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ON n-FOLD FILTERS IN BL-ALGEBRAS

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Abstract

The notions of n-fold fantastic basic logic and the related algebras, n-fold fantastic BL-algebras, are introduced. We also define n-fold fantastic filters and prove some relations between these filters and construct quotient algebras via these filters.

1. Introduction

The concept of BL-algebras was introduced by Hajek [5] in order to provide, an algebraic proof of the completeness theorem of "basic logic" (BL, for short). Soon after, Cignoli et al. in [2], proved that Hajek's logic really is the logic of continuous t-norms as conjectured by Hajek. At the same time started a systematic study of BL-algebras, too. Indeed,

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Turunen in [10], published, where BL-algebras were studied by deductive systems. Deductive systems correspond to subsets closed with respect to Modus Ponens and they are called filters, too. In [11], Boolean deductive systems and implicative deductive systems were introduced. Moreover, it was proved that these deductive systems coincide. In Haveshki et al. in [7], continued an algebraic analysis of BL-algebras and they introduced, e.g., implicative filters of BL-algebras. MV-algebras [1], product algebras and Gödel algebras are the most classes of BL-algebras. Filters theory play an important role in studying these algebras. From logical point of view, various filters correspond to various sets of provable formulae. Hajek in [5], introduced the concepts of (prime) filters of BL-algebras. Using prime filters of BL-algebras, he proved the completeness of basic logic.

The language of propositional Hajek basic logic [5] contains the binary connectives $\circ$ and $\Rightarrow$ and the constant $\overline{0}$. Axioms of BL are:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow w) \Rightarrow (\varphi \Rightarrow w))$,

(A2) $(\varphi \circ \psi) \Rightarrow \varphi$,

(A3) $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$,

(A4) $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$,

(A5a) $(\varphi \Rightarrow (\psi \Rightarrow w)) \Rightarrow ((\varphi \circ \psi) \Rightarrow w)$,

(A5b) $((\varphi \circ \psi) \Rightarrow w) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow w))$,

(A6) $((\varphi \Rightarrow \psi) \Rightarrow w) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow w) \Rightarrow w)$,

(A7) $\overline{0} \Rightarrow w$.

The above notion is generalized to an algebraic system, in which the required conditions are fulfilled (Definition 1). Filters in BL-algebras are also introduced in [5]. The notions of implicative and positive implicative filters were introduced in [7].

In Section 2, we give some definitions and theorems which are needed in the rest of the paper. We then introduce the notions of an n-fold fantastic basic logic and n-fold fantastic BL-algebra. We also introduce the notion of n-fold fantastic filters. We prove that, every n-fold fantastic BL-algebra is also an $(n+1)$-fold fantastic BL-algebra, but by an example, we show that the converse is not true. Some characterization for a BL-algebra to be n-fold fantastic, is given. We define n-fold fantastic filters, after that we state the equivalent conditions for n-fold fantastic BL-algebras. By [8], every n-fold positive implicative filter is an n-fold implicative filter, but the converse is not true. We show that, under some conditions, an n-fold implicative filter is an n-fold positive implicative filter.

2. *n*-Fold Fantastic BL-Algebras

Definition 2.1. A BL-algebra is an algebra $(A, \wedge, \vee, *, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms:

(BL1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(BL2) $(A, *, 1)$ is a commutative monoid,

(BL3) $*$ and $\to$ form an adjoint pair; i.e., $c \leq a \to b,$ if and only if $a * c \leq b$ for all $a, b, c \in a,$

(BL4) $a \wedge b = a * (a \to b),$

(BL5) $(a \to b) \vee (b \to a) = 1.$

Proposition 2.2 [3, 4, 5]**.** *In each BL-algebra A, the following relations hold for all $x, y, z \in A$:*

(p1) $x * (x \to y) \leq y,$

(p2) $x \leq (y \to (x * y)),$

(p3) $x \leq y,$ *if and only if* $x \to y = 1,$

(p4) $x \to (y \to z) = y \to (x \to z),$

(p5) *If* $x \leq y,$ *then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y,$

(p6) $y \leq (y \to x) \to x,$

(p7) $y \to x \leq (z \to y) \to (z \to x)$,

(p8) $x \to y \leq (y \to z) \to (x \to z)$,

(p9) $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$.

In what follows, let n denotes a positive integer and A a BL-algebra, unless otherwise specified. For any element x of A, let x^n denotes $x * \ldots * x$, in which x occurs n times, and $x^0 = 1$.

Definition 2.3. A filter of a BL-algebra A is a nonempty subset F of A such that for all $a, b \in A$,

(f1) $a, b \in F$ implies $a * b \in F$,

(f2) $a \in F$ and $a \leq b$ imply $b \in F$.

$1 \in F$, since $x \leq 1$ for every $x \in F$, and F is nonempty subset, $1 \in F$. It is proved in [11], that if F is a filter, then it satisfies:

(f3) $1 \in F$,

(f4) $x \in F$ and $x \to y \in F$ imply $y \in F$.

A subset D of A is called a *deductive system* in [11], if it satisfies the above two conditions. It is obvious that for a nonempty subset D, D is a deductive system, if and only if it is a filter.

Definition 2.4. A nonempty subset F of A is called an *n-fold implicative filter of* A, if it satisfies $1 \in F$ and:

(f5) $x^n \to (y \to z) \in F$, $x^n \to y \in F$ imply $x^n \to z \in F$, for all $x, y, z \in A$.

Theorem 2.5 [8, Theorem 4.6]**.** *Let F be a filter of A. Then for all $x, y, z \in A$, the following conditions are equivalent*:

(i) *F is an n-fold implicative filter of A*,

(ii) $x^n \to x^{2n} \in F$, *for all* $x \in A$,

(iii) $x^{n+1} \to y \in F$ *implies* $x^n \to y \in F$,

(iv) $x^n \to (y \to z) \in F$ *implies* $(x^n \to y) \to (x^n \to z) \in F$.

Definition 2.6. A nonempty subset F of A is called an *n-fold positive implicative filter*, if it satisfies $1 \in F$ and:

(f6) $x \to ((y^n \to z) \to y) \in F$ and $x \in F$ imply $y \in F$, for all $x, y, z \in A$.

Theorem 2.7 [8, Theorem 6.2]**.** *Every n-fold positive implicative filter of A is a filter of A.*

Theorem 2.8 [8, Theorem 6.3]**.** *Let F be a filter of A. Then for all* $x, y, z \in A$, *the following conditions are equivalent*:

(i) *F is an n-fold positive implicative filter*,

(ii) $(x^n \to 0) \to x \in F$ *implies* $x \in F$, *for all* $x \in A$,

(iii) $(x^n \to y) \to x \in F$ *implies* $x \in F$, *for all* $x, y \in A$.

Definition 2.9. A nonempty subset F of A is called a *fantastic filter*, if it satisfies $1 \in F$ and:

(f7) $\forall x, y, z \in A$, $z \to (y \to x) \in F$, $z \in F$ imply $((x \to y) \to y) \to x \in F$.

Theorem 2.10 [7, Theorem 4.2]**.** *Every fantastic filter of A is a filter of A.*

Definition 2.11. Axioms of an n-fold fantastic basic logic are those of BL plus

$$((\varphi^n \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \varphi \equiv \psi \Rightarrow \varphi,$$

where

$$\varphi^n = \varphi \circ \varphi \circ \ldots \circ \varphi \text{ for } n \text{ times},$$

and where $\varphi \equiv \psi$ denotes $\varphi \Rightarrow \psi$ plus $\psi \Rightarrow \varphi$.

Definition 2.12. A BL-algebra A is called to be *n-fold fantastic*, if it satisfies the equality $((x^n \to y) \to y) \to x = y \to x$ for all $x, y \in A$.

Example 2.13. Let $A = \{0, a, b, 1\}$. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

It is easy to see that A is an m-fold fantastic BL-algebra for every $m \geq 2$.

Theorem 2.14. *The n-fold fantastic basic logic is complete, for each formula* φ, *the following conditions are equivalent*:

(i) *The n-fold fantastic basic logic proves* φ,

(ii) φ *is an A-tautology for each linearly ordered n-fold fantastic BL-algebra A*,

(iii) φ *is an A-tautology for each n-fold fantastic BL-algebra A*.

Proof. It follows from [5, Theorem 2.3.22]. This is because n-fold fantastic basic logic is a schematic extension of BL. □

Theorem 2.15. *An n-fold fantastic BL-algebra is an m-fold fantastic BL-algebra for every* $m > n$.

Proof. Let A be an n-fold fantastic BL-algebra and $m > n$. Then, we have

$$((x^n \to y) \to y) \to x = y \to x,$$

for all $x, y \in A$. Since $x^m \leq x^n$, we get

$$y \to x = ((x^n \to y) \to y) \to x \leq ((x^m \to y) \to y) \to x.$$

On the other hand,

$$y \leq (x^m \to y) \to y,$$

and so

$$((x^m \to y) \to y) \to x \leq y \to x.$$

Thus

$$((x^{n+1} \to y) \to y) \to x = y \to x,$$

which shows that A is an m-fold fantastic BL-algebra. □

The following example shows that the converse of Theorem 2.15 is not

true.

Example 2.16. Let $B = \{0, a, b, c, 1\}$. Define $*$ and $\to$ as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	a	a	b
c	0	0	a	a	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	a	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

Then, $(B, \wedge, \vee, *, \to, 0, 1)$ is a 2-fold fantastic BL-algebra.

But, we have $((b \to 0) \to 0) \to b \neq 0 \to b,$ so B is not 1-fold fantastic BL-algebra.

Theorem 2.17. *For each BL-algebra A, the following conditions are equivalent*:

(i) *A is n-fold fantastic*,

(ii) $(x^n \to y) \to y \leq (y \to x) \to x,$ *for all* $x, y \in A$,

(iii) $x^n \to z \leq y \to z, z \leq x \Rightarrow y \leq x,$

(iv) $x^n \to z \leq y \to z, z \leq x, y \Rightarrow y \leq x,$

(v) $y \leq x \Rightarrow (x^n \to y) \to y \leq x.$

Proof. (i) $\Leftrightarrow$ (ii). Let A be n-fold fantastic. Then, we have

$$((x^n \to y) \to y) \to ((y \to x) \to x) = (y \to x) \to (((x^n \to y) \to y) \to x)$$

$$= (y \to x) \to (y \to x) = 1.$$

Hence, for all $x, y \in A$, $((x^n \to y) \to y) \leq (y \to x) \to x$. Conversely, assume that the inequality $((x^n \to y) \to y) \leq (y \to x) \to x$ holds for all $x, y \in A$. Then,

$$(y \to x) \to (((x^n \to y) \to y) \to x) = ((x^n \to y) \to y) \to ((y \to x) \to\to x)$$

$$\geq ((x^n \to y) \to y) \to ((x^n \to y) \to y)$$

$$= 1.$$

Hence, $(y \to x) \to (((x^n \to y) \to y) \to x) = 1$, i.e., $y \to x \leq ((x^n \to y) \to y) \to x$. Now, we have

$$(((x^n \to y) \to y) \to x) \to (y \to x) \geq y \to ((x^n \to y) \to y)$$

$$= (x^n \to y) \to (y \to y)$$

$$= (x^n \to y) \to 1$$

$$= 1,$$

and so $(((x^n \to y) \to y) \to x) \to (y \to x) = 1$, that is, $((x^n \to y) \to y) \to x \leq y \to x$. Hence, A is n-fold fantastic.

(ii) $\Rightarrow$ (iii). Let $x, y, z \in A$ be such that $x^n \to z \leq y \to z$ and $z \leq x$. Using (p4), (p5), and condition (ii), we have

$$1 = (x^n \to z) \to (y \to z) = y \to ((x^n \to z) \to z)$$

$$\leq y \to ((z \to x) \to x)$$

$$= y \to (1 \to x)$$

$$= (y \to x),$$

and so $y \to x = 1$, i.e., $y \leq x$.

(iii) $\Rightarrow$ (iv). It is trivial.

(iv) $\Rightarrow$ (v). Let $x, y \in A$ be such that $y \leq x$. Note that $y \leq (x^n \to y) \to y$ and $x^n \to y \leq ((x^n \to y) \to y) \to y$. It follows from (iv) that $(x^n \to y) \to y \leq x$.

(v) $\Rightarrow$ (ii). Since $x \leq (y \to x) \to x$, we have $((y \to x) \to x)^n \to y \leq x^n \to y$ by induction. Since, $y \leq (y \to x) \to x$, it follows from (p5) and (v) that

$$(x^n \to y) \to y \leq (((y \to x) \to x)^n \to y) \to y \leq (y \to x) \to x.$$

This completes the proof. □

3. *n*-Fold Fantastic Filter

Definition 3.1. A nonempty subset F of A is called an *n-fold fantastic filter*, if it satisfies $1 \in F$ and:

(f8) $\forall x, y, z \in A, z \to (y \to x) \in F, z \in F \Rightarrow ((x^n \to y) \to y) \to x \in F$.

Definition 3.2. A nonempty subset F of A is called a *weak n-fold fantastic filter* of A, if it satisfies $1 \in F$ and:

(f9) $\forall x, y, z \in A, z \to ((y^n \to x) \to x) \in F, z \in F$ imply $(x \to y) \to y \in F$.

Example 3.3. Let $C = \{0, a, b, 1\}$. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1.
b	0	a	1	1
1	0	a	b	1

Then C is a BL-algebra. It is easy to see that $F = \{b, 1\}$ is a 2-fold fantastic filter of C.

Theorem 3.4. *Any n-fold fantastic filter of a BL-algebra A is a filter of A.*

Proof. Let $x, x \to y \in F$. Hence, $x \to y = x \to (1 \to y) \in F$. Since, F is an n-fold fantastic filter, we get $((y^n \to 1) \to 1) \to y = y \in F$, that is, F is a filter. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. In Example 3.3, $\{1\}$ is a filter. Since, $1 \in \{1\}$ and $1 \to (a \to b) \in \{1\}$ and $((b^2 \to a) \to a) \to b = b \notin \{1\}$, $\{1\}$ is not a 2-fold fantastic filter.

Theorem 3.6. *Let F be a filter of a BL-algebra A. Then,*

(i) *F is an n-fold fantastic filter of A, if and only if $((x^n \to y) \to y) \to x \in F$, for all $x, y \in A$ with $y \to x \in F$.*

(ii) *F is a weak n-fold fantastic filter of A, if and only if $((x \to y) \to y) \to x \in F$, for all $x, y \in A$ with $(y^n \to x) \to x \in F$.*

Proof. Assume that F is an n-fold fantastic filter of A and let $x, y \in A$ be such that $y \to x \in F$. Then $1 \to (y \to x) = y \to x \in F$ and $1 \in F$. It follows from (f6) that $((x^n \to y) \to y) \to x \in F$. Conversely, let F be a filter of A such that $((x^n \to y) \to y) \to x \in F$, for all $x, y \in A$ with $y \to x \in F$. Let $x, y, z \in A$ be such that $z \to (y \to x) \in F$ and $z \in F$. Then $y \to x \in F$ by (f4), and hence $((x^n \to y) \to y) \to x \in F$ by assumption. Thus, F is an n-fold fantastic filter of A. Similar argument induces the second part. □

Theorem 3.7. *Let F and G be filters of a BL-algebra A such that $F \subseteq G$. If F is n-fold fantastic filter, then so is G.*

Proof. Let $x, y \in A$ be such that $y \to x \in G$. Setting $k := (y \to x) \to x$, then

$$y \to k = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in F.$$

Since, F is n-fold fantastic filter, it follows from Theorem 3.6 (i) and (p4) that

$$(y \to x) \to (((k^n \to y) \to y) \to x) = ((k^n \to y) \to y) \to ((y \to x) \to x)$$

$$= ((k^n \to y) \to y) \to k \in F \subseteq G.$$

This implies from (f4) that $((k^n \to y) \to y) \to x \in G$. Since $x \le k$, we have $k^n \to y \le x^n \to y$, so $((k^n \to y) \to y) \to x \le ((x^n \to y) \to y) \to x$. Using (f2), we know that $((x^n \to y) \to y) \to x \in G$, and hence G is n-fold fantastic filter of A, by Theorem 3.6 (i). □

Corollary 3.8. *Every filter of a BL-algebra A is n-fold fantastic filter, if and only if the filter* $\{1\}$ *is n-fold fantastic.*

Theorem 3.9. *An n-fold fantastic filter is an m-fold fantastic filter for every* $m > n$.

Proof. Let F be an n-fold fantastic filter of A and for $x, y \in A$, $y \to x \in F$ and $m > n$. Since, F is an n-fold fantastic filter, we get $((x^n \to y) \to y) \to x \in F$, and since $x^m \le x^n$, we have

$$((x^n \to y) \to y) \to x \le ((x^m \to y) \to y) \to x.$$

Hence by (f2), $((x^m \to y) \to y) \to x \in F$, i.e., F is an m-fold fantastic filter. □

The following example shows that the converse of Theorem 3.9 is not true.

Example 3.10. In Example 2.16, it is clear that $F = \{1\}$ is a 2-fold fantastic filter of B, but it is not a 1-fold fantastic filter. This is because $0 \to b = 1 \in F$. But $((b \to 0) \to 0) \to b = c \notin F$.

Let F be a filter of A. We define a binary relation "$\sim$" on A as follows: For every $x, y \in A, x \sim y$, if and only if $x \to y \in F$ and $y \to x \in F$.

Then "$\sim$" is a congruence relation on A. Denote $\mathrm{A} / F := \{[x] | x \in A\}$, where $[x] := \{y \in A | x \sim y\}$, and define binary operations "$\sqcup$", "$\sqcap$", "$\hookrightarrow$", and "$\star$" on A / F as follows:

$[x] \sqcup [y] = [x \vee y], [x] \sqcap [y] = [x \wedge y], [x] \hookrightarrow [y] = [x \to y]$, and $[x] \star [y] = [x * y]$, respectively. Then $(\mathrm{A} / F, \sqcup, \sqcap, \hookrightarrow, \star, 0, 1)$ is a BL-algebra.

Theorem 3.11. *A filter F of A is n-fold fantastic, if and only if every filter of the quotient algebra* A / F *is n-fold fantastic.*

Proof. Assume that F is an n-fold fantastic filter of A and let $x, y \in A$ be such that $[x] \hookrightarrow [y] = 1$. Then $x \to y \in F$, and so $((y^n \to x) \to x) \to y \in F$ by Theorem 3.6 (i). Hence,

$$(([y]^n \hookrightarrow [x]) \hookrightarrow [x]) \hookrightarrow [y] = [((y^n \to x) \to x) \to y] = [1],$$

which proves that $\{1\}$ is an n-fold fantastic filter of A / F. By Corollary 1, every filter of A / F is n-fold fantastic. Conversely, suppose that every filter of A / F is n-fold fantastic and let $x, y \in A$ be such that $y \to x \in F$. Then $[y] \hookrightarrow [x] = [y \to x] = 1$. Since, $\{1\}$ is an n-fold fantastic filter of A / F, it follows from Theorem 3.6 (i) that

$$[((x^n \to y) \to y) \to x] = (([x]^n \hookrightarrow [y]) \hookrightarrow [y]) \hookrightarrow [x] = [1],$$

that is, $((x^n \to y) \to y) \to x \in F$. Hence by Theorem 3.6 (i), F is an n-fold fantastic filter of A.

Theorem 3.12. *A BL-algebra A is n-fold fantastic, if and only if its trivial filter* $\{1\}$ *is n-fold fantastic filter.*

Proof. Let A be an n-fold fantastic and $1 \to (y \to x) = 1$, that is, $y \to x = 1$. Since A is n-fold fantastic, we have $((x^n \to y) \to y) \to x = 1$. Hence, $\{1\}$ is n-fold fantastic filter. Conversely, assume that $\{1\}$ is an n-fold fantastic filter of A and let $k = (y \to x) \to x$ for every $x, y \in A$. Then,

$$y \to k = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in \{1\},$$

and hence $((k^n \to y) \to y) \to k = 1$, that is, $(k^n \to y) \to y \leq k$. Now, $x \leq k$ implies $k^n \to y \leq x^n \to y$, and so $(x^n \to y) \to y \leq (k^n \to y) \to y$. It follows that

$$1 = ((k^n \to y) \to y) \to k \leq ((x^n \to y) \to y) \to k,$$

so that

$$((x^n \to y) \to y) \to ((y \to x) \to x) = ((x^n \to y) \to y) \to k = 1,$$

it means that, $(x^n \to y) \to y \leq (y \to x) \to x$. Hence by Theorem 2.17, A is an n-fold fantastic BL-algebra. □

Theorem 3.13. *Every n-fold fantastic filter is a weak n-fold fantastic filter.*

Proof. Let F be an n-fold fantastic filter of A. Then A / F is n-fold fantastic. Let $x, y \in A$ be such that $(y^n \to x) \to x \in F$. Then,

$$\begin{aligned}[(x \to y) \to y] &= ([x] \hookrightarrow [y]) \hookrightarrow [y] \\ &\geq ([y]^n \hookrightarrow [x]) \hookrightarrow [x] \\ &= [(y^n \to x) \to x] = [1],\end{aligned}$$

and so $[(x \to y) \to y] = [1]$, i.e., $(x \to y) \to y \in F$. It follows from Theorem 3.6 (i) that, F is a weak n-fold fantastic filter of A.

By [8], every n-fold positive implicative filter is an n-fold implicative filter, but the converse is not true. Now, we show under some conditions an n-fold implicative filter is an n-fold positive implicative filter. □

Theorem 3.14. *Let F be a filter of A. Then F is an n-fold positive implicative filter, if and only if F is an n-fold implicative and n-fold fantastic filter.*

Proof. Let F be an n-fold implicative and n-fold fantastic filter and assume that $(x^n \to 0) \to x \in F$. Since, $x^n \to x^{2n} \leq (x^{2n} \to 0) \to (x^n \to 0)$, by Theorem 2.5 and (f2), we have

$$(x^{2n} \to 0) \to (x^n \to 0) \in F.$$

Since, F is an n-fold fantastic filter and $(x^n \to 0) \to x \in F$, by Theorem 3.6 (i), we have

$$((x^n \to (x^n \to 0)) \to (x^n \to 0)) \to x = ((x^{2n} \to 0) \to (x^n \to 0)) \to x \in F.$$

On the other hands, since $(x^{2n} \to 0) \to (x^n \to 0) \in F$, by (f4), we have $x \in F$.

Therefore, by Theorem 2.8, F is an n-fold positive implicative filter of A. Conversely, let F be an n-fold positive implicative filter of A. Consider $x, y \in A$ and $y \to x \in F$. Putting $k = ((x^n \to y) \to y) \to x$, then by Proposition (2.2), we have

$$\begin{aligned}(k^n \to y) \to k &= (k^n \to y) \to (((x^n \to y) \to y) \to x)\\ &= ((x^n \to y) \to y) \to ((k^n \to y) \to x)\\ &\geq ((x^n \to y) \to y) \to ((x^n \to y) \to x)\\ &\geq y \to x \in F.\end{aligned}$$

Hence by (f2), we have $(k^n \to y) \to k \in F$, and since F is an n-fold positive implicative filter, by Theorem (2.8), $k = ((x^n \to y) \to y) \to x \in F$, i.e., F is an n-fold fantastic filter of A. □

The following example shows that every n-fold fantastic filter of A is not an n-fold positive implicative filter.

Example 3.15. In Example 2.16, $\{1\}$ is a 2-fold fantastic filter, but $(c^2 \to 0) \to c = 1$ and $c \neq 1$, hence $\{1\}$ is not 2-fold positive implicative filter.

The following example shows that every n-fold fantastic filter of A is not an n-fold implicative filter.

Example 3.16. In Example 3.3, $F = \{1, b\}$ is a 1-fold fantastic filter, but F is not 1-fold implicative filter, since $a \to (a \to 0) \in F$ and $a \to a \in F$, but $a \to 0 \notin F$.

The following example shows that every n-fold implicative filter of A is not an n-fold fantastic filter.

Example 3.17. Let $D = \{0, a, b, c, 1\}$. Define $*$ and $\to$ as follows:

$*$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then, $(D, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. It is clear that $F = \{b, 1\}$ is a 2-fold implicative filter, but it is not a 2-fold fantastic filter. We have $0 \to c \in F$, but $((c^2 \to 0) \to 0) \to c = c \notin F$.

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