Maximizing the Fundamental Natural Frequency of Triangular Composite Plates

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While many advances were made in the analysis of composite structures, it is generally recognized that the design of composite structures must be studied further in order to take full advantage of the mechanical properties of these materials. This study is concerned with maximizing the fundamental natural frequency of triangular, symmetrically laminated composite plates. The natural frequencies and mode shapes of composite plates of general triangular planform are determined using the Rayleigh-Ritz method. The plate constitutive equations are written in terms of stiffness invariants and nondimensional lamination parameters. Point supports are introduced in the formulation using the method of Lagrange multipliers. This formulation allows studying the free vibration of a wide range of triangular composite plates with any support condition along the edges and point supports. The boundary conditions are enforced at a number of points along the boundary. The effects of geometry, material properties and lamination on the natural frequencies of the plate are investigated.

With this stiffness invariant formulation, the effects of lamination are described by a finite number of parameters regardless of the number of plies in the laminate. We then determine the lay-up that will maximize the fundamental natural frequency of the plate. It is shown that the optimum design is relatively insensitive to the material properties for the commonly used material systems. Results are presented for several cases.

Introduction

Early studies on the vibrations of triangular plates were reviewed by Leissa (1969, 1977, 1981, 1987). Recently, Gorman (1986) presented an extensive study of free vibrations of isotropic right triangular plates with combinations of clamped and simply supported boundary conditions. A method of superposition, previously developed by Gorman, was utilized and, later, Gorman (1989a) used the same method to study the free vibration of isosceles, simply supported, triangular plates. Gorman (1989b) also studied right triangular plates with one free edge using his superposition technique. A comprehensive analysis of free vibration of right triangular isotropic plates, considering all combinations of boundary conditions was presented by Kim and Dickinson (1990). The Rayleigh-Ritz method with simple polynomial approximation functions was used. Application to orthotropic plates was illustrated by one example. Bhat (1987) used the Rayleigh-Ritz method with orthogonal polynomials to determine the natural frequencies of isotropic triangular plates of general planform. The Rayleigh-Ritz method was also applied to isotropic triangular plates of general shapes by Kim and Dickinson (1992). Extensive results are provided for isosceles plates with many combinations of boundary conditions. Plates of other planforms and orthotropic plates are also considered. The free vibration of right triangular and equilateral triangular plates with various support conditions along the edges was also studied by Singh and Chakraverty (1992). More recently, Leissa and Jaber (1992) presented a comprehensive study of the free vibration of completely free, isotropic, triangular plates. While in all the studies mentioned above the classical plate theory is used, Kitipornchai et al. (1993) analyzed the free vibration of isosceles triangular plates using the Mindlin shear deformable plate theory.

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received Feb. 1994. Associate Technical Editor: K. Wang.

The study of free vibration of laminated composite plates of triangular shapes has received little attention. Malhotra et al. (1989) used the finite element method to find the natural frequencies of single layer composite plates of isosceles triangular planform. Three sets of boundary conditions were considered. Liew, Lam, and Chow (1989) used the Rayleigh-Ritz method with orthogonal approximation functions to determine the natural vibration frequencies of right triangular single layer composite plates. Four combinations of boundary conditions were considered and the effect of fiber orientation was considered. Lam, Liew, and Chow (1990) also considered the free vibration of right triangular, single layer, composite plates using the Rayleigh-Ritz method and orthogonal polynomial approximation functions. The same combinations of boundary conditions as in Liew, Lam, and Chow (1989) were considered and mode shapes are presented for special cases.

The present investigation into the optimization of vibrating triangular plates has three objectives. The first objective is to develop a general approach for determining the natural frequencies and mode shapes of triangular plates of arbitrary planform and support conditions. Then, the constitutive equations will be written in terms of a finite number of lamination parameters in order to determine which lay-up will maximize the fundamental natural frequency. Finally, examples will be presented to verify the validity of the present formulation and to show sample results.

General Formulation

The transverse motion of symmetrically laminated plates is completely uncoupled from the in-plane motion and, according to the classical small deflection plate theory, the free vibration of such plates is governed by the equation of motion (Vinson and Sierakowski, 1987)

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} = \rho w_{,tt} \quad (1)$$

where ρ is the mass of the plate per unit area, w is the transverse

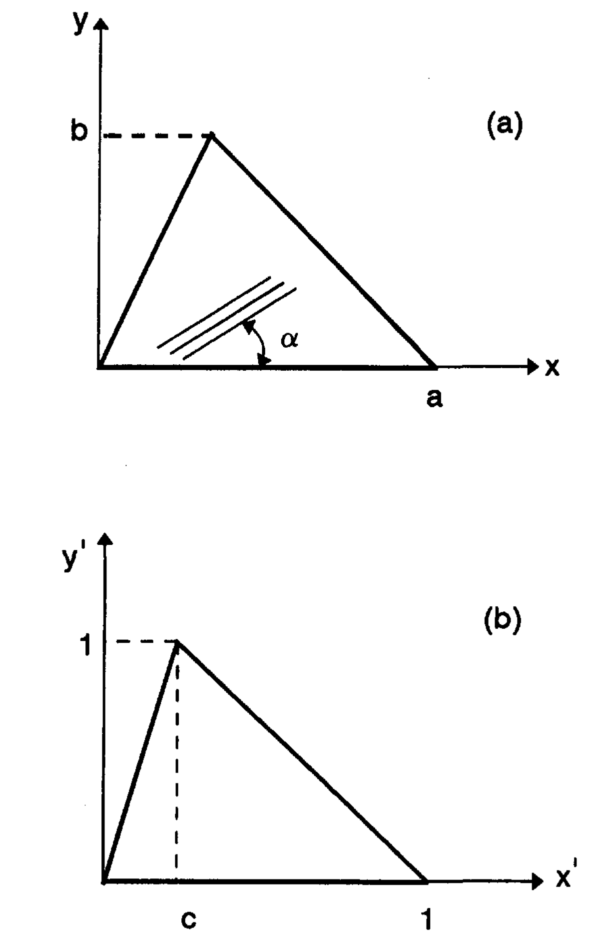

Fig. 1 Triangular plates: (*a*) physical plane, (*b*) nondimensionalized

displacement, and $M_{,x}$, $M_{,y}$ and $M_{,xy}$ are the moment resultants which are related to the plate curvatures as

$$\{M\} = [D]\{\kappa\} \tag{2}$$

or more explicitly by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix} \tag{3}$$

The D_{ij}'s in Eq. (3) are the bending stiffnesses of the plate. Multiplying Eq. (1) by a test function v, integrating over Ω, the domain occupied by the plate, and using the Gauss divergence theorem, the following variational eigenvalue problem is obtained

$$\int_\Omega \{\bar{\kappa}\}^T[D]\{\kappa\}d\Omega = \omega^2 \int_\Omega \rho v w d\Omega \tag{4}$$

where $\{\bar{\kappa}\} = [-v_{,xx}, -v_{,yy}, -2v_{,xy}]$ and ω is the natural frequency. The test function v must satisfy the essential boundary conditions of the problem. This weak formulation (Eq. 4) corresponds to the minimum of the quadratic functional

$$I(w) = \frac{1}{2}\int_\Omega \{\kappa\}^T[D]\{\kappa\}d\Omega - \frac{\omega^2}{2}\int_\Omega \rho w^2 d\Omega \tag{5}$$

where the first term represents the maximum strain energy and the second term the maximum kinetic energy.

In this investigation, triangular plates with one edge along the x-axis (Fig. 1) will be considered. An N-term approximation for the transverse displacement is taken as

$$w_N = \sum_{j=1}^{N} c_j \phi_j(x, y) \tag{6}$$

where the c_j's are constants to be determined and the approximation functions ϕ_j satisfy the essential (or displacement) boundary conditions along the edge $y = 0$. Using polynomial approximation functions

$$\phi_j(x, y) = x^m y^n \tag{7}$$

where $m \geq 0$ and n starts from 0, 1, or 2 depending on whether the edge $y = 0$ is free, simply supported or clamped. These approximation functions (Eq. 7) satisfy the essential boundary conditions along the edge $y = 0$; and can be used to study plates that are free along the other three edges. Polynomial functions that satisfy all essential boundary conditions can easily be found. However, as mentioned in Leissa (1987), they will necessarily be more complex and will make the evaluation of the mass and stiffness coefficients more difficult. Here, the functions given by Eq. (7) will be used, and the boundary conditions along the other two edges will be enforced at discrete points using the Lagrange Multiplier method. Requiring that the displacement or the slope normal to the boundary vanish at a number of points amounts to imposing p constraint equations

$$g_k(x_k, y_k) = 0 \quad k = 1 \ldots p \tag{8}$$

where $g_k(x_k, y_k) = w(x_k, y_k)$ when the displacement at point k is constrained, and $g_k(x_k, y_k) = (\partial/\partial\bar{n})w(x_k, y_k)$ when the slope in the direction of $\bar{n}$, the normal to the boundary, is forced to vanish. The modified functional for the problem is given by

$$\Pi(w, \bar{\lambda}) = I(w) + \sum_{k=1}^{p} \lambda_k g_k(x_k, y_k) \tag{9}$$

where $I(w)$ is the functional given in Eq. (5), and the λ_k's are the Lagrange multipliers which can be grouped in the vector $\bar{\lambda}$. If w is replaced by w_N, the following eigenvalue problem is obtained

$$\begin{bmatrix} K & K_M \\ K_M^T & 0 \end{bmatrix} \begin{Bmatrix} \underline{c_j} \\ \lambda_k \end{Bmatrix} = \omega^2 \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \underline{c_j} \\ \lambda_k \end{Bmatrix} \tag{10}$$

The matrix K_M has dimension $(N \times p)$ and each column corresponds to one of the constraints being enforced. For example, if constraint k is $w(x_k, y_k) = 0$, then $K_M(i, k) = \phi_i(x_k, y_k)$. This yields the $(N + p) \times (N + p)$ eigenvalue problem where the stiffness and mass matrices are given by

$$K_{ij} = \int_\Omega \{D_{11}\phi_{i,xx}\phi_{j,xx} + D_{22}\phi_{i,yy}\phi_{j,yy} + D_{12}(\phi_{i,xx}\phi_{j,yy} + \phi_{i,yy}\phi_{j,xx}) + 4D_{66}\phi_{i,xy}\phi_{i,xy} + 2D_{16}(\phi_{i,xx}\phi_{j,xy} + \phi_{i,xy}\phi_{j,xx}) + 2D_{26}(\phi_{i,yy}\phi_{j,xy} + \phi_{i,xy}\phi_{j,yy})\}d\Omega \tag{11}$$

$$M_{ij} = \int_\Omega \rho\phi_i\phi_j d\Omega \tag{12}$$

The eigenvalues and eigenvectors are obtained from Eq. (10) using the inverse iteration method (Bathe, 1982). The calculations are simplified after nondimensionalizing as shown in Fig. 1. To evaluate the elements of the stiffness and mass matrices defined by Eqs. (11, 12), integrals of the form

$$F(\alpha, \beta) = \int_0^1 (y')^\beta \left(\int_{cy'}^{1-(1-c)y'} (x')^\alpha dx' \right) dy' \tag{13}$$

where the powers α and β depend on the approximation functions used and the order of their derivatives in Eqs. (11, 12). The result

Table 1 Natural frequencies of isotropic FCF right triangular plates $\Omega = \omega a^2(\rho/D)^{1/2}$

b/a		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1.0	*	6.168	23.46	32.69	56.18	76.53	100.1
	6x6	6.168	23.46	32.69	56.18	76.53	100.1
1.5	*	2.876	12.11	17.82	29.66	42.76	55.98
	6x6	2.876	12.11	17.82	29.66	42.76	56.86
	6x8	2.875	12.10	17.82	29.66	42.75	55.95
2.0	*	1.656	7.111	12.35	17.42	29.24	33.30
	6x6	1.656	7.111	12.35	17.42	29.24	34.01
	6x8	1.656	7.109	12.35	17.41	29.20	33.27
2.5	*	1.074	4.634	9.449	11.36	20.58	23.37
	6x6	1.076	4.640	9.490	11.37	21.53	23.71
	6x8	1.073	4.632	9.444	11.35	20.57	23.31
3.0	*	.7514	3.248	7.570	8.040	14.59	18.50
	6x6	.7515	3.248	7.571	8.040	14.94	18.52
	6x8	.7510	3.247	7.566	8.034	14.59	18.47

* Kim, C. S., and Dickinson, S. M. (1990).

$$F(\alpha, \beta) = \left\{ \sum_{q=0}^{\alpha+1} \binom{\alpha+1}{q} \frac{(c-1)^q}{\beta+q+1} - \frac{c^{\alpha+1}}{\alpha+\beta+2} \right\} \Big/ (\alpha+1) \quad (14)$$

allows one to evaluate the mass and stiffness matrices exactly.

The bending stiffnesses in Eq. (3) can be written as

$$\begin{Bmatrix} D_{11} \\ D_{22} \\ D_{12} \\ D_{66} \\ D_{16} \\ D_{26} \end{Bmatrix} = \frac{h^3}{12} \begin{bmatrix} 1 & \zeta_9 & \zeta_{10} & 0 & 0 \\ 1 & -\zeta_9 & \zeta_{10} & 0 & 0 \\ 0 & 0 & -\zeta_{10} & 1 & 0 \\ 0 & 0 & -\zeta_{10} & 0 & 1 \\ 0 & +\zeta_{11}/2 & \zeta_{12} & 0 & 0 \\ 0 & -\zeta_{11}/2 & -\zeta_{12} & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} \quad (15)$$

where h is the thickness of the laminate. The stiffness invariants are defined as

$$U_1 = [Q_{11} + 2(Q_{12} + 2Q_{66}) + Q_{22}]/4$$
$$U_2 = [Q_{11} - Q_{22}]/2$$
$$U_3 = [Q_{11} - 2(Q_{12} + 2Q_{66}) + Q_{22}]/4$$
$$U_4 = [Q_{11} + 2(Q_{12} - 2Q_{66}) + Q_{22}]/4$$
$$U_5 = [Q_{11} - 2Q_{12} + Q_{22}]/4 \quad (16)$$

(Fukunaga, 1986, 1990; Fukunaga and Chou, 1988; Fukunaga and Vanderplaats, 1991), where the Q_{ij}'s are the reduced stiffnesses of a lamina which can be calculated knowing the engineering constants E_1, E_2, v_{12}, G_{12} for the material system used. The lamination parameters are defined as

$$\zeta_9 = \frac{12}{h^3}\int_{-h/2}^{+h/2} z^2 \cos 2\theta dz,$$
$$\zeta_{10} = \frac{12}{h^2}\int_{-h/2}^{+h/2} z^2 \cos^2 2\theta dz,$$
$$\zeta_{11} = \frac{12}{h^3}\int_{-h/2}^{+h/2} z^2 \sin 2\theta dz,$$
$$\zeta_{12} = \frac{12}{h^3}\int_{-h/2}^{+h/2} z^2 \sin 2\theta \cos 2\theta dz \quad (17)$$

where θ, the fiber orientation with respect to the x axis, is a function of z as it changes from ply to ply. For generally laminated plates, a maximum of twelve parameters are needed to describe the effect of lamination, but since we are considering only symmetric laminates, only the four parameters in Eq. (17) are needed. It can be shown that $-1 \le \zeta_9 \le 1$ and $\zeta_9^2 \le \zeta_{10} \le 1$ so that, for all laminates, the bending rigidities D_{11}, D_{22}, D_{12} and D_{66} depend only on the two parameters ζ_9 and ζ_{10} which vary in the domain between the parabola $\zeta_{10} = \zeta_9^2$ and the line $\zeta_{10} = 1$. Each symmetric angle-ply laminate is represented by a point on the parabola, cross-ply laminates are represented by a point on the line $\zeta_{10} = 1$. The advantages of using this stiffness invariant formulation are that the effect of laminate thickness, lay-up and material properties are separated, Eq. (15), and only 4 parameters are required to describe all symmetric laminates regardless of the number of plies.

Results

Boundary conditions along the edges are designated by three letters, starting with the left side, the bottom side ($y = 0$), and the right side. Letters C, S, F stand for clamped, simply supported, and free boundary conditions. In this study, results are presented in nondimensional form so that the densities of the materials are not needed. The first six natural frequencies of FCF, right triangular ($c = 0$), isotropic plates ($\nu = .3$) are in good agreement with the results given by Kim et al. (1990) for a wide range of aspect ratios (Table 1). The second column in Table 1 gives the source of the results. Those obtained in this study are denoted by two numbers indicating the number of terms in the displacement approximation. The first number is the number of terms in the x direction and the second number is the number of terms in the y direction. For example, 6×8 indicates that a 48 term approximation is used and, for a FCF plate, m and n in Eq. (7) vary from 0 to 5 and from 2 to 9, respectively. CFF, right triangular plates ($c = 0$) made out of unidirectional graphite-epoxy with fibers oriented in the x-direction, were also considered by Kim and Dickinson (1990). Nondimensional bending stiffnesses were given as

$$D_{11}/H = 10.548, \quad D_{22}/H = 0.6002,$$
$$D_{12}/H = 0.16806, \quad D_{66}/H = 0.41597, \quad (18)$$

and the present results (Table 2) agree well with those in Kim and Dickinson (1990). The boundary conditions on the edge $x = 0$ are satisfied taking approximation functions, Eq. (7), with $m \ge 2$ and, since the edge $y = 0$ is free, n starting from 0. The free vibration of graphite-epoxy, FCF, isosceles ($c = 0.5$), triangular plates with fibers oriented in the x-direction were studied by Kim and Dickinson (1992). In that case also, with the stiffnesses given by Eqs. (18), the present results (Table

Table 2 Natural frequencies of graphite-epoxy CFF right triangular ($c = 0$) plates $\Omega = \omega a^2(\rho/H)^{1/2}$

b/a	Source	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1.0	*	16.78	41.72	83.58	95.08	137.0	173.2
	6x6	16.78	41.72	84.58	95.13	140.0	183.4
	8x6	16.78	41.71	83.44	94.78	136.9	173.7
1/3	*	20.31	80.22	114.2	202.8	277.4	386.6
	6x6	20.31	80.23	114.2	202.8	277.4	386.6
3	*	14.18	23.28	34.44	48.36	69.40	81.54
	6x6	14.19	23.61	37.76	60.86	81.39	100.7
	9x6	14.18	23.28	34.55	48.80	71.92	81.11

* Kim, C. S., and Dickinson, S. M. (1990).

Table 3 Natural frequencies of graphite-epoxy, cantilever (FCF), isosceles (c = .5), triangular plates $\Omega = \omega a^2(\rho/H)^{1/2}$

b/a		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
.25	**	78.82	200.1	333.9	474.3	560.4	806.9
	6x4	78.82	200.1	335.3	484.4	564.1	850.4
	8x4	78.82	200.1	334.8	475.2	561.5	810.5
1.0	**	5.503	23.81	30.10	57.75	76.41	106.4
	4x4	5.505	23.88	30.15	59.15	78.03	151.1
	6x6	5.503	23.81	30.10	57.79	76.44	109.4
3.0	**	.6145	2.666	6.482	8.069	11.99	19.07
	4x4	.6149	2.677	6.668	8.110	18.32	19.42
	4x6	.6146	2.666	6.489	8.072	12.46	19.08
	4x8	.6146	2.666	6.482	8.070	12.00	19.08

** Kim, C. S., and Dickinson, S. M. (1992).

3) are again in good agreement with those previously reported. The first six natural frequencies of a SSS, right triangular (c = 0), isotropic plate are obtained by modeling the simple support boundary condition along the hypothenuse by enforcing the zero displacement constraint at a finite number of points equally spaced along that edge. The zero displacement constraints along the $x = 0$ and $y = 0$ edges are enforced by taking m and n in Eq. (7) to be strictly positive. Results (Table 4) show that good agreement with previous results (Kim and Dickinson, 1990) is obtained when the constraint is enforced at a large enough number of points and when enough terms are taken in the displacement approximation. The number of points p at which the zero displacement constraint is enforced along the hypothenuse is given in the third column of Table 4. As the number increases, so do the natural frequencies, while, as the number of terms in the approximation increases, the frequencies decrease. The results presented in Tables 1–4 validate the basic approach for calculating the natural frequencies of triangular plates and show that any symmetrically laminated plate can be studied.

When the number of plies in a laminate becomes larger than six, the effect of the bending-twisting coupling terms D_{16} and D_{26} becomes negligible (Ashton and Whitney, 1970). In the remaining examples to be considered here, only plates with many layers will be analyzed. This simplification reduces the number of independent parameters required to describe all symmetric laminates from four to two (ζ_9, ζ_{10}). In the following, the elastic properties of graphite-epoxy: E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = .28, G_{12} = 7.17 GPa, will be used, and the

Table 4 Natural frequencies of isotropic right triangular (c = 0) SSS plate $\Omega = \omega a^2(\rho/D)^{1/2}$

b/a	Source	p	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1	6x6	4	49.230	96.834	123.51	142.40	192.50	221.93
	6x6	5	49.353	99.081	128.48	166.61	186.99	230.91
	6x6	6	49.355	99.13	128.98	171.10	209.13	230.76
	6x6	7	49.356	99.140	129.00	171.41	210.24	245.33
	6x6	8	49.356	99.145	129.01	171.55	210.75	242.67
	*		49.35	98.76	128.4	169.1	200.3	249.8
2	6x6	8	27.78	50.58	77.54	91.20	118.34	149.17
	7x7	8	27.76	50.11	76.05	82.73	112.47	136.17
	*		27.76	49.91	74.85	81.84	107.4	122.2
3	5x10	8	21.85	35.61	51.45	66.69	71.14	90.80
	*		21.85	35.63	51.27	66.73	71.03	92.84

* Refers to Kim, C. S., and Dickinson, S. M. (1990).

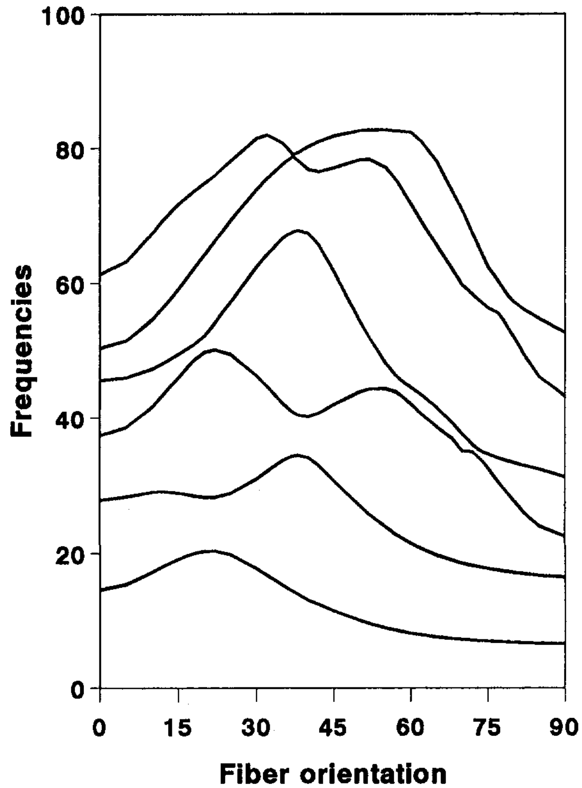

Fig. 2 Natural frequencies of FFF right triangular graphite-epoxy plate (a/b = 2)

frequencies are normalized as $\Omega = \omega a^2\,[12(1 - \nu_{12}\nu_{21})\rho/h^3E_1]^{1/2}$.

Figure 2 shows the evolution of the first six natural frequencies of an FFF, angle-ply laminated, right triangular (c = 0, a/b = 2) graphite-epoxy plate with many plies as a function of fiber orientation. These results were obtained using an 8 × 8 displacement approximation. Instances of curve veering and curve-crossing can be seen. The first natural frequency reaches a maximum with an angle ply laminate with fiber orientation

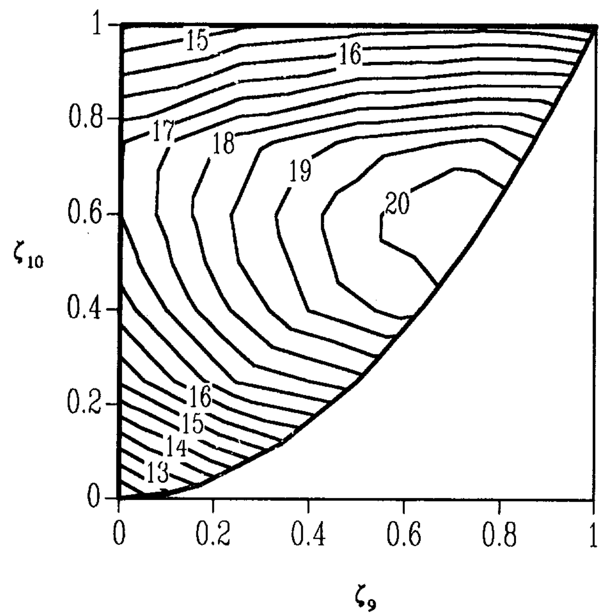

Fig. 3 Variation of first natural frequency of an FFF right triangular graphite-epoxy plate (a/b = 2) with lamination parameters ξ_9 and ξ_{10}

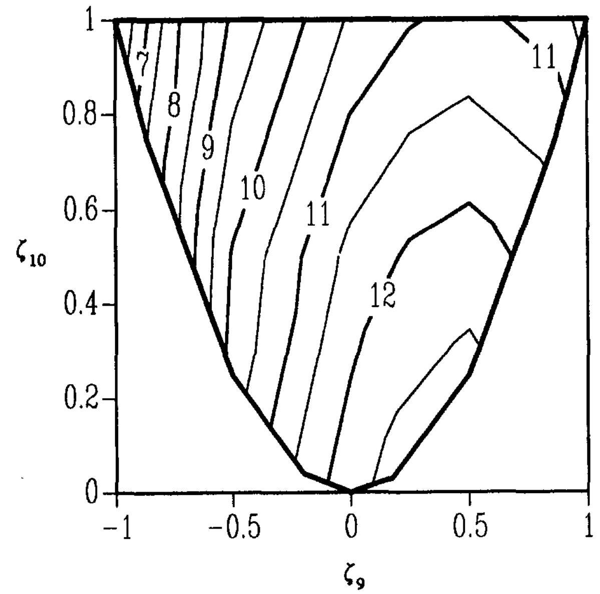

Fig. 4 Variation of first natural frequency of a CSF right triangular graphite-epoxy plate ($a/b = 1$) with lamination parameters ζ_9 and ζ_{10}

of ±21 degrees (Fig. 2). Figure 3 shows that, when a large range of values for the two lamination parameters are used, the maximum does occur very close to the parabola $\zeta_9^2 = \zeta_{10}$. The optimal values of the lamination parameters are $\zeta_9 = .75$ and $\zeta_{10} = .57$ and the fundamental natural frequency for that layup is only 0.08 percent higher than that of the ±21 deg. angle-ply laminate. For all practical purposes, the optimum lay-up is an angle-ply laminate.

Figure 4 shows that the fundamental natural frequency of a CSF right triangular plate ($c = 0$) with many plies and an aspect ratio of 1 is maximal when an angle-ply laminate is used. The variation of the first natural frequency of CSF, angle-ply laminated, right triangular plates with lamination angle is shown in Fig. 5 for 5 different values of a/b. For very small aspect ratios, the optimum lay-up is a unidirectional laminate with fibers oriented in the x-direction. As the aspect ratio increases, the optimum orientation angle tends towards ±45 deg. The variation of the optimum fiber orientation with aspect ratio is shown in Fig. 6. Notice that the optimum fiber orientation for boron-epoxy laminates (not shown here) with $E_1 = 209$ GPa, $E_2 = 19$ GPa, $\nu_{12} = .21$, $G_{12} = 6.4$ GPa is identical to that of graphite-epoxy laminates. With glass-epoxy laminates, $E_1 = 38.6$ GPa, $E_2 = 8.27$ GPa, $\nu_{12} = .26$, $G_{12} = 4.14$ GPa and a slight difference is observed as a/b becomes very small. Therefore, one can conclude that with commonly used material systems, the optimum fiber orientation is almost the same. Figure 7 shows the variation of the first natural frequency with fiber orientation for angle-ply laminates with 1, 6, 8, and many plies for a CSF right-angled triangular plate ($c = 0$, $a/b = 1$). These results indicate that as the number of plies in the laminate become large the effect of the bending-twisting coupling terms on the natural frequencies and the optimal design becomes small.

The last example is a graphite-epoxy, isosceles plate ($c = .5$, $a/b = 1$) that is simply supported along the edge $y = 0$ and has a point support at the opposite corner. Figure 8 indicates that, in that case, the optimum lay-up is not an angle-ply laminate even though the maximum is quite close to the parabola $\zeta_9^2 = \zeta_{10}$. It is to be noticed that, for this example, the first and second natural frequencies are equal at the optimum point whereas they are distinct for the previous examples. When the optimum does not lie on the boundary of the feasible domain

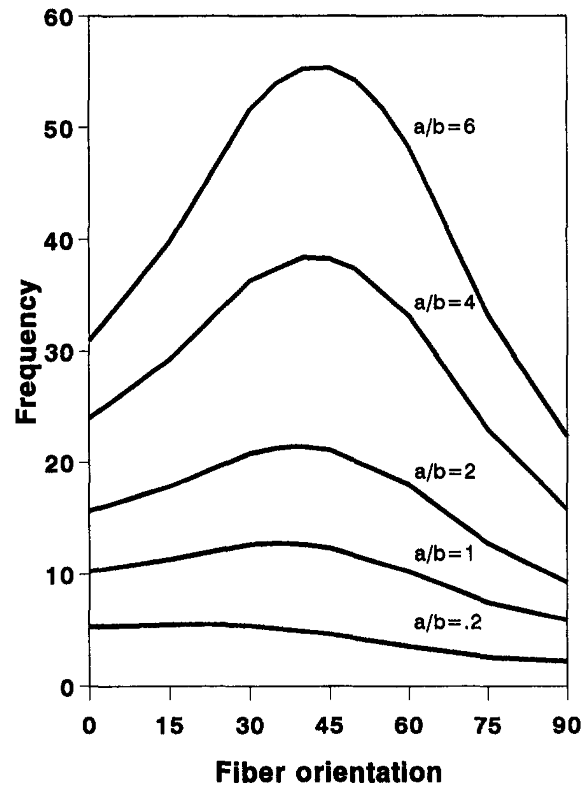

Fig. 5 Variation of the first natural frequency of an angle-ply laminated, graphite-epoxy, right triangular CSF plate with fiber orientation

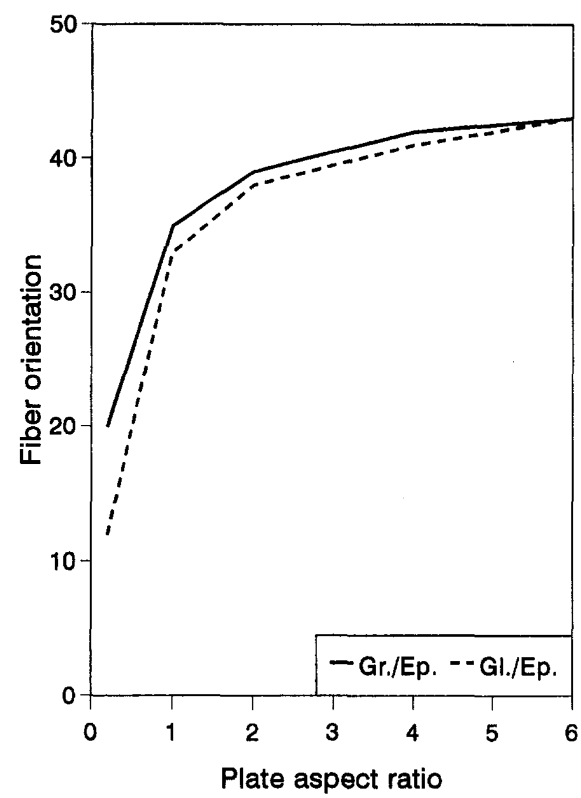

Fig. 6 Optimum fiber orientation for CSF right triangular plates as a function of aspect ratio

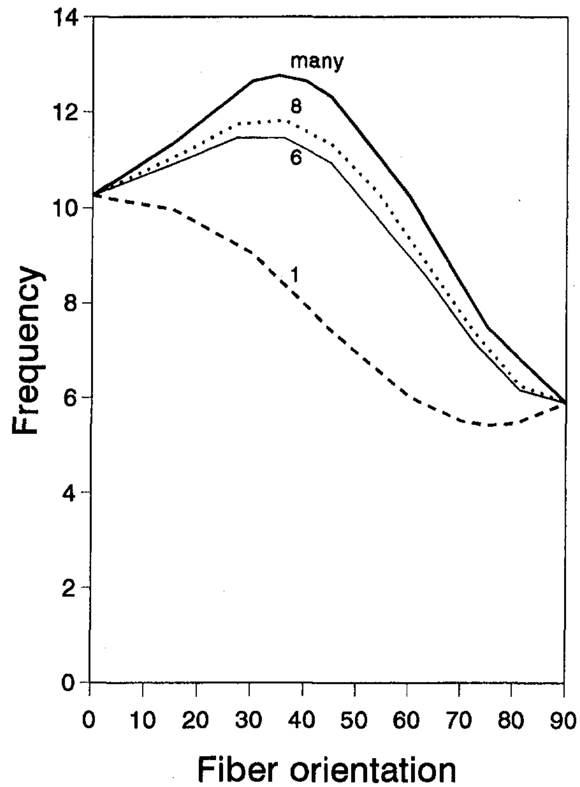

Fig. 7 First natural frequency of CSF right triangular angle-ply laminated plates as a function of fiber orientation and the number of plies

in the lamination parameter space, there are many lay-ups with the same combination of lamination parameters.

Conclusions

A general approach was presented to study the free vibration of triangular, symmetrically laminated plates of general planform and support conditions. The Rayleigh-Ritz method with simple polynomial approximation functions is used and additional displacement constraints are enforced using the method of Lagrange multipliers. For all symmetric laminates, the bending rigidities of the plate are expressed in terms of four nondimensional lamination parameters. As the number of plies becomes large, only two parameters are needed. Four examples are used to show the validity of the present formulation by comparison with highly accurate results presented by previous investigators.

With this model, the lamination scheme that will maximize the first natural frequency is determined for several examples. It is shown that, in most cases, the optimal lay-up is an angle ply laminate but, one example is presented to show that this is not always the case. The advantage of the present approach is that all triangular plates can be optimized and that the optimum combination of lamination parameters is determined without selecting a particular lamination scheme a-priori. The true optimum can thus be reached and, in some cases, many lay-ups will have the same combination of lamination parameters. For common material systems the optimal lay-ups has only a weak dependence on material properties.

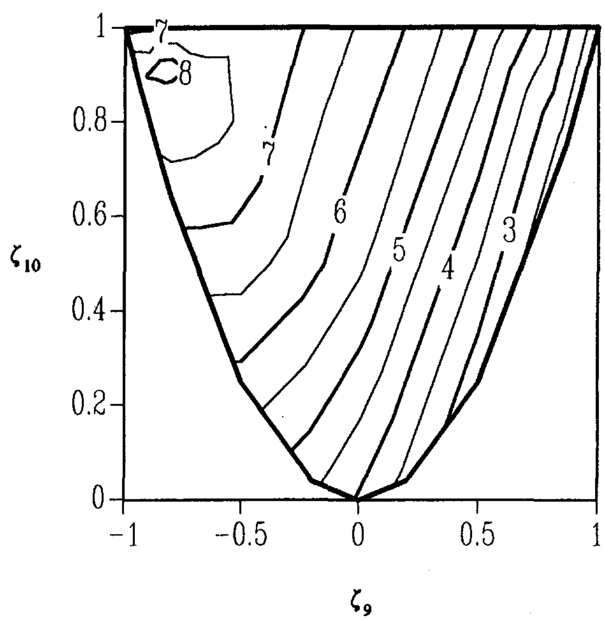

Fig. 8 Variation of first natural frequency of an isosceles triangular graphite-epoxy plate ($a/b = 1$) with lamination parameters ζ_9 and ζ_{10}

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