Numerical Solutions of Monic Chebyshev Polynomials on Large Scale Differentiation

M. El-Kady[1] and N. El-Sawy[2]

[1]Department of Mathematics, Faculty of Science, Helwan University,
Cairo, Egypt
E-mail: mam_el_kady@yahoo.com
[2]Department of Mathematics, Faculty of Science, Alazhar University (Girls),
Cairo, Egypt
E-mail: nabila_elsawy@yahoo.com

Abstract

In this paper, a new formula of the spectral differentiation matrices is presented. Therefore, the numerical solutions for higher-order differential equations are presented by expanding the unknown solution in terms of monic Chebyshev polynomials. The resulting systems of linear equations are solved directly for the values of the solution at the extreme points of the Chebyshev polynomial of order N. The round-off errors during the calculations of differentiation matrices elements are studied. A number of numerical examples are provided in order to show the advantages of the suggested differentiation matrices through comparisons with other works.

Keywords: *Monic Chebyshev polynomials, differentiation matrix, round-off error analysis.*

1   Introduction

The concept of the spectral methods is developed in the last few decades and it has proven to be a very useful tool in the numerical solution of differential equations (DEs) [6,18]. The high-order ordinary differential equations arise in many applications. Examples include the transverse vibration of a uniform beam that can be modeled by a fourth-order ODE; the vibration behavior of a ring-like structure by a sixth-order ODE [13,21]; the bending of a cylindrical barrel shell by an eighth-order ODE [14]; and the thermal instability of a horizontal layer of a fluid heated from below under the effect of rotation and a magnetic field by a higher order ODE [3,4,7]. In additional, many engineering problems, such as the deformation of a thin plate and the motion of a fluid, are governed by fourth-order partial differential equations (PDEs). Recently, such algorithms have been designed, both for higher-order differential equations and more general first-order systems [9,10]. Generally, problems involving high-order ODEs and PDEs are more difficult to solve than those with second-order ODEs and PDEs, respectively.

The differentiation matrices are now a popular tool for the solution of many types of ordinary differential equations [12]. In recent formulation, the basic idea is to represent the solution $f(x)$ by means of a truncated Chebyshev expansion, and to compute spatial derivatives of $f(x)$ by analytic differentiation of the series [1].

The difficulty with these problems lies not in the approximation of the differential equation, but in the fact that the resulting system of equations for the coefficients of the Chebyshev series of $du(x)/dx$ is dense. Gaussian elimination would require $O(N^3)$ operations, where N is the number of Chebyshev nodes used in the discretization. Unfortunately, for many situations of interest, complex behavior of the solution causes the condition number of the higher-order problem and the number of iterations to be large, so that direct methods would be preferable provided that an $O(N)$ or $O(N \log N)$ operation count could be maintained. Also, there are a number of difficulties associated with its use. For boundary value problems, the set of the collocation points is related to the set of basis functions as nodes of the quadrature formula which are used in the computations of the spectral coefficients. Furthermore, this approach involves the solution of very ill-condition linear systems of equations [12]. For instance, the condition number of the pseudospectral first-order operator is proportional with N^2; while the condition number for the second-order operator typically scales like N^4 [11].

The linear map $D^{(N)}$ is known as the spectral differentiation matrix. The process of obtaining approximations to the derivative of a function at collocation points x_i can be expressed as a matrix-vector multiplication. Spectral collocation/pseudospectral methods [5, 18] are known to have the capability to proven an exponential rate of convergence as the grid is refined or the degree of the interpolation polynomial is increased.

There are many works which have introduced an efficient differentiation matrices such as [1,2,17]. Recently, the authors in [10] introduce a new explicit expression of differentiation matrices by using an explicit formula for the derivatives of Chebyshev polynomials at Chebyshev Gauss- Lobatto (CGL) points. While the

authors in [9] have introduced an attempt to new differentiation matrix based on ultraspherical polynomials using two different sets of collocation points.

In matrix-vector multiplication, the total number of operations are $2N^2$, while, the matrix product can be computed in N^2 operations by using Solomonoff's algorithm [16]. It is important to remark that for large N the direct implementation of $D^{(N)}$ presented in [1] suffers from cancellation, causing large errors in the elements of the matrix $D^{(N)}$.

The authors in [1] show that the absolute error in the evaluation of element D_{01} is of order $O(N^4\epsilon)$, where ϵ is the machine precision. In this work, we provide an improvement of order $O(N^2\epsilon)$.

The aim of this work is to present a technique to reduce the condition number of higher-order differentiation matrices as well as rounding occurred in large scale of $D^{(N)}$. For this reason, this works present two main ideas: First, we investigate the monic Chebyshev approximations based on monic polynomials to avoid large quantity of the rounding in higher-order derivatives. Second, a direct implementation of $D^{(N)}$ are presented by using the previous implementation of $D^{(N-1)}$ without using matrix-matrix multiplication. Regarding to these, the monic Chebyshev polynomials are used here as a basis function to obtain the monic Chebyshev approximations at Chebyshev Gauss- Lobatto (CGL) points x_i. Moreover, the proposed method will be used to approximate the derivatives of some test functions and to obtain the numerical solutions of boundary value problems of higher-order differential equations.

The rest of the paper is organized as follows. In section 2, the definitions of monic Chebyshev polynomials and some properties are introduced; also the monic Chebyshev differentiation matrices are presented. A new implementation of computing the higher orders of differentiation matrices are presented in section 3. The round off errors resulting during computing the entries of the first three differentiation matrices are analyzed, also we introduce some techniques to reduce the round off errors and test functions in section 4. Some illustrative numerical results of boundary value problems of higher-order differential equations are presented in section 5. The final section is for some concluding remarks.

2   The Monic Chebyshev Differentiation Matrices

The classical expiration of differentiation matrix is: [15]

$$D_{ij}^{(1)} = \begin{cases} \dfrac{2N^2+1}{6}, & i=j=0, \\[2mm] -\dfrac{2N^2+1}{6}, & i=j=N, \\[2mm] -\dfrac{x_j}{2(1-x_j^2)}, & i=j\neq 0, \\[2mm] \dfrac{c_i}{c_j}\dfrac{1}{2(x_i-x_j)}, & i\neq j, \end{cases} \tag{1}$$

where $x_i = \cos\dfrac{i\pi}{N}$, $i = 0,1,\dots,N$ is the Chebyshev Gauss-Lobatto (CGL) points and $c_0 = c_N = 2$ and $c_i = 1$ for $i = 1,\dots,N-1$.

The monic Chebyshev polynomials of degree n $(n = 1,2,\dots)$ on $[-1,1]$ are defined by the formula:

$$Q_n(x) = 2^{1-n}\cos(n\cos^{-1}x).$$

Clearly, $|Q_n(x)| \le 1$ for $x \in [-1,1]$.

By the definition, and using elementary trigonometric identities, the recurrence relation are given by

$$Q_0(x) = 1, \quad Q_1(x) = x, \quad Q_2(x) = x^2 - \frac{1}{2},$$

and

$$Q_n(x) = x\,Q_{n-1}(x) - 1/4\,Q_{n-2}(x), \quad n > 2.$$

The first few monic Chebyshev polynomials are:

$$Q_3(x) = x^3 - 3/4$$
$$Q_4(x) = x^4 - x^2 + 1/8\,,$$
$$Q_5(x) = x^5 - \frac{5}{4}x^3 + 5/16\,x\,,$$
$$Q_6(x) = x^6 - 3/2\,x^4 + 9/16\,x^2 - 1/32\,.$$

Therefore, the monic Chebyshev polynomials form a complete orthogonal set on the interval $-1 \le x \le 1$ with respect to the weighting function $(x) = (1 - x^2)^{-\frac{1}{2}}$.

The Rodrigue's formula of the monic Chebyshev polynomials given by

$$Q_n(x) = \frac{2(-1)^n n!}{(2n)!}\sqrt{1 - x^2}\,\frac{d^n}{dx^n}(1 - x^2)^{n-\frac{1}{2}}.$$

Let $L^2(-1,1)$ be the space of square integrable function defined on $[-1,1]$. The monic Chebyshev polynomials constitute an orthogonal basis with respect to the inner product:

$$\langle Q_i, Q_j\rangle_w = \int_{-1}^{1} Q_i(x)Q_j(x)w(x)dx = \begin{cases} 0 & i \ne j \\ 2^{1-2i}\pi & i = j \ne 0 \\ \pi & i = j = 0 \end{cases} \qquad (3)$$

for the continuous orthogonality, and

$$\langle Q_i, Q_j\rangle_w = \sum_{k=0}^{N-1} Q_i\big(\xi_{k,N}\big)Q_j\big(\xi_{k,N}\big) = \begin{cases} 0 & i \ne j \\ 2^{1-2i}N & i = j \ne 0 \\ N & i = j = 0. \end{cases} \qquad (4)$$

for the discrete orthogonality, where $\xi_{k,N} = \cos\left(\frac{2k+1}{2N}\pi\right), k = 0,1,\ldots,N-1$ are the N zeros of $Q_N(x)$.

Now, let $f(x)$ is a smooth function approximated by the monic Chebyshev series as the following:

$$f(x) = \sum_{n=0}^{\infty} a_n Q_n(x), \tag{5}$$

Then the *m-th* derivatives of $f(x)$ has series expansion of the form:

$$f^{(m)}(x) = \sum_{n=0}^{\infty} a_n^{(m)} Q_n(x), \tag{6}$$

where the relation between the coefficients $a_n^{(m)}$ and a_n is given by Doha [8],

$$a_n^{(m)} = \frac{2^m}{(m-1)!\,c_n} \sum_{k=1}^{\infty} \frac{(k+m-2)!\,(n+k+m-2)!\,(n+2k+m-2)}{(k-1)!\,(n+k-1)!} a_{n+2k+m-2}, \tag{7}$$

where $c_0 = 2$, $c_m = 1$, $m \geq 1$, $n \geq 0$.

The relation between the monic Chebyshev polynomials and their derivatives are presented in the next theorem. However, the author in [8] has been introduced the similar results but he used the ultraspherical polynomials as a basis function.

Theorem 2.1 *The derivatives of the monic Chebyshev polynomials are given by*

$$Q_n^{(m)}(x) = \sum_{\substack{k=0 \\ (k+n-m)even}}^{n-m} b_{kn}^m \, Q_k(x), \quad k \geq m, \tag{8}$$

with

$$b_{kn}^m = \frac{2^m n}{(i-1)!\,c_k} \frac{(s-k+m-1)!\,(s+m-1)!}{(s)!\,(s-k)!} \tag{9}$$

where $2s = n + k - m$.

Proof.   The *m-th* derivative of (5) is given by

$$f^{(m)}(x) = \sum_{l=m}^{\infty} a_l Q_l^{(m)}(x). \tag{10}$$

From (7) and (6) we get:

$$f^{(m)}(x) =$$

$$\sum_{k=0}^{\infty} \frac{2^m}{(m-1)!\, c_k} \sum_{n=1}^{\infty} \frac{(n+m-2)!\,(k+n+m-2)!\,(k+2n+m-2)}{(n-1)!\,(k+n-1)!}\, a_{k+2n+m-2} Q_k(x).$$

If we set $l = k + 2n + m - 2$ and $2p = l + k - m$ we get

$$
\begin{aligned}
& f^{(m)}(x) \\
&= \sum_{k=0}^{\infty} \frac{2^m}{(m-1)!\, c_k} \sum_{\substack{l=k+m \\ (l+k-m)even}}^{\infty} r_{kl}^m Q_k(x),
\end{aligned}
\tag{11}
$$

where

$$r_{kl}^m = \frac{(p-k+m-1)!\,(p+m-1)!}{p!\,(p-k)!}\, l a_l.$$

We observe that

$$\frac{2^m}{(m-1)!\, c_k}\, r_{kn}^m = a_n b_{kn}^m.$$

Then from (10) and (11) we get

$$\sum_{k=0}^{\infty} \sum_{\substack{l=k+m \\ (l+k-m)even}}^{\infty} b_{kl}^m a_l\, Q_k(x) = \sum_{l=m}^{\infty} a_l Q_l^{(m)}(x).$$

By equating the coefficients of a_l, $l \geq m$, so we get the proof. $\quad\square$

In the next, the monic Chebyshev approximation has been introduced at (CGL) points to approximate the derivatives of the smooth functions $f(x)$. Consider the differential equation in the form:
$\mathcal{L}(f) = g$,
where $\mathcal{L}$ is the differential operator. The solution $f(x)$ can generally be approximated by a truncated expansion $f_N(x)$ given by

$$f_N(x) = \sum_{j=0}^{N} f_j \phi_j(x), \tag{12}$$

the functions $\phi_j(x)$ form a complete set of all orthogonal polynomial $\mathcal{P}_N$, If $\mathcal{L}_m$ corresponds to the projection of the operator $\mathcal{L}$ onto $\mathcal{P}_N$, then the monic Chebyshev approximation method consists in enforcing the equation

$$\mathcal{L}_N(f_N) = g$$

at a given set of (CGL) points. Then

$$\mathcal{L}_m f_i = \sum_{j=0}^{N} f_j \phi_j^{(m)}(x_i).$$

The interpolating polynomial $\mathcal{L}_m f_i$ can be expressed in terms of series expansion of the monic Chebyshev polynomials. By using theorem (1), the *m-th* derivatives of the interpolating polynomial will be as the follows:

$$\mathcal{L}_m f_i = \frac{1}{N} \sum_{j=0}^{N} \sum_{n=m}^{N} \sum_{\substack{k=0 \\ (k+n-m)even}}^{n-m} [2^{2n-1} \theta_j \theta_n b_{kn}^m Q_n(x_j) Q_k(x)] f_j.$$

We approximate the derivative of $f(x)$ at (CGL) points x_i, $i = 0,1, \dots, N$ by

$$[f^{(m)}] = D^{(m)}[f], \quad m = 1,2, \dots, N$$

where $D^{(m)} = \left[d_{ij}^{(m)}\right], i,j = 0,1, \dots, N$ are the square matrices of order $(N + 1)$ and their entries are given by:

$$d_{ij}^{(m)} = \frac{\theta_j}{N} \sum_{n=m}^{N} \sum_{\substack{k=0 \\ (k+n-m)even}}^{n-m} 2^{2n-1} \theta_n b_{k,n}^m Q_n(x_j) Q_k(x_i).$$

The first three derivatives of the matrix $D^{(m)}$ are given by:

$$d_{ij}^{(1)} = \frac{\theta_j}{N} \sum_{n=1}^{N} \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{2^{2n} \theta_n n}{c_k} Q_n(x_j) Q_k(x_i), \tag{13}$$

$$d_{ij}^{(2)} = \frac{\theta_j}{N} \sum_{n=2}^{N} \sum_{\substack{k=0 \\ (n+k)even}}^{n-2} \frac{2^{2n-1} \theta_n n(n^2 - k^2)}{c_k} Q_n(x_j) Q_k(x_i), \tag{14}$$

$$d_{ij}^{(3)} = \frac{\theta_j}{N} \sum_{n=3}^{N} \sum_{\substack{k=0 \\ (n+k)even}}^{n-3} \sum_{\substack{l=0 \\ (k+l)odd}}^{k-1} \frac{2^{2n} \theta_n nk(n^2 - k^2)}{c_k c_l} Q_n(x_j) Q_l(x_i), \tag{15}$$

where $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_j = 1$ for $j = 1,2, \dots, N - 1$.

3   Computing of $\mathbf{D^{(p)}}$

To get the higher orders of derivative matrices we can use explicit expressions

$$D^{(p)} = D_1^p,$$

which is not always satisfied, as was shown in [19]. In this section, a direct implementation of the higher derivatives [20] based on cardinal monic Chebyshev polynomial are investigated. The solution $f(x)$ of a differential equation $\mathcal{L}(f) = g$ can generally be approximated as the following expansion:

$$f_N(x) = \sum_{j=0}^{N} a_j \omega_{N,j}(x),$$

where

$$\omega_{N,j}(x) = \frac{\theta_j}{N} \sum_{n=0}^{N} \left[2^{2n-1} \theta_n Q_n(x_j) Q_n(x) \right], \tag{16}$$

is the cardinal basis functions of monic Chebyshev polynomial which form a complete set of the approximating space $\mathcal{P}_N$ and satisfy at (CGL) points the following relationship:

$$\omega_{N,j}(x_k) = \delta_{k,j}, \quad 0 \le j, k \le N$$

the coefficients a_j are simply associated with the values of f_N at x_j. The discrete operator $\mathcal{L}_N$ can be represented by $(N+1) \times (N+1)$ matrix $D^{(p)}$ whose coefficients are given by:

$$D_{kj}^{(p)} = \omega_{N,j}^{(p)}(x_k), \quad 0 \le k, j \le N.$$

We present a numerical method for computing $D^{(p)}$ based on the cardinal basis functions to avoid the rounding of errors as shown in the next theorem.

Theorem 3.1 *For $p \ge 0$ and $0 \le k, j \le N$ we have*

$$D_{kj}^{(p+1)} = \begin{cases} \dfrac{2^{N-1}}{b_j(p+2)} \, m_{p+2}(x_j), & k = j \\[3mm] \dfrac{p+1}{x_k - x_j} \left[(-1)^{j-k} \dfrac{b_k}{b_j} D_{kk}^{(p)} - D_{kj}^{(p)} \right], & k \ne j. \end{cases}$$

where

$$m_p(x) = \sum_{k=0}^{p} \binom{p}{k} (1-x^2)^{p-k} \frac{d^{k+1}}{dx^{k+1}} Q_N(x)$$

and $b_j = 2$ if $j = N$ and $b_j = 1$ otherwise.

Proof.   From [5] we get

$$\omega_{N,j}(x) = \frac{\theta_j}{N} \sum_{n=0}^{N} [2^{2n-1}\theta_n Q_n(x_j)Q_k(x)]$$

$$= (-1)^{j+1} \frac{2^{N-1}(1-x^2)}{b_j N^2 (x-x_j)} \frac{dQ_N(x)}{dx}$$

Let $0 \le j \le N$ and $p \ge 0$. Since

$$\frac{2^{N-1}(-1)^{j+1}}{b_j N^2}(1-x^2)\frac{dQ_N(x)}{dx} = (x-x_j)\omega_{N,j}(x),$$

we have

$$(x-x_j)\omega_{N,j}^{(p+2)}(x) + (p+2)\omega_{N,j}^{(p+1)}(x)$$

$$= \frac{2^{N-1}(-1)^{j+1}}{b_j N^2}\left[(1-x^2)\frac{dQ_N(x)}{dx}\right]^{(p+2)}$$

$$= \frac{2^{N-1}(-1)^{j+1}}{b_j N^2} \sum_{k=0}^{p+2} \binom{p+2}{k}(1-x^2)^{p-k+2}\frac{d^{k+1}}{dx^{k+1}}Q_N(x). \qquad (17)$$

For $x = x_j$ we get

$$\frac{2^{N-1}(-1)^{j+1}}{b_j N^2}m_{p+2}(x_j) = (p+2)\omega_{N,j}^{(p+1)}(x_j) = (p+2)D_{jj}^{(p+1)}, \qquad (18)$$

where

$$m_{p+2}(x) = \sum_{k=0}^{p+2}\binom{p+2}{k}(1-x^2)^{p-k+2}\frac{d^{k+1}}{dx^{k+1}}Q_N(x).$$

which proves the theorem for the diagonal elements of $D^{(p+1)}$.
If we apply (17) with $(p+1)$ instead of $(p+2)$ and $x = x_k$ with $k \ne j$ we have

$$\frac{2^{N-1}(-1)^{j+1}}{b_j N^2}m_{p+1}(x) = (x_k - x_j)\omega_{N,j}^{(p+1)}(x_k) + (p+1)\omega_{N,j}^{(p)}(x_k)$$

$$= (x_k - x_j)D_{kj}^{(p+1)} + (p+1)D_{kj}^{(p)}. \qquad (19)$$

By using (18) we have

$$(-1)^{j-k}(p+1)\frac{b_k}{b_j}D_{kk}^{(p)} = (x_k - x_j)D_{kj}^{(p+1)} + (p+1)D_{kj}^{(p)},$$

which proves the theorem for $k \neq j$.   □

4   Error Analysis and Performance Improvement

It is known that for the large N, the cancellation happened during the calculation of the elements of the differentiation matrix causing large errors [1]. The authors in [6] have presented the spacing of (CGL) points near the boundary of order $O(1/N^2)$. This spacing gives a small truncation error but leads to large round off errors. In this section, the main advantage of using monic Chebyshev polynomials is that the truncation leads to decrease of round-off values. In Ref. [1], Baltensperger and Trummer show that the error in the element D_{01}^1 has the large effect on the round-off error is of order $O(N^4 \delta)$.

While in this paper, the round-off error has been studied in case of monic Chebyshev polynomials and compared with [1].

Assume that

$$x_n^* = x_n + \delta_n,$$

is the exact (CGL) points where x_n is the computed value and δ_n is the corresponding error with δ_n approximately equals to machine precision with $\delta = max\{|\delta_n|\}$. The round-off error for the first three derivatives is summarized in Table (1) as the following:

Table 1: Round-off error for the first three derivatives

Error bound	Ref.[1].	Monic Chebyshev polynomials
For first derivatives	$D_{01}^{(1)*} - D_{01}^{(1)} = \dfrac{8N^4}{\pi^4}\delta$	$d_{01}^{(1)*} - d_{01}^{(1)} \leq \dfrac{\delta}{3}(1 - 3N + 2N^2)$
For second derivatives	$D_{01}^{(2)*} - D_{01}^{(2)} = \dfrac{64N^8}{\pi^8}\delta$	$d_{01}^{(2)*} - d_{01}^{(2)} \leq \dfrac{\delta}{15}(-2 + 5N - 5N^3 + 2N^4)$
For third derivatives	$D_{01}^{(3)*} - D_{01}^{(3)} = \dfrac{512N^{12}}{\pi^{12}}\delta$	$d_{01}^{(3)*} - d_{01}^{(3)} \leq \dfrac{2\delta}{105}(6 - 14N - \dfrac{7}{2}N^2 + \dfrac{35N^3}{2} - \dfrac{7N^4}{2} - \dfrac{7N^5}{2} + N^6)$

The construction of spectral differentiation matrices has been the subject of much discussion since it is sensitive to rounding errors. In order to reduce the effects of rounding errors, several modifications to the monic Chebyshev matrix (13) has been presented as the following [11].

Negative sum trick

This trick comes from the observation that the derivative of a constant must be zero. This means that:

$$\sum_{j=0}^{N} d_{ij}^{(1)} = 0, \quad i = 0,1,...,N$$

To enforce that condition we compute the diagonal entries of the matrix to satisfy

$$d_{ii}^{(1)} = -\sum_{j=0}^{N} d_{ij}^{(1)}. \tag{20}$$

The diagonal elements computed with (20) will not exactly be equal zero, but the overall effects of rounding errors are minimized.

Flipping trick

The error which incurred in the evaluation of differentiation matrix near $x = -1$ is significantly larger when $x = 1$, even if the matrix is symmetric. In other words, if i and j are small, then $d_{ij}^{(1)}$ can be computed accurately at the same time as if i and j are near N, then the evaluation of $d_{ij}^{(1)}$ is less accurate. This can be utilized by evaluating $d_{ij}^{(1)}$ in the upper half of the matrix and then by flipping it to take advantage of the following symmetry property [16].

$$d_{ij}^{(1)} = -d_{N-i,N-j}^{(1)}$$

Test functions

Consider the test functions $f_1(x) = cos(x)$ and $f_2(x) = x(1 - e^{2x})$, $f_3(x) = 1 + sin(2x^2)$, $x \in [-1,1]$. In the following tables ''maximum error'' always refers to the maximum difference between approximation and exact values at the Gauss–Lobatto points. Tables (2)-(4) represent the maximum absolute error obtained by using monic Chebyshev approximations for the first three derivatives of the three test functions compared with the classical method.

Table 2: The maximum absolute error of the first three derivatives of
$f_1(x) = cos(x)$

N	First derivative		Second derivative		Third derivative	
	Eq. 1	Eq.13	Eq.1	Eq.14	Eq.1	Eq.15
12	2.1D-13	1.1D-13	1.8D-11	1.2D-11	7.5D-10	5.7D-10
16	4.4D-14	2.1D-14	5.4D-13	1.2D-12	2.1D-10	8.7D-11
32	3.1D-12	1.7D-12	1.3D-10	6.8D-10	3.0D-07	8.9D-07
64	1.3D-11	1.3D-12	3.8D-08	7.2D-10	2.8D-05	1.1D-07

128	3.5D-10	9.3D-13	2.5D-10	3.2D-10	9.1D-03	5.8D-08
256	9.5D-09	1.6D-12	2.2D-04	2.3D-10	3.2D-00	4.3D-08
512	9.7D-08	4.8D-13	2.1D-03	2.0D-10	5.3D+02	3.0D-08

Table 3: The maximum absolute error of the first three derivatives of
$$f_2(x) = x(1 - e^{2x})$$

N	First derivative		Second derivative		Third derivative	
	Eq. 1	Eq.13	Eq.1	Eq.14	Eq.1	Eq.15
12	1.3D-07	1.3D-07	1.2D-05	1.2D-05	5.3D-04	5.3D-04
16	3.2D-12	3.5D-12	5.7D-10	6.2D-10	4.6D-08	4.8D-08
32	3.8D-11	2.3D-11	1.6D-08	7.0D-09	3.6D-06	8.8D-07
64	1.5D-10	5.1D-12	3.3D-07	3.1D-09	3.3D-04	4.5D-07
128	4.1D-09	4.5D-12	3.0D-05	1.3D-09	1.1D-01	2.1D-07
256	8.9D-08	6.5D-13	2.6D-03	1.2D-09	3.8D+01	1.9D-07
512	1.1D-06	4.9D-12	1.1D-01	1.4D-09	6.3D+03	2.0D-07

Table 4: The maximum absolute error of the first three derivatives of
$$f_3(x) = 1 + sin(2x^2)$$

N	First derivative		Second derivative		Third derivative	
	Eq. 1	Eq.13	Eq.1	Eq.14	Eq.1	Eq.15
12	1.2D-04	1.2D-04	1.2D-02	1.2D-02	5.1D-01	5.1D-01
16	6.1D-07	6.1D-07	1.0D-04	1.0D-04	8.0D-03	8.0D-03
32	1.1D-11	5.0D-12	4.7D-09	2.3D-09	1.1D-06	2.9D-07
64	4.5D-11	5.8D-12	9.8D-08	1.9D-09	9.8D-05	2.7D-07
128	1.2D-09	4.5D-12	9.0D-06	1.4D-09	3.2D-02	1.8D-07
256	1.3D-08	1.5D-12	3.7D-04	3.5D-10	5.4D-00	6.1D-08
512	1.6D-07	4.3D-13	1.6D-02	3.1D-10	9.0D+02	4.8D-08

From Tables (2)-(4) we observe that the use of the suggested matrix leads to highly accurate results better than those obtained by using classical matrix. Our proposed formulae show better results than those obtained via the classical formulae for large number of points. Also we note that, the higher derivatives approximated by using monic Chebyshev approximations more accurate than those computed by classical method.

5 Numerical Results

In this section, we present some numerical results for boundary value problems obtained by using the monic Chebyshev approximations to show that the suggested method is more accurate than the classical method.

Example 5.1 Consider the following linear second-order differential equation:

$$u''(x) - 4u'(x) + 4u(x) = exp(x) - \frac{4\,exp}{1 + exp^2},$$

with the Dirichlet boundary conditions

$$u(-1) = u(1) = 0.$$

The exact solution is

$$u(x) = exp(x) - \frac{\sinh(1)}{\sinh(2)} exp(2x) - \frac{exp}{1 + exp^2}.$$

The monic Chebyshev approximation of this equation is:

$$\sum_{j=0}^{N} d_{ij}^2 u(x_j) - 4 \sum_{j=0}^{N} d_{ij}^1 u(x_j) + 4u(x_i) = exp(x_i) - \frac{4\,exp}{1 + exp^2}.$$

Table (5) show that, the maximum absolute error of the monic Chebyshev approximations are compared with the result by using Eq. (1) and Ref. [10] and it is shown that our method is more accurate.

Table 5: The maximum absolute error for example (5.1)

N	Eq. (1)	Ref. [10]	monic Chebyshev
16	1.5D-07	1.5D-07	1.5D-07
32	1.4D-07	1.4D-07	1.5D-07
64	1.2D-06	7.4D-07	1.4D-07
128	7.3D-04	6.9D-04	1.4D-07
512	8.2D-02	2.7D-03	5.8D-09

The monic Chebyshev approximations are used to solve the example (5.1) at N =32 and obtain the results as shown in Fig.1 which confirms that the monic Chebyshev method gives almost the same solution as the analytic method
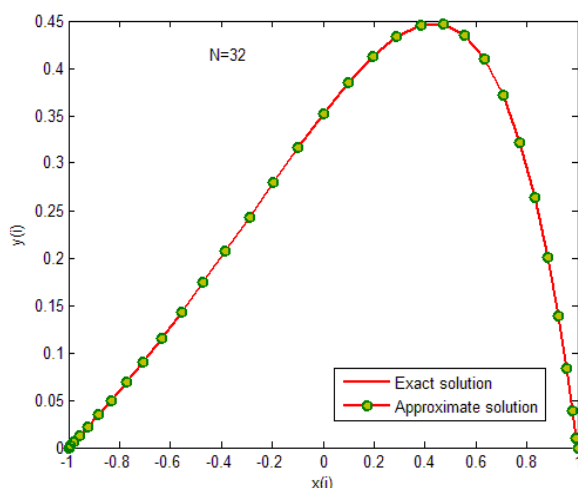
Fig.1. The solution of example (5.1) with N=32

Example 5.2 Consider the following Robin boundary value problem:

$$u''(x) + x \, exp \, (-x)u(x) = exp \, (x) + x \, , \quad -1 \le x \le 1 \, ,$$

$$u'(x) + u(x) = \frac{2}{exp} \, , \quad x = -1,$$

$$u'(x) - u(x) = 0 \, , \quad x = 1,$$

which has the exact solution $u(x) = exp \, (x) +.$ The monic Chebyshev approximation of this equation is:

$$\sum_{j=0}^{N} d_{ij}^2 u(x_j) + x_i exp \, (-x_i)u(x_i) = exp \, (x_i) + x_i \, , \quad -1 \le x \le 1 \, ,$$

and the boundary conditions is given by

$$\sum_{j=0}^{N} d_{0j}^1 u(x_j) + u(-1) = \frac{2}{exp} \, ,$$

$$\sum_{j=0}^{N} d_{Nj}^1 u(x_j) + u(1) = \frac{2}{exp} \, .$$

Fig.2. Presents the accuracy of the approximation solution with the exact for example (5.2) at N=32.
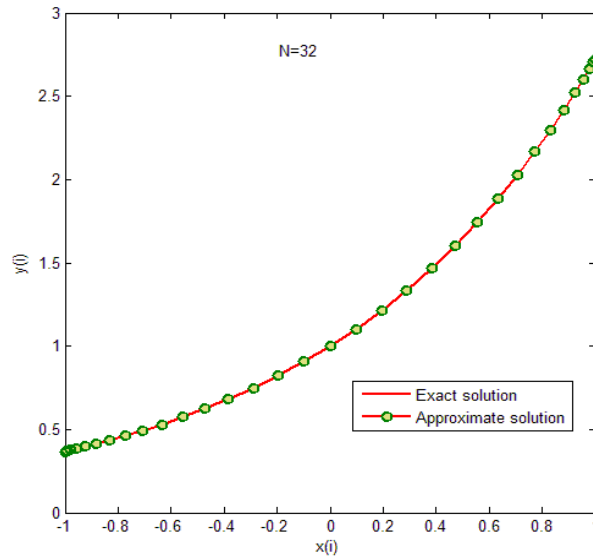
Fig.2. The solution of example (5.2) with N=32

Example 5.3   Consider the following boundary value problem:

$$64u^{(3)}(x) + 4(x + 1)u(x) = [x^3 - x^2 - 25x - 47]exp\,(0.5(x + 1)),$$
$$-1 \leq x \leq 1,$$
$$u(x) = u^{'}(x) - 1 = 0, \quad x = -1,$$
$$u^{'}(x) + exp = 0, \quad x = 1,$$

which has the exact solution

$$u(x) = 4(1 - x^2)exp\,(0.5(x + 1)).$$

The monic Chebyshev approximation of this equation is:

$$64\sum_{j=0}^{N} d_{ij}^3 u(x_j) + 4(x_i + 1)u(x_i) = [x_i^3 - x_i^2 - 25x_i - 47]exp\,(0.5(x_i + 1)),$$
$$-1 \leq x \leq 1,$$

the boundary conditions are given by

$$\sum_{j=0}^{N} d_{0j}^1 u(x_j) - 1 = 0,$$

and

$$\sum_{j=0}^{N} d_{Nj}^1 u(x_j) + exp = 0.$$

The problem has been solved using monic Chebyshev approximations, classic pseudospectral approximations and by using method in Ref. [10] at N=32. The results are shown in Fig.3.
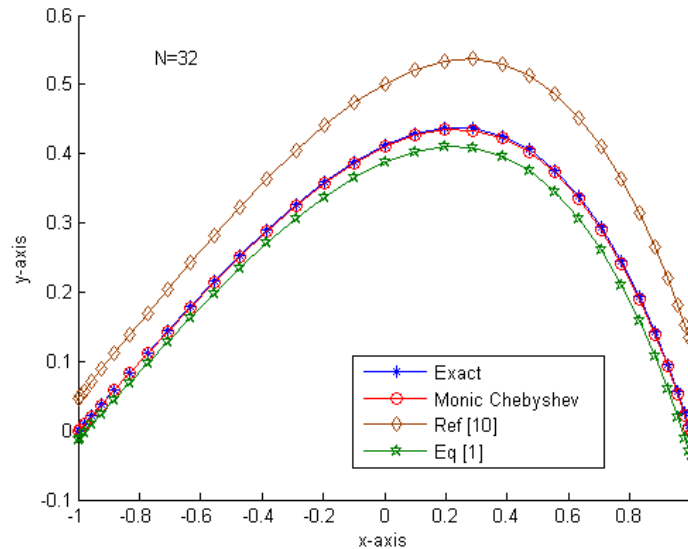

Fig.3. The solution of example (5.3) with N=32

6   Conclusion

In this paper the differentiation matrix based on the monic Chebyshev polynomials $Q_n(x)$ is presented. The main advantage of these polynomials is that the size of the monic polynomial is $1/2^{n-1}$, $n \geq 1$ and this becomes smaller as the degree n increases. The degree n monic polynomial with the smallest maximum on $[-1,1]$ is the modified Chebyshev polynomial $T_n(x)$. This result is used in approximate higher-order differential applications and can be applied to obtain an improvement interpolation scheme.

The tables present a comparison of the maximum absolute error resulting from the proposed method and those obtained by Eq.(1) and Ref.[10]. It shows favorable agreement and it is always more accurate than other treatments.

References

[1]   R. Baltensperger and M.R. Trummer, Spectral differencing with a twist, *SIAM J. Sci. Compu.*, 24(2003), 1465- 1487.

[2]   R. Barrio and J.M. Pena, Numerical evaluation of the p-th derivative of Jacobi series, *J. Appl. Numerical Math.*, 43(2003), 335-357.

[3]   A. Boutayeb and E.H. Twizell, Finite-difference method for twelfth-order boundary-value problems, *J. Comp. Appl. Math*, 35(1991), 133-138.

[4]   A. Boutayeb and E.H. Twizell, Finite-difference method for the solution of spectral eighth-order boundary-value problems, *Internat. J. Comput. Math.*, 48(1993), 63-75.

[5]   J.P. Boyd, *Chebyshev and Fourier Spectral Methods*, *(2nd ed.)*, Dover Publications, Inc., (2000).

[6]   C. Canuto, M.Y. Hussaini, A. Quarterini and T.A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, (1988).

[7]   S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford Univ. Press, Oxford, (1961).

[8]   E.H. Doha, The coefficients of differentiated expansions and derivatives of ultraspherical polynomials, *J. Comput. Math. Appl.*, 19(1991), 115-122.

[9]   M. el-kady and M.A. Ibrahim, Ultraspherical differentiation matrix method for solving systems of initial value differential algebraic equations, *Appl. Math and Comp. Sci.*, 9(2009), 1-12.

[10]   M.E. Elbarbary and M. El-Sayed, Higher order pseudospectral differentiation matrices, *Appl Numer Math*, 55(2005), 425–438.

[11]   E.A. Coutsias, T. Hagstrom and David Torres, An efficient spectral methods for ordinary differential equations with rational function coefficients, *Math. Comp.*, 65(1996), 611-635.

[12]   D. Funaro, *Polynomial Approximation of Differential Equations*, Springer-Verlag, Berlin, (1992).

[13]   R.H. Gutierrez and P.A.A. Laura, Vibrations on non-uniform rings studied by means of the differential quadrature method, *J. Sound Vib.*, 185(1995), 507-513.

[14]   V.S. Kelkar and R.T. Sewell, *Fundamentals of the Analysis and Design of Shell Structures New Jersey*, Prentice Hall Inc., (1987).

[15]   J.C. Mason and D.C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC, New York, (2003).

[16]   A. Solomonoff, A fast algorithm for spectral differentiation, *J. Comput. Phys.*, 98(1992), 174-177.

[17]   T. Tang and M.R. Trummer, Boundary layer resolving spectral methods for singular perturbation problems, *SIAM J. Sci. Comp.*, 17(1996), 430-438.

[18]   L.N. Trefethen, *Spectral Methods in Matlab*, SIAM, Philadelphia, (2000).

[19]   B.D. Welfert, *A Remark on Pseudospectral Differentiation Matrices*, Arizona State University, Tempe, AZ, (1992).

[20]   B.D. Welfert, Generation of pseudospectral differentiation matrices I, *SIAM J. Numer. Anal.*, 34(1997), 1640-1657.

[21]   T.Y. Wn and G.R. Liu, Application of generalized differential quadrature rule to sixth-order differential equations, *Commun. Numer. Methods Eng.*, 16(2000), 777-784.