Lost in Translation: Hybrid-Time Flows vs Real-Time Transitions

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Abstract. Recently, hybrid-time flow systems have been introduced as an extension to timed transition systems, hybrid automata, continuous time evolutions of differential equations etc. Furthermore, a number of notions of bisimulation have been defined on these flow systems reflecting abstraction from certain timing properties. In this paper, we research the difference in abstraction level between this new semantic model of flow systems, and the more traditional model of real-time transition systems. We explore translations between the old and new semantic models, and we give a necessary and sufficient condition, called *finite-set refutability*, for these translations to be without loss of information. Finally, we show that differential inclusions with an upper-semicontinuous, closed and convex right-hand side, are finite-set refutable, and easily extend this result to impuls differential inclusions and hybrid automata.

1 Introduction

In the literature on hybrid systems, a variety of semantic models is used to describe the combined discrete and continuous behavior of these systems. In Henzinger's early paper on hybrid automata [1], a real-time transition system semantics was used. Later, the timing on the transitions was replaced by flows, resulting in hybrid transition systems. This has enabled the definition of all kinds of compositions of hybrid systems in a more operational way by means of Hybrid I/O automata [2] and a wide range of hybrid process algebras and calculi [3–6]. Finally, following the behavioral approach of Polderman and Willems [7] and the evolutionary model of Aubin and Dordan [8], *flow systems over hybrid time-lines* have been proposed [9–11], which constitute a semantic formalism that is closer to the classical semantics of control theory.

Apart from a difference in ease of use depending on the application area, there is a difference in abstraction level that one should be concerned about when choosing between these semantic models. Perhaps not surprisingly, a hybrid-time flow model contains more detailed information regarding the behavior of a system than a real-time transition model. Furthermore, within the formalism of hybrid-time flow systems, three notions of bisimulation can be distinguished (see [12]) corresponding to different levels of abstraction at which timing can be regarded. The question arises what the exact difference in level of abstraction is when different formalisms and different notions of equivalence are used.

In this paper, we study the difference in abstraction level between hybrid-time flow systems and real-time transition systems. We start, in section 2, with some formal preliminaries on time, transitions and flows. In section 3, we discuss translations from hybrid-time flow systems to real-time transition systems and back. In the translation from transition systems to flow systems, which creates flows by 'pasting' transitions together, no information is lost. The translation in the other direction, which creates transitions based on the presence of a *witnessing flow* (as in hybrid automata theory [1]), turns out to be lossless if and only if the original hybrid-time flows are *finite-set refutable*. I.e. if and only if any flow that is not a valid behavior of the system can be refuted on the basis of observations at only a finite (but well-chosen) set of time-points. The notion of finite-set refutability seems to be connected to the physical intuition that a system can only be observed at a finite number of times, but we are not aware of any previous literature about it. It is likely that finite-set refutability has never been considered in isolation before, since it will usually be replaced by the stronger (topological) notion of compactness (see section 5).

In section 4, we recall three notions of bisimulation equivalence on hybrid-time flow systems [12], and give their corresponding notions on real-time transition systems (one of which is especially introduced in this paper for the purpose). We proceed by proving that the translations preserve these bisimulations. Furthermore, we prove that in case of finite-set refutable flow systems, bisimulation on the real-time transition system resulting from the translation implies bisimulation of the original flow systems. Finally, in section 5, we show that hybrid-time flow systems in which the continuous paths are generated through differential inclusions with an upper-semicontinuous, closed and convex, right-hand side, are finite-set refutable. In section 6, we conclude that the difference in abstraction level between hybrid-time flow systems and real-time transition systems is irrelevant for a very broad class of hybrid systems. We give some suggestions for further research and, amongst others, discuss how hybrid transition systems may come of use if the differential inclusions are not autonomous.

2 Time, Transitions and Flows

Two semantic approaches to the description of dynamical systems are still gaining popularity: (timed) transition system semantics, and flow system semantics. Both make use of a formal notion of time.

A *time-line* is usually defined to be a *linear order* that, depending on the theory to be developed, has certain additional properties. One of those properties, is that it is an *Abelian group*, i.e. it has an addition operator $+$ defined on it. In this paper, it also has a zero element 0, such that $x + 0 = x$, and for notational convenience it has an inverse $-$ such that $x + (-x) = 0$. To denote the passage of time, only the positive numbers are used, which are also referred to as the *future* time-line. The most often used future time-line in control theory is, arguably, that of the non-negative real numbers $\mathbb{R}^{\geq 0}$, with the natural numbers $\mathbb{N} = \mathbb{Z}^{\geq 0}$ at second place. Recently in [9, 10], a merge between those two time-lines has arisen known as *hybrid time* $\mathbb{H} = \mathbb{Z} \times \mathbb{R}$, in which the ordering of time-points is lexicographical, i.e. for $(z, r), (z', r') \in \mathbb{H}$ we have $(z, r) < (z', r')$ if and only if $z < z'$ or both $z = z'$ and $r < r'$, and the addition is pointwise, i.e. $(z, r) + (z', r') = (z + z', r + r')$. The future hybrid-time[1] line then consists of the positive quadrant $\mathbb{H}^{\geq 0} = \mathbb{N} \times \mathbb{R}^{\geq 0}$. Paths over this time-line are alternations of continuous changes, i.e. intervals over the 'real' part where the 'discrete' part stays constant, and discrete changes, i.e. changes in the 'discrete' part where the 'real' part stays constant. Thus, hybrid-time provides us with a mechanism to describe hybrid behavior efficiently. For the sake of completeness we mention that the general time-line theory of [10] allows flow systems over even more exotic time-lines, in order to support constructs like *meta hybrid-automata* (automata with hybrid-automata in their states) [13].

The earliest hybrid semantics did not make use of a mechanism like hybrid-time, but rather allowed discontinuous changes in the state of a system that take 0 time. Amongst others, the early hybrid automata frameworks used real-time transition systems [1] as their semantics. Note, that in certain hybrid process algebras and hybrid automata frameworks, the discontinuous changes are not directly associated with a 0-time transition, but rather with an action transition. Such an approach would not fundamentally change the results of this paper, except that the hybrid-time flows would somehow have to accommodate for such actions as well.

Definition 1. *A* real-time transition system *is a tuple* $\mathcal{T} = \langle X, \mathbb{R}^{\geq 0}, \rightarrow \rangle$*, with* X *a* valuation space*,* $\mathbb{R}^{\geq 0}$ *the future real-time line, and* $\rightarrow \subseteq X \times \mathbb{R}^{\geq 0} \times X$ *the* time transition relation*. A transition* $(x, t, x') \in \rightarrow$ *will be denoted by* $x \xrightarrow{t} x'$*.*

- $\mathcal{T}$ *is* non-zero prefix-closed *if every transition* $x \xrightarrow{t} x''$*, with* $t > 0$*, can be split into transitions* $x \xrightarrow{t'} x'$ *and* $x' \xrightarrow{t''} x''$ *with* $t = t' + t''$*, and* $t', t'' > 0$*;*
- $\mathcal{T}$ *is* non-zero concatenation-closed *if for every two transitions* $x \xrightarrow{t} x'$ *and* $x' \xrightarrow{t'} x''$*, with* $t, t' > 0$*, there is also a transition* $x \xrightarrow{t+t'} x''$*.*

From here on, we will always assume real-time transition systems to be non-zero prefix closed and non-zero concatenation closed.

The latest hybrid semantics use flows over hybrid-time to describe system behaviors [9, 10]. This gives a more expressive semantics than obtained by using real-time transition systems, as we will see further on. However, in hybrid time, the usual model of a path being a function from some time-interval to a valuation space no longer applies. An interval $[t_0, t_1] = \{r \mid t_0 \leq r \leq t_1\}$ in future hybrid-time is a 'square' containing all possible ways in which time can proceed from $t_0 = (n_0, r_0)$ to $t_1 = (n_1, r_1)$. An interval does not yet specify in which order the discrete and continuous time-steps are taken. As a result, a path over hybrid-time has a more complicated domain than the interval-domain used in classical control theory. A formal definition is given below.

[1] For ease of notation, we will write 0 in stead of $(0, 0)$ whenever this is convenient.

Definition 2. *A* hybrid-time path[2] *through a valuation space X is a partial function $\phi : \mathbb{H}^{\geq 0} \to X$ such that $dom(\phi) = \bigcup_{i \leq N} \{i\} \times [r_i, r'_i]$ with $r'_i = r_{i+1}$ for all $i < N$. The set of all hybrid-time paths over X is denoted* $\mathrm{Path}(\mathbb{H}^{\geq 0}, X)$.

Definition 3. *On a hybrid-time path $\phi \in \mathrm{Path}(\mathbb{H}^{\geq 0}, X)$ we define the post-fix operation*

$$\phi^{\geq t}(\tau) \triangleq \phi(\tau + t) \text{ for } \tau + t \in dom(\phi),$$

and the prefix-operation

$$\phi^{\leq t}(\tau) \triangleq \phi(\tau) \text{ for } \tau \leq t \in dom(\phi).$$

On hybrid-time paths ϕ and ϕ', with $t \in dom(\phi)$ and $\phi(t) = \phi'(0)$, we define the concatenation

$$(\phi \cdot_t \phi')(\tau) \triangleq \begin{cases} \phi(\tau) & ; \text{ for } \tau \leq t \\ \phi'(\tau - t) & ; \text{ for } \tau \geq t \end{cases}$$

Finally, we define the progress operator which returns the domain of ϕ up to the first time instance at which a discrete step is taken:

$$\mathrm{Pro}(\phi) = \{t \in dom(\phi) \mid t > (0,0) \wedge t \leq \min\{(1,r) \mid (1,r) \in dom(\phi)\}\}.$$

Definition 4. *A* hybrid-time flow system *is a tuple $\mathcal{F} = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$, with X a* valuation space, *$\mathbb{H}^{\geq 0}$ the future hybrid-time line, and $\Phi : X \to 2^{\mathrm{Path}(\mathbb{H}^{\geq 0}, X)}$ a map from valuations to sets of hybrid-time paths.*

- *$\mathcal{F}$ has* initialization *if the flows associated with a state actually start in that state i.e. $\phi(0) = x$ for all $x \in \mathrm{dom}(\Phi)$ and $\phi \in \Phi(x)$;*
- *$\mathcal{F}$ is* time-invariant *if the allowed flows do not depend on the current time, i.e. $\phi^{\geq t} \in \Phi(\phi(t))$ for all $x \in \mathrm{dom}(\Phi)$, $\phi \in \Phi(x)$ and $t \in dom(\phi)$;*
- *$\mathcal{F}$ is* prefix-closed *if breaking off a flow is allowed, i.e. $\phi^{\leq t} \in \Phi(x)$ for all $x \in \mathrm{dom}(\Phi)$, $\phi \in \Phi(x)$ and $t \in dom(\phi)$;*
- *$\mathcal{F}$ has* property of state *if future flows only depend on the current valuation, and not on the past of the flow, i.e. $\phi \cdot_t \phi' \in \Phi(x)$ for all $x \in \mathrm{dom}(\Phi)$, $\phi \in \Phi(x)$, $t \in dom(\phi)$ and $\phi' \in \Phi(\phi(t))$;*

From here on, we will always assume hybrid-time flow systems to have initialization, to be time-invariant and prefix-closed, and to have the property of state.

3 Translations

To obtain insight in the difference in abstraction level between real-time transition systems and hybrid-time flow systems, we define straightforward translations between the two. We verify that these translations preserve the desired closure properties, and at the end of the section we study the information that is lost in these translations.

[2] The definitions in [10, 12] are more general. They start from a more general notion of time-line, of which $\mathbb{H}^{\geq 0}$ is a particular instance.

We start out with a translation from hybrid-time flow systems to real-time transition systems, which creates a transition whenever there is a hybrid-time flow witnessing this transition. Note that in the definition below, only witnesses starting in $(0, 0)$ are considered. Furthermore, we only consider witnesses for a single discrete change and for a single continuous flow. Due to the assumptions of time-invariance and property of state, any witness can be reduced to a sequence of such 'elementary' witnesses. An alternative translation using 'full' witnesses would not change the theorems obtained in this paper, but would complicate their proofs.

Definition 5. *Given a hybrid-time flow system* $\mathcal{F} = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$ *we define the associated real-time transition system* $\mathbf{T}(\mathcal{F}) = \langle X, \mathbb{R}^{\geq 0}, \rightarrow \rangle$ *such that there is a transition* $x \xrightarrow{t} x'$ *of duration* t *from state* $x \in X$ *to state* $x' \in X$ *if and only if:*

- $t = 0$ *and there is a* $\phi \in \Phi(x)$ *such that* $x' = \phi((1, 0))$*, or*
- $t > 0$ *and there is a* $\phi \in \Phi(x)$ *such that* $x' = \phi((0, t))$*.*

Naturally, we must verify that the standard closure properties on real-time transition systems are preserved.

Theorem 1. $\mathbf{T}(\mathcal{F})$ *is non-zero prefix closed and non-zero concatenation closed.*

Proof. Straightforward, but using the general assumptions that $\mathcal{F}$ has initialization, is time-invariant and prefix-closed and has the property of state.

Next, we give a translation from real-time transition systems to hybrid-time flow systems, which creates a hybrid-time flow by pasting real-time transitions in a suitable manner. A hybrid-time flow is only constructed (i.e. extracted from the real-time transition system) if every change of state that appears in the flow is mimicked by some real-time transition.

Definition 6. *Given a real-time transition system* $\mathcal{T} = \langle X, \mathbb{R}^{\geq 0}, \rightarrow \rangle$*, a hybrid-time path* $\phi \in \mathrm{Path}(\mathbb{H}^{\geq 0}, X)$ *is an* extracted path *of* $\mathcal{T}$ *if*

- *for every* $(n, r), (n, r') \in dom(\phi)$ *with* $r < r'$ *there is a transition* $\phi(n, r) \xrightarrow{r'-r} \phi(n, r')$*,*
- *for every* $(n, r) \in dom(\phi)$ *with also* $(n+1, r) \in dom(\phi)$*, there is a transition* $\phi(n, r) \xrightarrow{0} \phi(n+1, r)$*.*

Definition 7. *Given a real-time transition system* $\mathcal{T} = \langle X, \mathbb{R}^{\geq 0}, \rightarrow \rangle$ *we define the associated hybrid-time flow system* $\mathbf{F}(\mathcal{T}) = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$ *as the set of all extracted paths of* $\mathcal{T}$*. More precisely, for an initial valuation* $x \in X$ *we have* $\phi \in \Phi(x)$ *if and only if* ϕ *is an extracted path of* $\mathcal{T}$ *with* $\phi(0) = x$*.*

Of course, we verify that the standard closure properties of hybrid-time flow systems are preserved.

Theorem 2. $\mathbf{F}(\mathcal{T})$ *has initialization, is time-invariant and prefix-closed and has the property of state.*

Proof. Straightforward, but using the general assumptions that $\mathcal{T}$ is non-zero prefix closed and non-zero concatenation closed.

In the translation from real-time transition system to hybrid-time flow system no abstraction is applied; the translation is without loss of information. For the translation in the other direction this is not the case. Next, we prove that the abstraction resulting from translating a hybrid-time flow system into a real-time transition system, is that we only observe the behavior of the system at a finite number of points in time. In other words, if a proposed hybrid-time path $\phi \in \mathrm{Path}(\mathbb{H}^{\geq 0}, X)$ cannot be refuted on the basis of a finite set of time-points, then a real-time transition system cannot distinguish it from an actual behavior of the system.

As an example, consider the differential inclusion $\dot{x} \in [-1, 1]$ and the differential inclusion $\dot{x} \in \{-1, 1\}$ which switches between slope -1 and 1 arbitrarily fast. As we prove further on in section 5, the set of solutions of the first inclusion is finite-set refutable, while the set of solutions of the second is not. Furthermore, the behavior defined by the second inclusion is a strict subset of the behavior defined by the first. In particular, the function $x(t) = 0$ is a solution of the first inclusion, but not of the second. Still, given any finite set of time points D, there is a solution $y(t)$ of $\dot{y} \in \{-1, 1\}$ such that $y(d) = x(d) = 0$ for all $d \in D$ (just find an appropriate zig-zag line). In fact, any solution of the first inclusion can be approximated by the second in this way. As a result, the real-time transition systems generated by the two differential inclusions are identical.

Definition 8. *A hybrid-time flow system $\mathcal{F} = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$ is* finite-set refutable *if for every path $\psi \in \mathrm{Path}(\mathbb{H}^{\geq 0}, X)$ such that $\psi \notin \Phi(\psi(0))$, there is a finite set $T_\psi \subseteq dom(\psi)$ such that for every $\phi \in \Phi(\psi(0))$ with $dom(\phi) = dom(\psi)$ there is a $t \in T_\psi$ with $\phi(t) \neq \psi(t)$.*

Theorem 3. *For any real-time transition system $\mathcal{T}$, $\mathbf{F}(\mathcal{T})$ is finite-set refutable.*

Proof. Let $\mathcal{T} = \langle X, \mathbb{R}^{\geq 0}, \rightarrow \rangle$. Let $x \in X$, and assume that we have a hybrid-time path $\phi \notin \Phi(x)$, with $\phi(0) = x$. Then, by construction of $\mathbf{F}(\mathcal{T})$, ϕ is not an extracted path of $\mathcal{T}$. Hence, there exist $t_1, t_2 \in dom(\phi)$ with $t_1 = (n_1, r_1)$ and $t_2 = (n_2, r_2)$ such that the transition $\phi(t_1) \overset{r_2 - r_1}{\rightarrow} \phi(t_2)$ is not in $\mathcal{T}$. But then, no extracted path of $\mathcal{T}$ can coincide with ϕ at both t_1 and t_2, and hence the finite set $\{t_1, t_2\}$ is a witness on the basis of which ϕ can be refuted[3].

Theorem 4. *For finite-set refutable hybrid-time flow system $\mathcal{F}$, $\mathbf{F}(\mathbf{T}(\mathcal{F})) = \mathcal{F}$.*

Proof. Let $\mathcal{F} = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$. We use $\mathbf{F}(\mathbf{T}(\mathcal{F})) = \langle X, \mathbb{H}^{\geq 0}, \Phi' \rangle$ to denote the result of the translation forwards and backwards. It is trivial to see, for any $x \in X$, that $\phi \in \Phi(x)$ implies $\phi \in \Phi'(x)$. Hence, we focus on the other direction. Assume that $\phi \in \Phi'(x)$ and that $dom(\phi) = \bigcup_{i \leq N} \{i\} \times [r_i, r'_i]$, for some $N \in \mathbb{N}$. Let $t_j = (m_j, s_j) \in dom(\phi)$, with $0 \leq j \leq M \leq 2N$, be any (finite) sequence

[3] Indeed, $\mathbf{F}(\mathcal{T})$ is even 2-point refutable. But, 2-point refutability and finite-set refutability coincide for flow-systems with property of state [14].

of times including, at least, all the beginning and end-points of the real-time intervals. I.e. let t_j be a sequence such that for every $i \leq N$ there are $j, k \leq M$ with $t_j = (i, r_i)$ and $t_k = (i, r'_i)$. Now, by construction of $\mathbf{F}(\mathbf{T}(\mathcal{F}))$, there are transitions $t_j \overset{s_{j+1}-s_j}{\rightarrow} t_{j+1}$ in $\mathbf{T}(\mathcal{F})$, for each $0 \leq j < M-1$. Hence, by construction of $\mathbf{T}(\mathcal{F})$, there is a path $\psi_j \in \Phi(\phi(t_j))$ with $\psi_j(0, s_{j+1}) = \phi(t_{j+1})$ when $m_j = m_{j+1}$, and with $\psi_j(1, 0) = \phi(t_{j+1})$ when $r_j = r_{j+1}$. The concatenation of these paths ψ_j gives a path $\psi \in \Phi(\phi(0)) = \Phi(x)$ with $dom(\psi) = dom(\phi)$ that furthermore coincides with ϕ at every t_j. In conclusion, for every finite set of times D, we can find a sequence t_j visiting all points in D and all switching points of ϕ. Furthermore, we can construct a path $\psi \in \Phi(x)$ that coincides with ϕ at every t_j, and hence at every $d \in D$. Since $\mathcal{F}$ is assumed to be finite-set refutable, we conclude $\phi \in \Phi(x)$.

Corollary 1. *For any hybrid-time flow system* $\mathcal{F}$, $\mathbf{F}(\mathbf{T}(\mathcal{F})) = \mathcal{F}$ *if and only if* $\mathcal{F}$ *is finite-set refutable.*

Proof. Straightforward from the previous two theorems.

Finally, we observe that indeed no information is lost if we start from a real-time transition system.

Theorem 5. *For any real-time transition system* $\mathcal{T}$, $\mathbf{T}(\mathbf{F}(\mathcal{T})) = \mathcal{T}$.

Proof. Straightforward, using the prefix-closure of real-time transition systems to ensure that each transition of $\mathcal{T}$ is represented by some flow in $\mathbf{F}(\mathcal{T})$.

4 Bisimulation equivalence

In [12], Davoren and Tabuada introduced three notions of bisimulation equivalence on hybrid-time flow systems, in an attempt to preserve properties in the temporal logic GFL* [10]. In this paper, we discuss the relation between these three notions, and three similar notions of bisimulation defined on real-time transition systems (one of which is especially introduced in this paper for the purpose of comparison). The most important topic we adres in this section, is that finite-set refutability as a necessary and sufficient condition for lossless translation, is no guarantee that the notions of bisimulation on real-time transition systems will not abstract from more information than the respective notions of bisimulation on hybrid-time flow systems. Below, we prove that finite-set refutability indeed guarantees that bisimulations on hybrid-time flow systems correspond to their companion bisimulations on real-time transition systems.

The intuition on the definitions give below, is that t-bisimulation preserves the exact timing properties of paths, while p-bisimulation allows paths to be ‘compressed’, ‘stretched’, or in some other way cast to a different time-line. The notion of r-bisimulation is not concerned with timing at all, and only preserves the order in which states are reached.

Definition 9. *Given hybrid-time flow systems* $\mathcal{F}_1 = \langle X_1, \mathbb{H}^{\geq 0}, \Phi_1 \rangle$ *and* $\mathcal{F}_2 = \langle X_2, \mathbb{H}^{\geq 0}, \Phi_2 \rangle$*, a relation* $\mathcal{R} \subseteq X_1 \times X_2$ *is called a*

- timed simulation *or* t-simulation[4] *if for every* $x_1 \in X_1$*,* $x_2 \in X_2$ *and* $\phi_1 \in \Phi_1(x_1)$ *with* $x_1 \mathcal{R} x_2$ *there exists* $\phi_2 \in \Phi_2(x_2)$ *with* $dom(\phi_1) = dom(\phi_2)$ *such that for every* $t \in dom(\phi_1)$ *we have* $\phi_1(t) \mathcal{R} \phi_2(t)$*;*
- progress simulation *or* p-simulation *if for every* $x_1 \in X_1$*,* $x_2 \in X_2$*,* $\phi_1 \in \Phi_1(x_1)$ *and* $t_1 \in \text{Pro}(\phi_1)$ *with* $x_1 \mathcal{R} x_2$ *there exists* $\phi_2 \in \Phi_2(x_2)$ *with* $t_2 \in \text{Pro}(\phi_2)$ *such that* $\phi_1(t_1) \mathcal{R} \phi_2(t_2)$ *and for every* $s_2 \in dom(\phi_2)$ *with* $0 < s_2 \leq t_2$ *there is a* $s_1 \in dom(\phi_1)$ *with* $0 < s_1 \leq t_1$ *such that* $\phi_1(s_1) \mathcal{R} \phi_2(s_2)$*;*
- reachable simulation *or* r-simulation *if for every* $x_1 \in X_1$*,* $x_2 \in X_2$*,* $\phi_1 \in \Phi_1(x_1)$ *and* $0 < t_1 \in dom(\phi_1)$ *with* $x_1 \mathcal{R} x_2$ *there exists* $\phi_2 \in \Phi_2(x_2)$ *and* $0 < t_2 \in dom(\phi_2)$ *such that* $\phi_1(t_1) \mathcal{R} \phi_2(t_2)$*.*

In general, a relation $\mathcal{R}$ *is called a* bisimulation *if* $\mathcal{R}$ *and* $\mathcal{R}^{-1}$ *are simulations.*

Next, we give the companion bisimulations defined on (relative-time) real-time transition systems, and show the relation with their hybrid-time flow system originals. As was already pointed out in [12] the notion of t-simulation is, in fact, the usual notion of simulation on real-time transition systems as used, for example, in timed process algebras [15].

Definition 10. *Given real-time transition systems* $\mathcal{T}_1 = \langle X_1, \mathbb{R}^{\geq 0}, \rightarrow_1 \rangle$ *and* $\mathcal{T}_2 = \langle X_2, \mathbb{R}^{\geq 0}, \rightarrow_2 \rangle$*, a relation* $\mathcal{R} \subseteq X_1 \times X_2$ *is called a*

- timed simulation *or* t-simulation *if for every* $x_1, x_1' \in X_1$*,* $x_2 \in X_2$ *and* $t \in \mathbb{R}$ *with* $x_1 \xrightarrow{t}_1 x_1'$ *and* $x_1 \mathcal{R} x_2$ *there exists* $x_2' \in X_2$ *such that* $x_2 \xrightarrow{t}_2 x_2'$ *and* $x_1' \mathcal{R} x_2'$*;*
- progress simulation*, or* p-simulation *if for every* $x_1 \in X_1$*,* $x_2 \in X_2$ *and extracted hybrid-time path* ϕ_1 *from* $\mathcal{T}_1$ *with* $t_1 \in \text{Pro}(dom(\phi_1))$*,* $\phi_1(0) = x_1$ *and* $x_1 \mathcal{R} x_2$*, there exists an extracted hybrid-time path* ϕ_2 *from* $\mathcal{T}_2$ *with* $t_2 \in \text{Pro}(dom(\phi_2))$*,* $\phi_2(0) = x_2$*,* $\phi(t_1) \mathcal{R} \phi(t_2)$ *and for every* $s_2 \in dom(\phi_2)$ *with* $0 < s_2 \leq t_2$ *there is a* $s_1 \in dom(\phi_1)$ *with* $0 < s_1 \leq t_1$ *such that* $\phi(s_1) \mathcal{R} \phi(s_2)$*.*
- reachable simulation *or* r-simulation *if for every* $x_1, x_1' \in X_1$*,* $x_2 \in X_2$ *and* $t \in \mathbb{R}$ *with* $x_1 \xrightarrow{t}_1 x_1'$ *and* $x_1 \mathcal{R} x_2$ *there exists* $x_2' \in X_2$ *and* $t' \in \mathbb{R}$ *such that* $x_2 \xrightarrow{t'}_2 x_2'$ *and* $x_1' \mathcal{R} x_2'$*;*

As before, a relation $\mathcal{R}$ *is called a* bisimulation *if* $\mathcal{R}$ *and* $\mathcal{R}^{-1}$ *are simulations.*

One should note, that in the literature the bisimulation relation $\mathcal{R}$ is also required to preserve other observable aspects of a system, such as the atomic propositions on the state-space in logic [16], and the result of the observation function $y : X \rightarrow Y$ in control theory [17]. The proofs below are robust against adding such observables.

The following theorem shows that the translation from real-time transitions to hybrid-time flows preserves simulations, and consequently preserves bisimulation equivalence.

[4] Despite the more compact formulation we use here, the notions of t- and p- simulation coincide with those of [12]

Theorem 6. *Given real-time transition systems* $\mathcal{T}_1 = \langle X_1, \mathbb{R}^{\geq 0}, \rightarrow_1 \rangle$ *and* $\mathcal{T}_2 = \langle X_2, \mathbb{R}^{\geq 0}, \rightarrow_2 \rangle$*, a relation* $\mathcal{R} \subseteq X_1 \times X_2$

- *is a t-simulation of* $\mathcal{T}_1$ *by* $\mathcal{T}_2$ *if and only if it is a t-simulation of the translated hybrid-time flow system* $\mathbf{F}(\mathcal{T}_1)$ *by* $\mathbf{F}(\mathcal{T}_2)$*,*
- *is a p-simulation of* $\mathcal{T}_1$ *by* $\mathcal{T}_2$ *if and only if it is a p-simulation of the translated hybrid-time flow system* $\mathbf{F}(\mathcal{T}_1)$ *by* $\mathbf{F}(\mathcal{T}_2)$*,*
- *is a r-simulation of* $\mathcal{T}_1$ *by* $\mathcal{T}_2$ *if and only if it is a r-simulation of the translated hybrid-time flow system* $\mathbf{F}(\mathcal{T}_1)$ *by* $\mathbf{F}(\mathcal{T}_2)$*.*

Proof. The 'only if' direction in the above theorems is trivial, since having a path ϕ in the translation implies having all transitions $\phi(n_1, r_1) \stackrel{r_2 - r_1}{\rightarrow} \phi(n_2, r_2)$ for $(n_1, r_1), (n_2, r_2) \in dom(\phi)$. The 'if' direction becomes straightforward after observing that with each transition $x \stackrel{t}{\rightarrow} x'$ with $t > 0$ there is also a path $\phi \in \Phi(x)$ such that $\phi(0, t) = x'$, due to the non-zero prefix closedness of $\mathcal{T}$.

For the translation from hybrid-time flows to real-time transition systems we have only the 'only if' direction. The reason for not having the 'if' direction is that simulating a transition in $\mathbf{T}(\mathcal{F}_1)$ by a transition in $\mathbf{T}(\mathcal{F}_2)$ does not guarantee that these transitions were generated by similar paths in $\mathcal{F}_1$ and $\mathcal{F}_2$. For r-simulation, only the actual states that are reached are of importance, not the paths leading to them, which is why we have both directions for r-simulation.

Theorem 7. *For any hybrid-time flow systems* $\mathcal{F}_1 = \langle X_1, \mathbb{H}^{\geq 0}, \Phi_1 \rangle$ *and* $\mathcal{F}_2 = \langle X_2, \mathbb{H}^{\geq 0}, \Phi_2 \rangle$*, and a relation* $\mathcal{R} \subseteq X_1 \times X_2$

- *is a t-simulation of* $\mathcal{F}_1$ *by* $\mathcal{F}_2$*, only if it is a t-simulation of the real-time transition system* $\mathbf{T}(\mathcal{F}_1)$ *by* $\mathbf{T}(\mathcal{F}_2)$*,*
- *is a p-simulation of* $\mathcal{F}_1$ *by* $\mathcal{F}_2$*, only if it is a p-simulation of the real-time transition system* $\mathbf{T}(\mathcal{F}_1)$ *by* $\mathbf{T}(\mathcal{F}_2)$*,*
- *is a r-simulation of* $\mathcal{F}_1$ *by* $\mathcal{F}_2$*, if and only if it is a r-simulation of the real-time transition system* $\mathbf{T}(\mathcal{F}_1)$ *by* $\mathbf{T}(\mathcal{F}_2)$*.*

Proof. By construction of the translation.

For the other direction in t-simulation and p-simulation, we need finite-set refutability.

Theorem 8. *For any hybrid-time flow systems* $\mathcal{F}_1 = \langle X_1, \mathbb{H}^{\geq 0}, \Phi_1 \rangle$ *and* $\mathcal{F}_2 = \langle X_2, \mathbb{H}^{\geq 0}, \Phi_2 \rangle$*, with* $\mathcal{F}_2$ *finite-set refutable, and given a relation* $\mathcal{R} \subseteq X_1 \times X_2$

- $\mathcal{R}$ *is a t-simulation of the hybrid time flow system* $\mathcal{F}_1$ *by* $\mathcal{F}_2$*, if it is a t-simulation of the real-time transition system* $\mathbf{T}(\mathcal{F}_1)$ *by* $\mathbf{T}(\mathcal{F}_2)$*,*
- $\mathcal{R}$ *is a p-simulation of the hybrid time flow system* $\mathcal{F}_1$ *by* $\mathcal{F}_2$*, if it is a p-simulation of the real-time transition system* $\mathbf{T}(\mathcal{F}_1)$ *by* $\mathbf{T}(\mathcal{F}_2)$*.*

Proof. The proof for the notion of p-simulation is rather straightforward, since by theorem 4 we know for finite-set refutable systems that every extracted path of $\mathbf{T}(\mathcal{F}_2)$ is in fact a path of $\mathcal{F}_2$. In our definition of t-simulation on real-time transition systems we did not make use of the notion of paths, hence we must reconstruct them. This is done in a similar fashion as in theorem 4. Assume that $\mathcal{R}$ is a t-simulation for $\mathbf{T}(\mathcal{F}_1)$ and $\mathbf{T}(\mathcal{F}_2)$. Furthermore, let $x_1 \in X_1$, $x_2 \in X_2$ and $\phi_1 \in \Phi_1(x_1)$ with $x_1 \,\mathcal{R}\, x_2$. Now, like in the proof of theorem 4, we create a sequence $t_j = (m_j, s_j) \subseteq dom(\phi_1)$ of length $M + 1$ that at least contains all the discrete steps in ϕ_1. Then, using the simulation relation $\mathcal{R}$ we mimic the transitions $\phi_1(t_j) \overset{s_{j+1}-s_j}{\rightarrow}_1 \phi_1(t_{j+1})$ by transitions $x_j \overset{s_{j+1}-s_j}{\rightarrow}_2 x_{j+1}$ such that $\phi_1(t_j) \,\mathcal{R}\, x_j$ for all $j \leq M$. By construction of $\mathbf{T}(\mathcal{F}_2)$ we find paths $\psi_j \in \Phi_2(\phi_1(t_j))$, and concatenating these gives us a path $\phi_D \in \Phi_2(x_2)$ such that for every $t \in dom(\phi_1) \cap D$ we have $\phi_1(t) \,\mathcal{R}\, \phi_D(t)$. The fact that Φ_2 is finite-set refutable then leads to the conclusion that there is also a $\phi_2 \in \Phi_2(x_2)$ such that for every $t \in dom(\phi_1) = dom(\phi_2)$ we have $\phi_1(t) \,\mathcal{R}\, \phi_2(t)$.

5 Impuls differential inclusions

Now that we have shown that finite-set refutability captures the loss of information between hybrid-time flow systems and real-time transition systems, we should ask ourselves which systems are finite-set refutable. Our first intuition is that most physical systems should have a finite-set refutable representation, because finite-set refutability is a consequence of the physical principle that we can only distinguish systems on the basis of a well-chosen but finite set of observations. Below, we find mathematical confirmation of this intuition, because the long-standing modeling of physical behavior through continuous differential equations (and more generally, differential inclusions with an upper-semicontinuous, closed and convex, right-hand side) leads to compact, and hence finite-set refutable, real-time flow systems (i.e. hybrid-time flow systems without discrete behavior). The definitions below are taken from [11] and require some formal background in topology and in the theory of (impuls) differential inclusions to understand. See for example also [18].

Definition 11. *An* impulse differential inclusion *is a tuple $H = \langle X, F, R, J \rangle$, consisting of a finite dimensional vector space X, a set valued map $F : X \to 2^X$, regarded as a differential inclusion $\dot{x} \in F(x)$, a set valued map $R : X \to 2^X$, regarded as a reset map, and a set $J \subseteq X$, regarded as a forced transition set.*

Definition 12. *A hybrid-time flow system $\mathcal{F}_H = \langle X, \mathbb{H}^{\geq 0}, \Phi_F \rangle$ is the* solution *of an impulse differential inclusion, $H = \langle X, F, R, J \rangle$ when for all $x \in X$ and all paths $\phi \in \mathrm{Path}(\mathbb{H}^{\geq 0}, X)$ we find $\phi \in \Phi_F(x)$ if and only if*

- *$\phi(0, 0) = x$,*
- *for all $(n, r) \in dom(\phi)$ with $(n + 1, r) \in dom(\phi)$ we have $\phi(n + 1, r) \in R(\phi(n, r))$,*
- *for all $(n, r), (n, r') \in dom(\phi)$ we have that $\phi(\cdot)$ is a solution of the differential inclusion $\dot{x} \in F(x)$ over the interval $[(n, r), (n, r')]$, in the sense of [18][5], and $\phi(t) \notin J$ for $(n, r) \leq t < (n, r')$.*

[5] To explain the complete solution concept on differential inclusions would be out of the scope of this paper.

The solutions of impuls differential inclusions indeed satisfy the properties we require of hybrid-time flow systems in this paper.

Theorem 9. *A hybrid-time flow system* $\mathcal{F}_H$ *that is the solution of an impuls differential inclusion* $H = \langle X, F, R, J \rangle$ *has initialization, is time-invariant and prefix-closed and has the property of state.*

Proof. That $\mathcal{F}_H$ has initialization follows from its construction above. Time-invariant, prefix-closedness and the property of state are well-known properties of differential inclusions which, amongst others, follows straightforwardly from the theory explained in [18]. The extension with discontinuous behavior using a reset map R, and the restriction using a forced jump-set J do not influence these properties. The full proof of this claim is omitted for reasons of space.

Next, we show that compactness of the solution to an impuls differential inclusion, is sufficient to guarantee finite-set refutability.

Theorem 10. *Let* $\mathcal{F}_H = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$ *be the solution to an impuls differential inclusion* $H = \langle X, F, R, J \rangle$*, and furthermore let* $\Phi(x)$ *be a* compact set *for every* $x \in X$*. Then* $\mathcal{F}_H$ *is finite-set refutable.*

Proof. Assume that a path $\psi \in \text{Path}(\mathbb{H}^{\geq 0}, X)$ cannot be refuted on the basis of any finite set of time points, then we must prove $\psi \in \Phi(\psi(0))$. We will first prove that ψ can be approximated by a continuous solution $\phi_\omega \in \Phi(\psi(0))$, and secondly, we prove that ψ is in fact continuous itself (hence equal to ϕ_ω).

We start out by observing that the hybrid-time axis has a countable topology. Therefore, we can construct a sequence $D_i \subseteq dom(\psi)$ of finite sets, which converges to a set D_ω, that is dense in $dom(\psi)$. Also assume that $0 \in D_i$ for each i. Because ψ cannot be refuted on the basis of any of the sets D_i, there exists an associated sequence $\phi_i \in \Phi(\psi(0))$ such that $\phi_i(d) = \psi(d)$ for each i and each $d \in D_i$. Using the assumed compactness of $\Phi(\psi(0))$, we know that this sequence ϕ_i has a subsequence converging in a solution $\phi_\omega \in \Phi(\psi(0))$. This solution coincides with ψ on the dense set D_ω, and furthermore $\phi_\omega \in \Phi(\psi(0))$. Hence we have the promised approximation.

Finally, as the solutions to impuls differential inclusions are continuous (between the countably many jumps due to resets), we know in particular that the approximation ϕ_ω is continuous, regardless of the initial choice of D_ω. This is sufficient to prove by contradiction that ψ is also continuous. Namely, should ψ poses a discontinuity at t_0, then we can start out with $t_0 \in D_\omega$, and we would have found the same discontinuity in ϕ_ω. In conclusion, ϕ_ω and ψ are both found to be continuous, and to coincide on a dense set. Hence, $\psi = \phi_\omega \in \Phi(\psi(0))$.

It is a classical result, that compactness is obtained for differential inclusions (without reset maps) of which the function F is upper-semicontinuous and has a closed and convex right-hand side. (In [11], the strictly stronger condition of F being *Marchaud* is used throughout the whole paper.)

Definition 13. *A function* $F : X \to 2^X$ *is* upper-semicontinuous *at* $x_0 \in X$ *if for any open set* U *containing* $F(x_0)$ *there exists an open set* V *containing* x_0 *such that* $F(V) \subseteq U$*. The function* F *is upper-semicontinuous if it is upper-semicontinuous at every* $x_0 \in X$*.*

Theorem 11. *Let* $\mathcal{F}_H = \langle X, \mathbb{H}^{\geq 0}, \Phi \rangle$ *be the solution to an impuls differential inclusion* $H = \langle X, F, \emptyset, \emptyset \rangle$*, with* F *upper-semicontinuous, and* $F(x)$ *closed and convex for every* $x \in X$*. Under these conditions* $\Phi(x)$ *is a compact set of paths for every* x*.*

Proof. Transliterate corollary 4.5 of [18].

Adding a reset map R of forced transition set J to a finite-set refutable differential inclusion will not render it finite-set irrefutable.

Theorem 12. *Let* $\mathcal{F}_H = \langle X, \mathbb{H}^{\geq 0}, \Phi_H \rangle$ *be the solution to an impuls differential inclusion* $H = \langle X, F, R, J \rangle$*, with* F *upper-semicontinuous, and* $F(x)$ *closed and convex for every* $x \in X$*. Under these conditions* $\mathcal{F}_H$ *is finite-set refutable (but not necessarily compact).*

Proof. The proof of this theorem is too long to be presented here completely, but it relies on the observation that $\mathcal{F}_H$ can be constructed by first building the solution $\mathcal{F}_G$ of $G = \langle X, F, \emptyset, \emptyset \rangle$. We build $\mathcal{F}_G$ and translate this solution to a real-time transition system $\mathbf{T}(\mathcal{F}_G)$, which is a lossless translation according to theorems 4, 10 and 11. Then we add transitions $x \xrightarrow{0} x'$ to $\mathbf{T}(\mathcal{F}_G)$ whenever $(x, x') \in R$, and we remove transitions $x \xrightarrow{r} x'$ whenever $r > 0$ and $x \in J$. Thus we obtain a real-time transition system $\mathcal{T}_H$, which we translate back to the hybrid-time flow system $\mathbf{F}(\mathcal{T}_H)$. The omitted part of the proof consists of showing that indeed $\mathbf{F}(\mathcal{T}_H) = \mathcal{F}_H$. Finally, it follows from theorem 3 that this hybrid-time flow system is finite-set refutable.

As a corollary, we now see that the behavior of hybrid automata is indeed finite-set refutable.

Corollary 2. *A hybrid-time flow system* $\mathcal{F}$ *generated by a hybrid automaton with differential inclusions that satisfy the conditions of the previous theorem, is finite-set refutable.*

6 Conclusions

We have compared the semantic frameworks of hybrid-time flow systems and real-time transition systems in order to obtain insight in the difference in abstraction level between the two. We have captured this difference in the notion of finite-set refutability, which captures a necessary and sufficient condition for lossless translation, even in the context of bisimulations. We have argued that finite-set refutability is a very reasonable condition to impose on models, since it is a result of the physical intuition that we can only distinguish systems on the basis of a finite number of observations. Finally, we have proven that a broad class of differential equations and (impuls) differential inclusions, namely those that are upper-semicontinuous and closed and convex, have a finite-set refutable set of solutions.

These results suggest that the use of real-time transition systems as a model for autonomous physical systems does not introduce additional abstractions compared to hybrid-time flows. But, when the model is still 'open' to inputs and other types of compositions, hybrid-time flow systems may lead to more precise models. In this latter case, however, another alternative is to use hybrid transition systems, with real-time paths as transition-labels. As we show in an earlier technical report [14] on this subject, the definition of finite-set refutability can be adapted to suit the translation to such hybrid transition systems, which means that no unwanted abstractions arise in the hybrid automaton theory of [2] and in the hybrid process algebras of [4, 3, 5].

A natural question that arises for future research, is whether a given hybrid-time flow system can be made finite-set refutable. In other words, whether there is a convenient operator that closes a system under finite-set refutability. In our counter-example of section 3, we used a non-finite-set refutable differential inclusion $\dot{x} \in \{-1, 1\}$ and its finite-set refutable closure $\dot{x} \in [-1, 1]$. Here, the closure was obtained by taking the convex hull, but in general this approach is likely to add spurious solutions as well. Note, that the given conditions on differential inclusions are only sufficient conditions. Upper-semicontinuity and closedness may not be necessary. As an example, the solutions of the differential inclusion $\dot{x} \in (-1, 1)$, with its right-hand side upper-semicontinuous and convex, but not closed, are not compact, but are still finite-set refutable. It is a consequence of theorem 7, that reachability will not be affected by finite-set refutable closure.

Based on the results in [14], we claimed that the notion of finite-set refutability is still a necessary and sufficient condition for lossless translation when the hybrid-time flow systems are not time-invariant and the real-time transition systems use absolute rather than relative timing. Naturally, the actual translations are different in that case. Note, however, that the conditions for compactness of time-variant differential inclusions are rather complex, as some of the theorems in [18] show.

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