} u_{tx} + uu_{xx} = &-\frac{1}{2}u_x^2 + u^2 + \frac{1}{2}\gamma^2 \\ &- \frac{1}{2}\gamma_x^2 - G * \left(\frac{1}{2}u_x^2 + u^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right). \end{aligned} \tag{11}$$

Let $0 < T < T^*$. Recalling that $u \in C^1([0,T), H^2)$, we show that u and u_x are continuous on $[0,T) \times \mathbb{R}$ and $x \to u(t,x)$ is Lipschitz, uniformly with respect to t in any compact time interval in $[0,T)$.

We get

$$\begin{aligned} &\frac{d}{dt}u_x(q(x_0,t),t) \\ &= (u_{tx} + uu_{xx})(q(x_0,t),t) \\ &= \left(-\frac{1}{2}u_x^2 + u^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right)(t, q(t,x_0)) \\ &\quad - G * \left(\frac{1}{2}u_x^2 + u^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right) \\ &\leq -\frac{1}{2}u_x^2 + \frac{1}{2}u^2, \end{aligned} \tag{12}$$

where we used $G*(u^2+(1/2)u_x^2) \geq (1/2)u^2$, $\gamma_x^2(x,t)-\gamma^2(x,t) \leq \gamma_x^2(q(x_0,t),t) - \gamma^2(q(x_0,t),t)$, and $\rho(q(x_0,t),t) = 0$.

As

$$\frac{d}{dt}\rho(q(x,t),t)\,q_x(x,t) = 0, \tag{13}$$

we get

$$\rho(q(x_0,t),t)\,q_x(x_0,t) = \rho_0(x_0) = 0; \tag{14}$$

it is easy to get $q_x(x_0,t) > 0$ in (8), so $\rho(q(x_0,t),t) = 0$.

Consider $\gamma_x^2(x,t)-\gamma^2(x,t) \leq \gamma_x^2(q(x_0,t),t)-\gamma^2(q(x_0,t),t)$; we can refer to [3].

The obvious factorization $u^2 - u_x^2 = (u-u_x)(u+u_x)$; this leads us to study the functions of the form:

$$\begin{aligned} I(x_0,t) &= e^{q(x_0,t)}(u - u_x)(q(x_0,t),t), \\ II(x_0,t) &= e^{-q(x_0,t)}(u + u_x)(q(x_0,t),t). \end{aligned} \tag{15}$$

Computing the derivatives with respect to t using the definition of the flow map (6) gives

$$\begin{aligned} I_t(x_0,t) &= e^{q(x_0,t)}\left[u^2 - uu_x + (u_t + uu_x) - (u_{xt} + uu_{xx})\right](q(x_0,t),t) \\ &= e^{q(x_0,t)}\left[-uu_x + \frac{1}{2}u_x^2 - \frac{1}{2}(\gamma^2 - \gamma_x^2) + (G - \partial_x G) * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}(\gamma^2 - \gamma_x^2)\right)\right] \\ &\geq e^{q(x_0,t)}\left(\frac{1}{2}u^2 - uu_x + \frac{1}{2}u_x^2\right) \\ &= \frac{1}{2}e^{q(x_0,t)}(u - u_x)^2 \geq 0. \end{aligned} \tag{16}$$

In fact, the next lemma will be used. □

Lemma 4. *Consider*

$$(G \pm \partial_x G) * \left(u^2 + \frac{1}{2}u_x^2\right) \geq \frac{1}{2}u^2. \tag{17}$$

Proof. Consider

$$\begin{aligned} &\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}\left(u^2 + u_x^2\right)(\xi)\,d\xi \\ &\quad \geq e^{-x}\int_{-\infty}^{x} e^{\xi} uu_x d\xi = \frac{1}{2}u^2(x) - \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}u^2(\xi)\,d\xi. \end{aligned} \tag{18}$$

So we get

$$\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}\left(u^2 + \frac{1}{2}u_x^2\right)(\xi)\,d\xi \geq \frac{1}{4}u^2. \tag{19}$$

The same computations also obtain that

$$\frac{1}{2}e^{x}\int_{-\infty}^{x} e^{-\xi}\left(u^2 + \frac{1}{2}u_x^2\right)(\xi)\,d\xi \geq \frac{1}{4}u^2. \tag{20}$$

We have

$$\begin{aligned} (G - \partial_x G) &= e^{-x}\int_{-\infty}^{x} e^{\xi}\left(u^2 + \frac{1}{2}u_x^2\right)(\xi)\,d\xi, \\ (G + \partial_x G) &= \frac{1}{2}e^{x}\int_{-\infty}^{x} e^{-\xi}\left(u^2 + \frac{1}{2}u_x^2\right)(\xi)\,d\xi; \end{aligned} \tag{21}$$

taking the linear combination in the two last inequalities implies estimate (17). □

Similarly,

$$II_t(x_0,t) = -\frac{1}{2}e^{-q(x_0,t)}(u + u_x)^2 \leq 0. \tag{22}$$

It is convenient to establish the following fundamental proposition.

Proposition 5. *u as in Theorem 2. Set*

$$\begin{aligned} I(x_0,t) &= e^{q(x_0,t)}(u - u_x)(q(x_0,t),t), \\ II(x_0,t) &= e^{-q(x_0,t)}(u + u_x)(q(x_0,t),t). \end{aligned} \tag{23}$$

Then, for all $x \in \mathbb{R}$, the function $t \to I(x_0,t)$ is monotonically increasing and $t \to II(t,x_0)$ is monotonically decreasing.

It is easy to factorize

$$\left(u^2 - u_x^2\right)(q(x_0,t),t) = I(x_0,t)\,II(x_0,t); \tag{24}$$

from inequality (12) we get

$$\frac{d}{dt}u_x(q(x_0,t),t) \leq \frac{1}{2}I(x_0,t)\,II(x_0,t). \tag{25}$$

Now let x_0 be such that $u_0'(x_0) < -|u_0(x_0)|$. Proposition 5 yields, for all $t \in [0,T)$,

$$I(x_0,t) \geq I_0(x_0) > 0, \qquad II(x_0,t) \leq II_0(x_0) < 0, \tag{26}$$

where we used $u_0'(x_0) < -|u_0(x_0)|$, then we get $I_0(x_0) > 0$ and $II_0(x_0) < 0$.

Assume, by contradiction, $T = \infty$; set $A(t) = u_x(q(x_0,t),t)$; thus we get

$$A'(t) \leq \frac{1}{2}I(x_0,t)\,II(x_0,t) \leq \frac{1}{2}I_0(x_0)\,II_0(x_0) < 0. \tag{27}$$

Set $\beta_0 = (1/2)(u_0'^2 - u_0^2)(x_0)$; then $A(t) \leq A(0) - \beta_0 t$; we can find t_0 such that $(A(0) - \beta_0 t_0)^2 \geq E_1(E_1 = \|u(t) + \gamma(t)\|_{H^1}^2 = \|u_0 + \gamma_0\|_{H^1}^2)$. For $t \geq t_0$, then $A(t) \leq A(t_0)$; we obtain

$$\begin{aligned} A'(t) &\leq \frac{1}{2}I(x_0,t)\,II(x_0,t) = \frac{1}{2}\left(u^2 - u_x^2\right)(q(x_0,t),t) \\ &\leq \frac{1}{2}\left(\frac{1}{2}E_1 - A(t)^2\right) \\ &\leq -\frac{1}{4}A(t)^2. \end{aligned} \tag{28}$$

This implies that, for $t \geq t_0$,

$$A(t) \leq \frac{4A(t_0)}{4 - (t - t_0)A(t_0)}. \tag{29}$$

From above, $u_x(q(x_0,t),t)$ must blow up in finite time, and $T^* = t_0 + 4/A(t_0) < \infty$, so the condition of the blowup scenario (5) is fulfilled.

4. Blowup for the CH2 System

In this section, we consider the following two-component Camassa-Holm system:

$$\begin{aligned} u_t + uu_x + \partial_x\left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{\delta}{2}\rho^2\right)\right) &= 0, \\ t > 0, \quad x \in \mathbb{R}&, \\ \rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}&. \end{aligned} \tag{30}$$

The CH2 system appears initially in [11]. Wave breaking mechanism was discussed in [3, 12–14]. The existence of global solutions was analyzed in [6, 15, 16]. This system also has the following conservation laws [17]:

$$E_1 = \int_{\mathbb{R}} \left(u^2 + u_x^2 + \delta\rho^2\right) dx,$$
$$E_2 = \int_{\mathbb{R}} \left(u^3 + uu_x^2 + \delta u\rho^2\right) dx. \quad (31)$$

In [6], a blow-up condition is established as $y_0(x_0) = 0$, $\int_{-\infty}^{x_0} e^{\xi} y_0(\xi)d\xi \geq 0$ and $\int_{x_0}^{\infty} e^{-\xi} y_0(\xi)d\xi \leq 0$; here $y_0(x_0) = (1-\partial_x^2)u_0(x_0)$. Similar to Theorem 2, we can do the following improvement.

Theorem 6. *Suppose* $X_0 = (u_0, \rho_0)^T \in H^s \times H^{s-1}$ *to system* (30), $s \geq 3/2$, *and* $\rho(x_0) = 0$; *furthermore*

$$u_0'(x_0) < -\left|u_0(x_0)\right|, \quad (32)$$

for some point $x_0 \in \mathbb{R}$. *Then the solution to our system* (30) *with initial value* X_0 *blows up in finite time.*

The proof is similar to Theorem 2 and we omit it.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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