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Kinetic Energy and Complementary Kinetic Energy in Gyrodynamics

The concepts of kinetic energy and complementary kinetic energy permit to distinguish between two different formulations of what happens to be the same quantity in Newtonian mechanics. These formulations turn out to play a significant role in gyrodynamics in that they can be used very effectively to establish fundamental equations. It is relatively straightforward to establish a complementary kinetic energy expression. The establishment of the kinetic energy expression is, in spite of its unambiguous definition, more complicated, and a way is presented how to obtain it without too much difficulty. Equations are presented for the more common angle systems, i.e., for Euler angles of the first kind, Cardan angles of the first kind, both because of their fundamental importance, and Cardan angles of the fifth kind because of its prevalence in aeronautics.

Fundamental Concepts

Following Crandall (1957) and Tabarrok (1981), we adopt the concepts of kinetic energy T, and complementary kinetic energy T^*, as analogies to potential energy V and complementary potential energy V^*. Using generalized coordinates q_i and generalized momenta p_i of analytical mechanics, the kinetic energy of a scleronomic (i.e., where the time t does not appear explicitly) system with n-degrees-of-freedom is defined by

$$T=\int \dot{q}_i(q,p)dp_i \tag{1}$$

while its complementary kinetic energy is

$$T^*=\int p_i(q,\dot{q})d\dot{q}_i \tag{2}$$

with $i = 1, 2, 3, \ldots, n$.

For a point mass m on a straight path q, as an example,

$$T=\frac{1}{2}\frac{p^2}{m} \quad T^*=\frac{1}{2}m\dot{q}^2. \tag{3a,b}$$

Generalized velocities and generalized momenta are obtained by partial differentiation and defined by

$$\dot{q}_i=\frac{\partial}{\partial p_i}T(q,p) \tag{4}$$

$$p_i=\frac{\partial}{\partial \dot{q}_i}T^*(q,\dot{q}) \tag{5}$$

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For a point mass m on a straight path q, as an example,

$$\dot{q}=\frac{\partial}{\partial p}\left(\frac{1}{2}\frac{p^2}{m}\right)=\frac{p}{m} \quad p=\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{q}^2\right)=m\dot{q}. \tag{6a,b}$$

In Newtonian mechanics it so happens that kinetic energy T and complementary kinetic energy T^* have the same magnitude (Fig. 1). This circumstance can often be exploited to great advantage. But in order to be able to apply formulas (4) and (5), the kinetic energy T *must* be expressed in terms of q and p, while the complementary kinetic energy T^* *must* be expressed in terms of q and $\dot{q}$. Thus it is possible to establish $T^* = T^*(q, \dot{q})$ which, in the case of gyrodynamics of rigid bodies, is usually readily done. Partial differentiation with respect to the generalized velocities $\dot{q}_i$ then results in the generalized momenta p_i. With the generalized momenta p_i known, it becomes then possible to form $T = T(q, p)$, which can subsequently be differentiated partially with respect to the

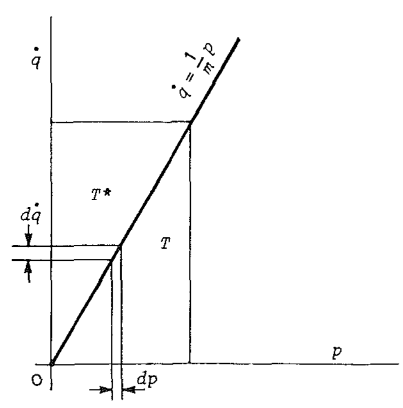

Fig. 1 Kinetic energy T and complementary kinetic energy T^* for a single point mass in Newtonian mechanics

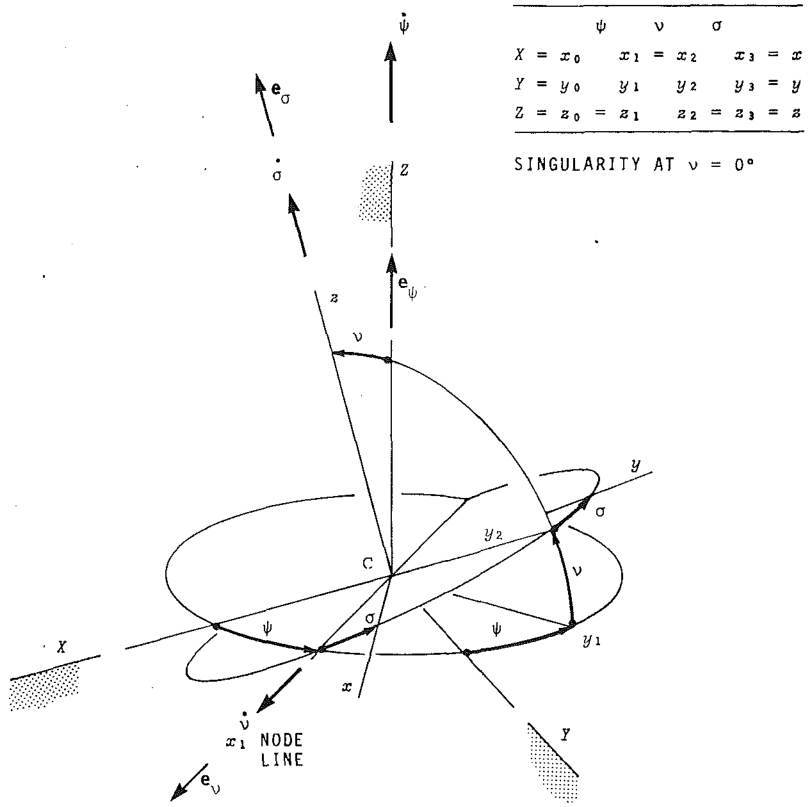

Fig. 2 Euler angles of the first kind

generalized momenta p_i, to obtain the generalized velocities $\dot{q}_i$. This process is used in the present paper to establish the fundamental relationships for a single rigid gyro, first with Euler angles as generalized coordinates, and then with Cardan angles as generalized coordinates.

Kinetic Energy of Rotation. A suitable ansatz for the complementary kinetic energy of rotation of a single rigid gyro is

$$T^* = \frac{1}{2}\,(A\omega_x^2 + B\omega_y^2 + C\omega_z^2) \tag{7}$$

where the angular velocity is

$$\boldsymbol{\omega} = [\mathbf{e}_x \quad \mathbf{e}_y \quad \mathbf{e}_x] \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{8}$$

and the inertia tensor is

$$[I] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}. \tag{9}$$

The $Cxyz$ coordinate system is body fixed, coincides with the gyro's principal axes, and originates at the gyro's mass center C.

Since angular momentum components and angular velocity components are related by

$$H_x = A\omega_x$$
$$H_y = B\omega_y$$
$$H_z = C\omega_z, \tag{10}$$

Eq. (7) can then be rearranged to supply an ansatz for the kinetic energy

$$T = \frac{1}{2}\left(\frac{H_x^2}{A} + \frac{H_y^2}{B} + \frac{H_z^2}{C}\right). \tag{11}$$

Expressions (7) and (11) are, however, neither specifically a kinetic energy T, nor specifically a complementary kinetic energy T^*, in the sense of Eqs. (1) and (2). In order to express them properly, we need generalized coordinates, for which Euler angles are suitable, as well as Cardan angles.

It is well known (Magnus, 1971) that Cartesian coordinates of the angular velocity are nonholonomic quantities (nonintegrable with respect to time) and therefore cannot be used as generalized velocities. However, as long as the kinetic energies as defined in Eqs. (1) and (2) are not explicitly dependent on generalized coordinates themselves, they can be adopted for nonholomic velocities as well. This is specifically the case in Eq. (7), and by analogy in Eq. (11). Distinguishing between the terms complementary kinetic (7) and kinetic energy (11) is justified here solely by the role the expression can play in further application in the present paper.

Euler Angles of the First Kind

In order to establish the angular position, often called the *attitude* of a gyro, three angular coordinates, the so-called *Euler angles* may be used. In Fig. 2 a $Cxyz$ coordinate system is shown at angles ψ, ν, σ with respect to a $CXYZ$ coordinate system. The $CXYZ$ coordinate system is fixed in space, while the $Cxyz$ coordinate system is body fixed (i.e., fixed to the gyro). The x_1-line, representing the intersection of the xy-plane with the XY-plane, is called the *node line*. The time derivatives of the Euler angles are the *Euler frequencies* (or *Euler rates*), viz. precession $\dot{\psi}$, nutation $\dot{\nu}$, and spin $\dot{\sigma}$. The Euler frequencies can readily be integrated to yield the Euler angles

$$\psi = \psi_0 + \int \dot{\psi}\,dt$$
$$\nu = \nu_0 + \int \dot{\nu}\,dt$$
$$\sigma = \sigma_0 + \int \dot{\sigma}\,dt \tag{12}$$

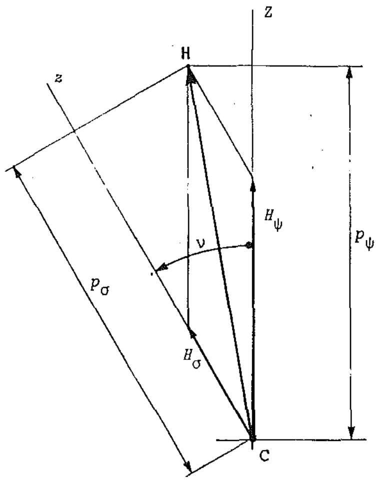

Fig. 3 Angular momentum components and generalized momenta for Euler angles of the first kind (when $H_\nu = 0$)

and thus give the attitude of the gyro as a function of time. This property makes Euler angles suitable as generalized coordinates.

The angular velocity ω of the gyro can be written (Rimrott, 1988)

$$\omega = [\mathbf{e}_x \quad \mathbf{e}_y \quad \mathbf{e}_z]\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = [\mathbf{e}_\psi \quad \mathbf{e}_\nu \quad \mathbf{e}_\sigma]\begin{bmatrix}\dot\psi\\ \dot\nu\\ \dot\sigma\end{bmatrix} \quad (13)$$

where $Cxyz$ are gyro-fixed Cartesian coordinate axes, and $q_1 = \psi$, $q_2 = \nu$, $q_3 = \sigma$ are the generalized position coordinates of analytical mechanics.

The transition is given by

$$\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\sin\nu\sin\sigma & \cos\sigma & 0\\ \sin\nu\cos\sigma & -\sin\sigma & 0\\ \cos\nu & 0 & 1\end{bmatrix}\begin{bmatrix}\dot\psi\\ \dot\nu\\ \dot\sigma\end{bmatrix}. \quad (14)$$

The angular momentum components follow the same transition (14), i.e.,

$$\begin{bmatrix}H_x\\ H_y\\ H_z\end{bmatrix} = \begin{bmatrix}\sin\nu\sin\sigma & \cos\sigma & 0\\ \sin\nu\cos\sigma & -\sin\sigma & 0\\ \cos\nu & 0 & 1\end{bmatrix}\begin{bmatrix}H_\psi\\ H_\nu\\ H_\sigma\end{bmatrix}. \quad (15)$$

Complementary Kinetic Energy. Combining Eqs. (2), (7), and (14) gives us now the complementary kinetic energy, properly expressed in terms of generalized coordinates and generalized velocities,

$$T^* = \frac{1}{2}\,[A(\dot\psi\sin\nu\sin\sigma + \dot\nu\cos\sigma)^2 + B(\dot\psi\sin\nu\cos\sigma - \dot\nu\sin\sigma)^2 + C(\dot\psi\cos\nu + \dot\sigma)^2], \quad (16)$$

where the Euler angles ψ, ν, σ are the generalized coordinates q_i of analytical mechanics.

Generalized Momenta. According to Eq. (5), the generalized momenta p_i are obtained by partial differentiation of the complementary kinetic energy T^* with respect to the generalized velocities $\dot q_i$, with $\dot q_1 = \dot\psi$, $\dot q_2 = \dot\nu$, and $\dot q_3 = \dot\sigma$. Thus (Greenwood, 1988),

$$p_\psi = \frac{\partial T^*}{\partial\dot\psi} = A(\dot\psi\sin\nu\sin\sigma + \dot\nu\cos\sigma)\sin\nu\sin\sigma + B(\dot\psi\sin\nu\cos\sigma - \dot\nu\sin\sigma)\sin\nu\cos\sigma + C(\dot\psi\cos\nu + \dot\sigma)\cos\nu \quad (17a)$$

$$p_\nu = \frac{\partial T^*}{\partial\dot\nu} = A(\dot\psi\sin\nu\sin\sigma + \dot\nu\cos\sigma)\cos\sigma - B(\dot\psi\sin\nu\cos\sigma - \dot\nu\sin\sigma)\sin\sigma \quad (17b)$$

$$p_\sigma = \frac{\partial T^*}{\partial\dot\sigma} = C(\dot\psi\cos\nu + \dot\sigma). \quad (17c)$$

Note that the generalized momenta p_i are angular and represent the *projections* (covariant components) of the angular momentum vector $\mathbf{H}$ onto the $\dot\psi$, $\dot\nu$, and $\dot\sigma$ directions. They do not represent the (contravariant) components H_ψ, H_ν, and H_σ of the angular momentum vector, to which they are related by (Fig. 3)

$$\begin{bmatrix}p_\psi\\ p_\nu\\ p_\sigma\end{bmatrix} = \begin{bmatrix}1 & 0 & \cos\nu\\ 0 & 1 & 0\\ \cos\nu & 0 & 1\end{bmatrix}\begin{bmatrix}H_\psi\\ H_\nu\\ H_\sigma\end{bmatrix}. \quad (18)$$

The inverse relationship is

$$\begin{bmatrix}H_\psi\\ H_\nu\\ H_\sigma\end{bmatrix} = \begin{bmatrix}\frac{1}{\sin^2\nu} & 0 & \frac{-\cos\nu}{\sin^2\nu}\\ 0 & 1 & 0\\ \frac{-\cos\nu}{\sin^2\nu} & 0 & \frac{1}{\sin^2\nu}\end{bmatrix}\begin{bmatrix}p_\psi\\ p_\nu\\ p_\sigma\end{bmatrix} \quad (19)$$

Equations (17) can be written in matrix form

$$\begin{bmatrix}p_\psi\\ p_\nu\\ p_\sigma\end{bmatrix} = \begin{bmatrix}(A\sin^2\sigma + B\cos^2\sigma)\sin^2\nu + C\cos^2\nu & (A-B)\sin\nu\sin\sigma\cos\sigma & C\cos\nu\\ (A-B)\sin\nu\sin\sigma\cos\sigma & A\cos^2\sigma + B\sin^2\sigma & 0\\ C\cos\nu & 0 & C\end{bmatrix}\begin{bmatrix}\dot\psi\\ \dot\nu\\ \dot\sigma\end{bmatrix} \quad (20)$$

or, in shorthand notation,

$$\{p\} = [A]\{\dot q\}. \quad (21)$$

It is seen that the inertia coefficient matrix $[A]$ is a symmetric transformation matrix which maps the generalized velocities into the generalized momenta.

The transformation matrix contains partial derivatives as elements.

$$[A] = \begin{bmatrix}\frac{\partial p_\psi}{\partial\dot\psi} & \frac{\partial p_\psi}{\partial\dot\nu} & \frac{\partial p_\psi}{\partial\dot\sigma}\\ \frac{\partial p_\nu}{\partial\dot\psi} & \frac{\partial p_\nu}{\partial\dot\nu} & \frac{\partial p_\nu}{\partial\dot\sigma}\\ \frac{\partial p_\sigma}{\partial\dot\psi} & \frac{\partial p_\sigma}{\partial\dot\nu} & \frac{\partial p_\sigma}{\partial\dot\sigma}\end{bmatrix} = \left[\frac{\partial p_i}{\partial\dot q_j}\right] \quad (22)$$

Symmetry means that, e.g.,

$$\frac{\partial p_\psi}{\partial\dot\nu} = \frac{\partial p_\nu}{\partial\dot\psi}, \quad (23)$$

etc., or in general we have that

$$\frac{\partial p_i}{\partial\dot q_j} = \frac{\partial p_j}{\partial\dot q_i} \quad i,j = \psi,\nu,\sigma. \quad (24)$$

With the help of Eq. (21) the complementary kinetic energy (16) can now be expressed as

$$T^* = \frac{1}{2}\,\{\dot q\}^T[A]\{\dot q\}. \quad (25)$$

Lagrange Equation. All loading cases for a single rigid gyro are covered by Lagrange's equation in its fundamental form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i}-\frac{\partial T^*}{\partial q_i}=Q_i \qquad (26)$$

With the generalized momenta (17) and the complementary kinetic energy (16), the three Lagrange equations become

$$\frac{d}{dt}p_\psi-\frac{\partial T^*}{\partial \psi}=Q_\psi \qquad (27a)$$

$$\frac{d}{dt}p_\nu-\frac{\partial T^*}{\partial \nu}=Q_\nu \qquad (27b)$$

$$\frac{d}{dt}p_\sigma-\frac{\partial T^*}{\partial \sigma}=Q_\sigma. \qquad (27c)$$

An inspection of Eq. (16) shows that the complementary kinetic energy T^* does not contain the generalized coordinate ψ explicitly, thus $\partial T^*/\partial\psi = 0$ in Eq. (27a). Note that the Q_i are covariant torque components, which are related to the contravariant torque components M_i by the same transformation as in Eq. (19), i.e.,

$$\begin{bmatrix} M_\psi \\ M_\nu \\ M_\sigma \end{bmatrix}=\begin{bmatrix} \frac{1}{\sin^2\nu} & 0 & \frac{-\cos\nu}{\sin^2\nu} \\ 0 & 1 & 0 \\ \frac{-\cos\nu}{\sin^2\nu} & 0 & \frac{1}{\sin^2\nu} \end{bmatrix}\begin{bmatrix} Q_\psi \\ Q_\nu \\ Q_\sigma \end{bmatrix} \qquad (28)$$

The square of the magnitude of the torque applied can be shown to be

$$M^2=M_x^2+M_y^2+M_z^2 \qquad (29a)$$

$$M^2=M_\psi^2+M_\nu^2+M_\sigma^2+2M_\psi M_\sigma\cos\nu \qquad (29b)$$

$$M^2=\frac{1}{\sin^2\nu}\left(Q_\psi^2+Q_\nu^2\sin^2\nu+Q_\sigma^2\right)-\frac{2\cos\nu}{\sin^2\nu}Q_\psi Q_\sigma \qquad (29c)$$

D'Alembert's Principle. D'Alembert's principle, involving the vanishing of the virtual work of applied torques and inertial torques, reads

$$\delta W=(\mathbf{M}-\dot{\mathbf{H}})\cdot\delta\theta=0. \qquad (30)$$

Using Eulerian angles as generalized coordinates, D'Alembert's principle can now be written as

$$\delta W=\left(Q_\psi-\left(\dot{p}_\psi-\frac{\partial T^*}{\partial\psi}\right)\right)\delta\psi+\left(Q_\nu-\left(\dot{p}_\nu-\frac{\partial T^*}{\partial\nu}\right)\right)\delta\nu+\left(Q_\sigma-\left(\dot{p}_\sigma-\frac{\partial T^*}{\partial\sigma}\right)\right)\delta\sigma=0. \qquad (31)$$

Since the three variations $\delta\psi$, $\delta\nu$, and $\delta\sigma$ do not vanish and are independent, their coefficients must vanish, i.e., they must satisfy the Lagrange Eqs. (27).

Kinetic Energy. Inverting Eq. (21) results in

$$\{\dot{q}\}=[A]^{-1}\{p\} \qquad (32)$$

and the kinetic energy (1) can then be obtained as

$$T=\frac{1}{2}\{p\}^T[A]^{-1}\{p\}. \qquad (33)$$

The inversion of the matrix $[A]$ is, however, cumbersome, and a more tractable approach for obtaining the kinetic energy is the following: beginning with Eqs. (11) and (15) we obtain, as an intermediate expression,

$$T=\frac{1}{2}\left(\frac{1}{A}\left(H_\psi\sin\nu\sin\sigma+H_\nu\cos\sigma\right)^2+\frac{1}{B}\left(H_\psi\sin\nu\cos\sigma-H_\nu\sin\sigma\right)^2+\frac{1}{C}\left(H_\psi\cos\nu+H_\sigma\right)^2\right). \qquad (34)$$

The reader is asked to compare Eqs. (34) and (16) and to note the similarity of the terms in round brackets.

Properly expressed in terms of the generalized momenta p_ψ, p_ν, p_σ, the kinetic energy (1) is, from Eqs. (19) and (34),

$$T=\frac{1}{2}\left[\frac{1}{\sin^2\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)p_\psi^2+\left(\frac{\cos^2\sigma}{A}+\frac{\sin^2\sigma}{B}\right)p_\nu^2+\left(\frac{1}{C}+\frac{1}{\tan^2\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\right)p_\sigma^2+\frac{2\sin\sigma\cos\sigma}{\sin\nu}\left(\frac{1}{A}-\frac{1}{B}\right)p_\psi p_\nu-\frac{2\sin\sigma\cos\sigma}{\tan\nu}\left(\frac{1}{A}-\frac{1}{B}\right)p_\nu p_\sigma-\frac{2}{\sin\nu\tan\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)p_\psi p_\sigma\right]. \qquad (35)$$

Generalized Velocities. According to definition (4), the generalized velocities are obtained by partial differentiation of the kinetic energy (35) with respect to the generalized momenta.

$$\dot{\psi}=\frac{\partial T}{\partial p_\psi}=\frac{1}{\sin^2\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)p_\psi+\frac{\sin\sigma\cos\sigma}{\sin\nu}\left(\frac{1}{A}-\frac{1}{B}\right)p_\nu-\frac{1}{\sin\nu\tan\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)p_\sigma \qquad (36a)$$

$$\dot{\nu}=\frac{\partial T}{\partial p_\nu}=\frac{\sin\sigma\cos\sigma}{\sin\nu}\left(\frac{1}{A}-\frac{1}{B}\right)p_\psi+\left(\frac{\cos^2\sigma}{A}+\frac{\sin^2\sigma}{B}\right)p_\nu-\frac{\sin\sigma\cos\sigma}{\tan\nu}\left(\frac{1}{A}-\frac{1}{B}\right)p_\sigma \qquad (36b)$$

$$\dot{\sigma}=\frac{\partial T}{\partial p_\sigma}=-\frac{1}{\sin\nu\tan\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)p_\psi-\frac{\sin\sigma\cos\sigma}{\tan\nu}\left(\frac{1}{A}-\frac{1}{B}\right)p_\nu+\left(\frac{1}{C}+\frac{1}{\tan^2\nu}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\right)p_\sigma \qquad (36c)$$

Writing Eqs. (36) in matrix form gives

$$\begin{bmatrix}\dot{\psi}\\ \dot{\nu}\\ \dot{\sigma}\end{bmatrix}=\begin{bmatrix}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\frac{1}{\sin^2\nu} & \left(\frac{1}{A}-\frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\sin\nu} & -\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\frac{1}{\sin\nu\tan\nu}\\ \left(\frac{1}{A}-\frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\sin\nu} & \frac{\cos^2\sigma}{A}+\frac{\sin^2\sigma}{B} & -\left(\frac{1}{A}-\frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\tan\nu}\\ -\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\frac{1}{\sin\nu\tan\nu} & -\left(\frac{1}{A}-\frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\tan\nu} & \frac{1}{C}+\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\frac{1}{\tan^2\nu}\end{bmatrix}\begin{bmatrix}p_\psi\\ p_\nu\\ p_\sigma\end{bmatrix} \qquad (37)$$

or, in shorthand notation,

$$\{\dot{q}\}=[A]^{-1}\{p\}, \qquad (38)$$

i.e., the same as Eq. (32). The (symmetric) matrix $[A]^{-1}$ is the inverse of the transformation matrix of Eq. (20) and maps the generalized momenta into the generalized velocities.

Cardan Angles of the First Kind

Just as was the case with Euler angles, Cardan frequencies can be integrated to yield the Cardan angles

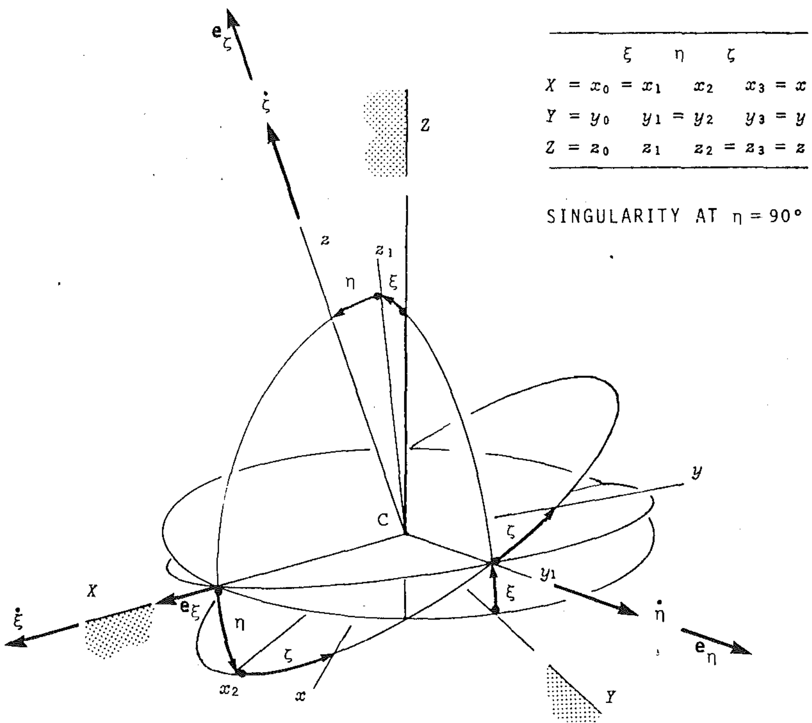

Fig. 4 Cardan angles of the first kind

$$\xi = \xi_0 + \int \dot{\xi}\, dt$$
$$\eta = \eta_0 + \int \dot{\eta}\, dt$$
$$\zeta = \zeta_0 + \int \dot{\zeta}\, dt, \qquad (39)$$

and thus give the attitude of the gyro as function of time. This property makes Cardan angles suitable as generalized coordinates.

When terms such as *yaw* (or *heading*) angle, *pitch* (or *flight path*) angle, and *roll* (or *bank*) angle are used, then Cardan angles are involved. Depending on the choice of axes, altogether six kinds of Cardan angle systems can be employed (Rimrott, 1988). The subsequent equations are for Cardan angles of the first kind (Fig. 4), consisting of ξ (about the space-fixed X-axis), η (about the once carried y_1-axis), and ζ (about the twice-carried z-axis).

The angular velocity ω of the gyro can either be expressed in terms of components along the body-fixed $Cxyz$ coordinate system, or in terms of the *Cardan frequencies* along the pertinent (nonorthogonal) carried coordinates:

$$\boldsymbol{\omega} = [\mathbf{e}_x \quad \mathbf{e}_y \quad \mathbf{e}_z] \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = [\mathbf{e}_\xi \quad \mathbf{e}_\eta \quad \mathbf{e}_\zeta] \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{bmatrix}. \qquad (40)$$

The transition between the two coordinate system can be effected by

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \cos\eta \cos\zeta & \sin\zeta & 0 \\ -\cos\eta \sin\zeta & \cos\zeta & 0 \\ \sin\eta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{bmatrix}. \qquad (41)$$

Transition (41) is also valid for the angular momentum components, i.e.,

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\eta \cos\zeta & \sin\zeta & 0 \\ -\cos\eta \sin\zeta & \cos\zeta & 0 \\ \sin\eta & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix}. \qquad (42)$$

Complementary Kinetic Energy. Combining Eqs. (2), (7), and (41) gives us now, for the complementary kinetic energy,

$$T^* = \frac{1}{2}\,[A(\dot{\xi}\cos\eta\cos\zeta + \dot{\eta}\sin\zeta)^2 + B(-\dot{\xi}\cos\eta\sin\zeta + \dot{\eta}\cos\zeta)^2 + C(\dot{\xi}\sin\eta + \dot{\zeta})^2], \qquad (43)$$

where the Cardan angles ξ, η, and ζ are the generalized coordinates q_i of analytical mechanics.

Generalized Momenta. The generalized momenta p_i are obtained by partial differentiation of the complementary kinetic energy T^* with respect to the generalized velocities $\dot{q}_i$, with $\dot{q}_1 = \dot{\xi}$, $\dot{q}_2 = \dot{\eta}$, $\dot{q}_3 = \dot{\zeta}$. Thus,

$$p_\xi = \frac{\partial T^*}{\partial \dot{\xi}} = A(\dot{\xi}\cos\eta\cos\zeta + \dot{\eta}\sin\zeta)\cos\eta\cos\zeta + B(\dot{\xi}\cos\eta\sin\zeta - \dot{\eta}\cos\zeta)\cos\eta\sin\zeta + C(\dot{\xi}\sin\eta + \dot{\zeta})\sin\eta \qquad (44a)$$

$$p_\eta = \frac{\partial T^*}{\partial \dot{\eta}} = A(\dot{\xi}\cos\eta\cos\zeta + \dot{\eta}\sin\zeta)\sin\zeta + B(-\dot{\xi}\cos\eta\sin\zeta + \dot{\eta}\cos\zeta)\cos\zeta \qquad (44b)$$

$$p_\zeta = \frac{\partial T^*}{\partial \dot{\zeta}} = C(\dot{\xi}\sin\eta + \dot{\zeta}). \qquad (44c)$$

The generalized momenta p_i are related to the components H_ξ, H_η, and H_ζ of the angular momentum vector (Fig. 5),

$$\begin{bmatrix} p_\xi \\ p_\eta \\ p_\zeta \end{bmatrix} = \begin{bmatrix} 1 & 0 & \sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix}. \qquad (45)$$

The inverse relationship is

$$\begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos^2\eta} & 0 & \frac{-\sin\eta}{\cos^2\eta} \\ 0 & 1 & 0 \\ \frac{-\sin\eta}{\cos^2\eta} & 0 & \frac{1}{\cos^2\eta} \end{bmatrix} \begin{bmatrix} p_\xi \\ p_\eta \\ p_\zeta \end{bmatrix}. \qquad (46)$$

Lagrange Equation. With the generalized momenta (44) and the complementary kinetic energy (43), the three Lagrange equations become

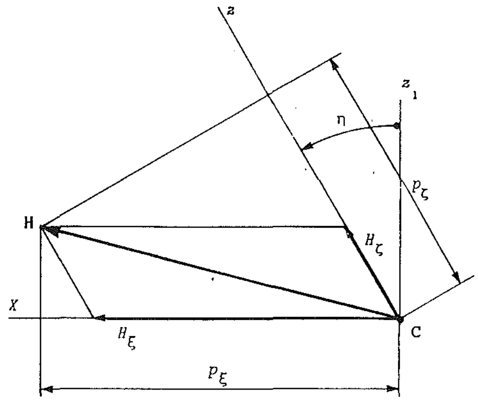

Fig. 5 Angular momentum components and generalized momenta for Cardan angles of the first kind (when $H_\eta = 0$)

$$\frac{d}{dt}p_\xi - \frac{\partial T^*}{\partial \xi} = Q_\xi \tag{47a}$$

$$\frac{d}{dt}p_\eta - \frac{\partial T^*}{\partial \eta} = Q_\eta \tag{47b}$$

$$\frac{d}{dt}p_\zeta - \frac{\partial T^*}{\partial \zeta} = Q_\zeta. \tag{47c}$$

For Cardan angles of the first kind, the angle ξ is taken about a space-fixed axis, thus $\partial T^*/\partial\xi = 0$ in Eq. (47*a*). D'Alembert's principle (30) can be expressed in terms of Cardan angles of the first kind by using the Lagrange Eqs. (47).

Note that the generalized forces Q_i are covariant components of the applied torque similar to Eqs. (29) and the square of the magnitude of the torque applied can be shown to be

$$M^2 = M_x^2 + M_y^2 + M_z^2 \tag{48a}$$

$$M^2 = M_\xi^2 + M_\eta^2 + M_\zeta^2 + 2M_\xi M_\zeta \sin\eta \tag{48b}$$

$$M^2 = \frac{1}{\cos^2\eta}\left(Q_\xi^2 + Q_\eta^2\cos^2\eta + Q_\zeta^2\right) - \frac{2\sin\eta}{\cos^2\eta}Q_\xi Q_\zeta, \tag{48c}$$

where M_x, M_y, M_z are the torque components along gyro-fixed Cartesian axes, and M_ξ, M_η, M_ζ are the contravariant torque components.

Kinetic Energy. Using Eqs. (11) and (42) an intermediate expression is obtained:

$$T = \frac{1}{2}\left(\frac{1}{A}\left(H_\xi\cos\eta\cos\zeta + H_\eta\sin\zeta\right)^2 + \frac{1}{B}\left(-H_\xi\cos\eta\sin\zeta + H_\eta\cos\zeta\right)^2 + \frac{1}{C}\left(H_\xi\sin\eta + H_\zeta\right)^2\right). \tag{49}$$

Note again the similarities, of the terms in round brackets, between Eqs. (49) and (43).

The kinetic energy (1) in terms of Cardan angles and Cardan momenta is obtained by combining Eqs. (49) and (46), and is

$$T = \frac{1}{2}\left[\frac{1}{\cos^2\eta}\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)p_\xi^2 + \left(\frac{\sin^2\zeta}{A}+\frac{\cos^2\zeta}{B}\right)p_\eta^2 + \left(\frac{1}{C}+\tan^2\eta\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)\right)p_\zeta^2 + \frac{2\sin\zeta\cos\zeta}{\cos\eta}\left(\frac{1}{A}-\frac{1}{B}\right)p_\xi p_\eta - 2\tan\eta\sin\zeta\cos\zeta\left(\frac{1}{A}-\frac{1}{B}\right)p_\eta p_\zeta - \frac{2\tan\eta}{\cos\eta}\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)p_\xi p_\zeta\right]. \tag{50}$$

Generalized Velocities. The generalized velocities are obtained by forming the appropriate partial derivatives

$$\dot\xi = \frac{\partial T}{\partial p_\xi} = \frac{1}{\cos^2\eta}\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)p_\xi + \frac{\sin\zeta\cos\zeta}{\cos\eta}\left(\frac{1}{A}-\frac{1}{B}\right)p_\eta - \frac{\tan\eta}{\cos\eta}\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)p_\zeta \tag{51a}$$

$$\dot\eta = \frac{\partial T}{\partial p_\eta} = \frac{\sin\zeta\cos\zeta}{\cos\eta}\left(\frac{1}{A}-\frac{1}{B}\right)p_\xi + \left(\frac{\sin^2\zeta}{A}+\frac{\cos^2\zeta}{B}\right)p_\eta - \tan\eta\sin\zeta\cos\zeta\left(\frac{1}{A}-\frac{1}{B}\right)p_\zeta \tag{51b}$$

$$\dot\zeta = \frac{\partial T}{\partial p_\zeta} = -\frac{\tan\eta}{\cos\eta}\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)p_\xi - \tan\eta\sin\zeta\cos\zeta\left(\frac{1}{A}-\frac{1}{B}\right)p_\eta + \left(\frac{1}{C}+\tan^2\eta\left(\frac{\cos^2\zeta}{A}+\frac{\sin^2\zeta}{B}\right)\right)p_\zeta \tag{51c}$$

Equations for Cardan angles of the second and third kinds (Rimrott, 1988) can be obtained by appropriate cyclic interchanges.

Cardan Angles of the Fifth Kind

Of importance in airplane flight dynamics, Euler angles of the fifth kind (Rimrott, 1988) consist of a *yaw* (heading) angle ζ about a space-fixed line to the zenith, a *pitch* angle η about a once-carried (through ζ) line to the horizon, and a *roll* (bank) angle ξ about a twice-carried (through ζ and η) gyro-fixed line (Fig. 6).

The angular velocity can be expressed in two ways:

$$\boldsymbol{\omega} = [\mathbf{e}_x \quad \mathbf{e}_y \quad \mathbf{e}_z]\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = [\mathbf{e}_\xi \quad \mathbf{e}_\eta \quad \mathbf{e}_\zeta]\begin{bmatrix}\dot\xi\\ \dot\eta\\ \dot\zeta\end{bmatrix}. \tag{52}$$

The transition relation is

$$\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}1 & 0 & -\sin\eta\\ 0 & \cos\xi & \sin\xi\cos\eta\\ 0 & -\sin\xi & \cos\xi\cos\eta\end{bmatrix}\begin{bmatrix}\dot\xi\\ \dot\eta\\ \dot\zeta\end{bmatrix}. \tag{53}$$

Similarly, for the angular momentum components,

$$\begin{bmatrix}H_x\\ H_y\\ H_z\end{bmatrix} = \begin{bmatrix}1 & 0 & -\sin\eta\\ 0 & \cos\xi & \sin\xi\cos\eta\\ 0 & -\sin\xi & \cos\xi\cos\eta\end{bmatrix}\begin{bmatrix}H_\xi\\ H_\eta\\ H_\zeta\end{bmatrix}. \tag{54}$$

Complementary Kinetic Energy. Combining Eqs. (1) and (53) gives the complementary kinetic energy

$$T^* = \frac{1}{2}\left(A(\dot\xi - \dot\zeta\sin\eta)^2 + B(\dot\eta\cos\xi + \dot\zeta\sin\xi\cos\eta)^2 + C(-\dot\eta\sin\xi + \dot\zeta\cos\xi\cos\eta)^2\right). \tag{55}$$

Generalized Momenta. Forming partial derivatives leads to the generalized momenta (5)

$$p_\xi = \frac{\partial T^*}{\partial\dot\xi} = A(\dot\xi - \dot\zeta\sin\eta) \tag{56a}$$

$$p_\eta = \frac{\partial T^*}{\partial\dot\eta} = B(\dot\eta\cos\xi + \dot\zeta\sin\xi\cos\eta)\cos\xi + C(\dot\eta\sin\xi - \dot\zeta\cos\xi\cos\eta)\sin\xi \tag{56b}$$

$$p_\zeta = \frac{\partial T^*}{\partial\dot\zeta} = -A(\dot\xi - \dot\zeta\sin\eta)\sin\eta + B(\dot\eta\cos\xi + \dot\zeta\sin\xi\cos\eta)\sin\xi\cos\eta + C(-\dot\eta\sin\xi + \dot\zeta\cos\xi\cos\eta)\cos\xi\cos\eta. \tag{56c}$$

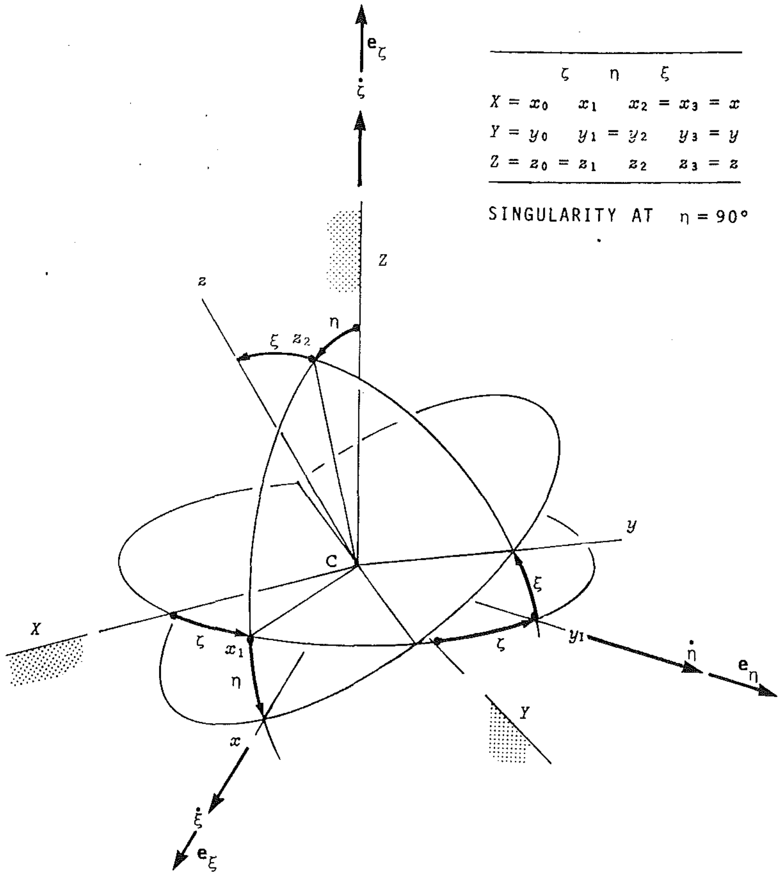

Fig. 6 Cardan angles of the fifth kind

The generalized momenta p_j are related to the angular momentum components H_i (Fig. 7) by

$$\begin{bmatrix} p_\xi \\ p_\eta \\ p_\zeta \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\eta \\ 0 & 1 & 0 \\ -\sin\eta & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix}. \tag{57}$$

The inverse relationship is

$$\begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos^2\eta} & 0 & \frac{\sin\eta}{\cos^2\eta} \\ 0 & 1 & 0 \\ \frac{\sin\eta}{\cos^2\eta} & 0 & \frac{1}{\cos^2\eta} \end{bmatrix} \begin{bmatrix} p_\xi \\ p_\eta \\ p_\zeta \end{bmatrix}. \tag{58}$$

Kinetic Energy. The kinetic energy can be obtained by first using the angular momentum components (54) and Eq. (11), resulting in

$$T = \frac{1}{2}\left(\frac{1}{A}\,(H_\xi - H_\zeta \sin\eta)^2 + \frac{1}{B}\,(H_\eta \cos\xi + H_\zeta \sin\xi\, \cos\eta)^2 + \frac{1}{C}\,(-H_\eta \sin\xi + H_\zeta \cos\xi\, \cos\eta)^2\right). \tag{59}$$

Note again the similarity of the terms in round brackets of Eqs. (59) and (55). But Eq. (59) is not yet in terms of generalized momenta. To achieve this we call upon Eq. (59), with the help of which we eventually obtain

$$T = \frac{1}{2}\Bigg[\left(\frac{1}{A} + \left(\frac{\sin^2\xi}{B} + \frac{\cos^2\xi}{C}\right)\tan^2\eta\right)p_\xi^2 + \left(\frac{\cos^2\xi}{B} + \frac{\sin^2\xi}{C}\right)p_\eta^2 + \frac{1}{\cos^2\eta}\left(\frac{\sin^2\xi}{B} + \frac{\cos^2\xi}{C}\right)p_\zeta^2$$

$$+ 2\sin\xi\,\cos\xi\,\tan\eta\left(\frac{1}{B} - \frac{1}{C}\right)p_\xi p_\eta + \frac{2\sin\xi\,\cos\xi}{\cos\eta} + \left(\frac{1}{B} - \frac{1}{C}\right)p_\eta p_\zeta + 2\,\frac{\tan\eta}{\cos\eta}\left(\frac{\sin^2\xi}{B} + \frac{\cos^2\xi}{C}\right)p_\xi p_\zeta\Bigg]. \tag{60}$$

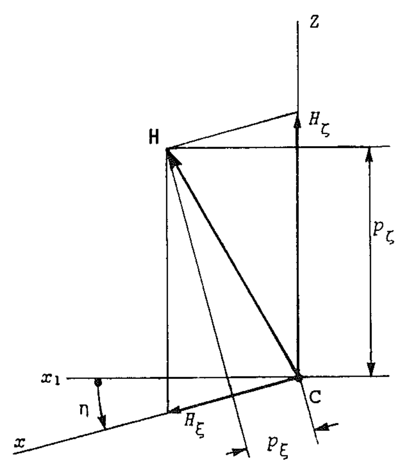

Fig. 7 Angular momentum components and generalized momenta for Cardan angles of the fifth kind (when $H_\eta = 0$)

Generalized Velocities. By partial differentiation we obtain the generalized velocities (4).

$$\dot{\xi}=\frac{\partial T}{\partial p_\xi}=\left[\frac{1}{A}+\left(\frac{\sin^2\xi}{B}+\frac{\cos^2\xi}{C}\right)\tan^2\eta\right]p_\xi$$

$$+\left(\frac{1}{B}-\frac{1}{C}\right)\sin\xi\,\cos\xi\,\tan\eta\,p_\eta+\left(\frac{\sin^2\xi}{B}+\frac{\cos^2\xi}{C}\right)\frac{\tan\eta}{\cos\eta}p_\zeta \quad (61a)$$

$$\dot{\eta}=\frac{\partial T}{\partial p_\eta}=\left(\frac{1}{B}-\frac{1}{C}\right)\sin\xi\,\cos\xi\,\tan\eta\,p_\xi+\left(\frac{\cos^2\xi}{B}+\frac{\sin^2\xi}{C}\right)p_\eta$$

$$+\left(\frac{1}{B}-\frac{1}{C}\right)\frac{\sin\xi\,\cos\xi}{\cos\eta}p_\zeta \quad (61b)$$

$$\dot{\zeta}=\frac{\partial T}{\partial p_\zeta}=\left(\frac{\sin^2\xi}{B}+\frac{\cos^2\xi}{C}\right)\frac{\tan\eta}{\cos\eta}p_\xi+\left(\frac{1}{B}-\frac{1}{C}\right)\frac{\sin\xi\,\cos\xi}{\cos\eta}p_\eta$$

$$+\left(\frac{\sin^2\xi}{B}+\frac{\cos^2\xi}{C}\right)\frac{1}{\cos^2\eta}p_\zeta \quad (61c)$$

The Lagrange equations are valid in the form given by Eqs. (47), and D'Alembert's principle (30) can also readily be expressed in terms of Cardan angles of the fifth kind. Equations for Cardan angles of the fourth and sixth kinds (Rimrott, 1988) can be obtained by appropriate cyclic interchanges.

Conclusions

For a proper analytical mechanics treatment of gyrodynamic problems, generalized coordinates are essential. Depending on circumstances, either Euler angles or Cardan angles are suitable. In the present paper the fundamental equations for generalized momenta and generalized velocities are given. For their derivation, the concepts of kinetic energy and complementary kinetic energy have been employed.

References

Crandall, S., 1957, "Complementary Extremum Principles for Dynamics," *Proceedings, Ninth International Congress of Applied Mechanics*, Vol. 5, pp. 80–87.

Greenwood, Donald T., 1988, *Principles of Dynamics*, 2nd ed., Prentice-Hall, Englewood Cliffs, N.J.

Magnus, K., 1971, *Kreisel*, Springer-Verlag, Berlin.

Rimrott, F. P. J., 1988, *Introductory Attitude Dynamics*, Springer-Verlag, New York.

Tabarrok, B., 1981, "On the Principle of Least Action and its Complementary Form," *Solid Mechanics Archives*, Vol. 7, pp.309–329.