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*Research Article*

# A Meshless Method Based on the Fundamental Solution and Radial Basis Function for Solving an Inverse Heat Conduction Problem

**Muhammad Arghand and Majid Amirfakhrian**

*Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran*

Correspondence should be addressed to Majid Amirfakhrian; amirfakhrian@iauctb.ac.ir

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We propose a new meshless method to solve a backward inverse heat conduction problem. The numerical scheme, based on the fundamental solution of the heat equation and radial basis functions (RBFs), is used to obtain a numerical solution. Since the coefficients matrix is ill-conditioned, the Tikhonov regularization (TR) method is employed to solve the resulted system of linear equations. Also, the generalized cross-validation (GCV) criterion is applied to choose a regularization parameter. A test problem demonstrates the stability, accuracy, and efficiency of the proposed method.

## 1. Introduction

Transient heat conduction phenomena are generally described by the parabolic heat conduction equation, and if the initial temperature distribution and the boundary conditions are specified, then this, in general, leads to a well-posed problem which may easily be solved numerically by using various numerical methods. However, in many practical situations when dealing with a heat conducting body it is not always possible to specify the boundary conditions or the initial temperature. Hence, we are faced with an inverse heat conduction problem. Inverse heat conduction problems (IHCPs) occur in many branches of engineering and science. Mechanical and chemical engineers, mathematicians, and specialists in other sciences branches are interested in inverse problems. From another point of view, since the existence, uniqueness, and stability of the solutions of these problems are not usually confirmed, they are generally identified as ill-posed [1–4]. According to the fact that unknown solutions of inverse problems are determined through indirect observable data which contain measurement errors, such problems are naturally unstable. In other words, the main difficulty in the treatment of inverse problems is the unstability of their solution in the presence of noise in the measured data. Hence, several numerical methods have been proposed for solving the various kinds of inverse problems. In addition to, ill-posedness of these kinds of problems, ill-conditioning of the resulting discretized matrix from the traditional methods like the finite differences method (FDM) [5], the finite element method (FEM), and so forth [6, 7], is the main problem making all numerical algorithms for determining the solution of these kinds of problems. Accordingly, within recent years, meshless methods, as the method of fundamental solution (MFS), radial basis functions (RBFs) method, and some other methods, have been applied by many scientists in the field of applied sciences and engineering [8–15]. Kupradze and Aleksidze [16] first introduced MFS which defines the solution of the problem as a linear combination of fundamental solutions. Hon et al. [17–20] applied the MFS to solve some inverse heat conduction problems. In 1990s, Kansa applied RBFs method to solve the different types of partial differential equations [21, 22]. After that, Kansa and many scientists regarded RBFs method to solve different types of mathematical problems from partial or ordinary differential equations to integral equations [23–26]. Following their works, during recent years, many researchers have made some changes in RBFs and MFS methods and have developed advance methods to solve some of these kinds problems [27, 28]. Consequently, in this work, we will present a meshless numerical scheme, based on combining the radial basis function and the fundamental solution of the heat equation, in order to approximate the solution of a backward inverse heat conduction problem (BIHCP), the problem in which an unknown initial condition or/and temperature distribution in previous time will be determined. This kind of problem may emerge in many practical application areas such as archeology and mantle plumes [29]. On the other hand, since the system of the linear equations obtained from discretizing the problem in the presented method is ill-conditioned, Tikhonov regularization (TR) method is applied in order to solve it. The generalized cross-validation (GCV) criterion has been assigned to adopt an optimum amount of the regularization parameter. The structure of the rest of this work is organized as follows: In Section 2, we represent the mathematical formulation of the problem. The method of fundamental solutions, radial basis functions method, and method of fundamental solution-radial basis functions (MFSRBF) are described in Section 3. Section 4 embraces Tikhonov regularization method with a rule for choosing an appropriate regularization parameter. In Section 5, we present the obtained numerical results of solving a test problem. Section 6 ends in a brief conclusion and some suggestions.

## 2. Mathematical Formulation of the Problem

In this section, we consider the following one-dimensional inverse heat conduction problem:

$$u_t - a^2 u_{xx} = 0, \quad (x,t) \in \Omega = (0,l) \times (0,t_{\max}), \tag{1}$$

with the following initial and boundary conditions:

$$\begin{gathered} u(x,0) = \begin{cases} \varphi(x), & x \in [0,x_0), \\ f(x), & x \in [x_0,l], \end{cases} \\ u(0,t) = p(t), \quad t \in [0,t_{\max}], \\ u(l,t) = q(t), \quad t \in [0,t_{\max}], \end{gathered} \tag{2}$$

where $\varphi(x)$ and $q(t)$ are considered as known functions and $t_{\max}$ is a given positive constant, while $f(x)$ and $p(t)$ are regarded as unknown functions. So, in order to estimate $f(x)$ and $p(t)$, we consider additional temperature measurements and heat flux given at a point $x_0$, $x_0 \in (0,1)$, as overspecified conditions:

$$\begin{gathered} u_x(x_0,t) = g(t), \quad t \in [0,t_{\max}], \\ u(x_0,t) = h(t), \quad t \in [0,t_{\max}]. \end{gathered} \tag{3}$$

To solve the above problem, at first, we divide the problem (1)–(3) into two separate problems. The problem $A$ is as follows:

$$\begin{gathered} u_t - a^2 u_{xx} = 0, \quad (x,t) \in \Omega_A = (0,x_0) \times (0,t_{\max}), \\ u(x,0) = \varphi(x), \quad x \in [0,x_0], \\ u(0,t) = p(t), \quad t \in [0,t_{\max}], \\ u_x(x_0,t) = g(t), \quad t \in [0,t_{\max}], \\ u(x_0,t) = h(t), \quad t \in [0,t_{\max}], \end{gathered} \tag{4}$$

and the problem $B$ is considered as follows:

$$\begin{gathered} u_t - a^2 u_{xx} = 0, \quad (x,t) \in \Omega_B = (x_0,l) \times (0,t_{\max}), \\ u(x,0) = f(x), \quad x \in [x_0,l], \\ u_x(x_0,t) = g(t), \quad t \in [0,t_{\max}], \\ u(x_0,t) = h(t), \quad t \in [0,t_{\max}], \\ u(l,t) = q(t), \quad t \in [0,t_{\max}]. \end{gathered} \tag{5}$$

Obviously, the problems A and B are considered as IHCP, where $p(t)$, $u(x,t)$ and $f(x)$, $u(x,t)$ are unknown functions in the problems A and B, respectively.

## 3. Method of Fundamental Solutions and Method of Radial Basis Functions

In this section, we introduce the numerical scheme for solving the problem (1)–(3) using the fundamental solutions and radial basis functions.

*3.1. Method of Fundamental Solutions.* The fundamental solution of (1) is presented as below:

$$k(x,t) = \frac{1}{2a\sqrt{\pi t}} e^{-x^2/(4a^2 t)} H(t), \tag{6}$$

where $H(t)$ is Heaviside unit function. Assuming that $T > t_{\max}$ is a constant, it can be demonstrated that the time shift function

$$\phi(x,t) = k(x,t+T) \tag{7}$$

is also a nonsingular solution of (1) in the domain $\Omega$.

In order to solve an IHCP by MFS, as the problem $A$, since the basis function $\phi$ satisfies the heat equation (1) automatically, we assume that $\{(x_k,t_k)\}_{k=1}^{N}$ is a given set of scattered points on the boundary $\partial\Omega_A$. An approximate solution is defined as a linear combination of $\phi$ as follows:

$$\Phi(x,t) = \sum_{k=1}^{N} \lambda_k \phi(x - x_k, t - t_k), \tag{8}$$

where $\phi(x,t)$ is given by (7) and $\lambda_k$'s are unknown coefficients which can be determined by solving the following matrix equation:

$$\mathbf{A}\Lambda = \mathbf{b}, \tag{9}$$

where $\Lambda = [\lambda_1,\dots,\lambda_N]^T$ and $\mathbf{b}$ is a $N \times 1$ known vector. Also, as the fundamental functions are the solution of the heat equation, only the initial and boundary conditions are practiced to make the system of linear equations; that is, $\mathbf{A}$ is

Table 1: Some well-known radial basis functions.

| Infinitely smooth RBFs | $\psi(r)$ |
|---|---|
| Gaussian (GA) | $e^{(-cr^2)}$ |
| Multiquadrics (MQ) | $\sqrt{r^2+c^2}$ |
| Inverse multiquadrics (IMQ) | $\left(\sqrt{r^2+c^2}\right)^{-1}$ |
| Inverse quadric (IQ) | $(r^2+c^2)^{-1}$ |

a $N \times N$ square matrix which is defined using the initial and the boundary conditions as follows:

$$\mathbf{A} = \left[a_{ij}\right], \quad a_{ij} = \Phi(x,t), \ (x,t) \in \partial\Omega_A, \ i,j = 1,\dots,N. \tag{10}$$

For more details, see [13, 18].

*3.2. Method of Radial Basis Functions.* In this section, we consider RBF method for interpolation of scattered data. Suppose that $\mathbf{x}^*$ and $\mathbf{x}$ are a fixed point and an arbitrary point in $\mathbb{R}^d$, respectively. A radial function $\psi^*$ is defined via $\psi^* = \psi^*(r)$, where $r = \|\mathbf{x} - \mathbf{x}^*\|_2$. That is, the radial function $\psi^*$ depends only on the distance between $\mathbf{x}$ and $\mathbf{x}^*$. This property implies that the RBFs $\psi^*$ are radially symmetric about $\mathbf{x}^*$. Some well-known infinitely smooth RBFs are given in Table 1. As it is observed, these functions depend on a free parameter $c$, known as the shape parameter, which has an important role in approximation theory using RBFs.

Let $\{\mathbf{x}_k\}_{k=1}^N$ be a given set of distinct points in domain $\Omega$ in $\mathbb{R}^d$. The main idea of using the RBFs is interpolation with a linear combination of RBFs of the same types as follows:

$$\Psi(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k \psi_k(\mathbf{x}), \tag{11}$$

where $\psi_k(\mathbf{x}) = \psi(\|\mathbf{x} - \mathbf{x}_k\|)$ and $\lambda_k$'s are unknown coefficients for $k = 1,2,\dots,N$. Assume that we want to interpolate the given values $f_k = f(\mathbf{x}_k)$, $k = 1,2,\dots,N$. The unknown coefficients $\lambda_k$'s are obtained, so that $\Psi(\mathbf{x}_k) = f_k$, $k = 1,2,\dots,N$, which results in the following system of linear equations:

$$\mathbf{A}\Lambda = \mathbf{b}, \tag{12}$$

where $\Lambda = [\lambda_1,\dots,\lambda_N]^T$, $\mathbf{b} = [f_1,\dots,f_N]^T$, and $\mathbf{A} = [a_{ij}]$ for $i,j = 1,2,\dots,N$ with entries $a_{ij} = \psi_j(\mathbf{x}_i)$. Generally, the matrix $\mathbf{A}$ has been shown to be positive definite (and therefore nonsingular) for distinct interpolation points for infinitely smooth RBFs by Schoenberg's theorem [30]. Also, using Micchelli's theorem [31], it is shown that $\mathbf{A}$ is an invertible matrix for a distinct set of scattered nodes in the case of MQ-RBF.

## 4. Method of Fundamental Solutions-Radial Basis Functions

This section is specified to introduce the numerical scheme for solving the problem (1)–(3) using the fundamental solutions and radial basis functions. At first, we are going to figure out how radial basis functions and the fundamental solutions are applied to approximate the solution of the problem $A$, 2. Similarly, this method will be used for solving the problem $B$ on domain $\Omega_B$. Because the one-dimensional heat conduction equation depends on both parameters of $x$ and $t$, $N$ scattered nodes are considered in the domain $\Omega_A = [0,x_0] \times [0,t_{\max}+T]$. We assume that

$$\Xi_I = \{(x_k,t_k)\}_{k=1}^{N_I}, \qquad \Xi_b = \{(x_k,t_k)\}_{k=1}^{N_B} \tag{13}$$

are two sets of scattered points in the domain $\Omega_A$, where $N = N_I + N_B$ and $\Xi_I$ and $\Xi_b$ are the interior and the boundary points in domains $\Omega_A = [0,x_0] \times [0,t_{\max}+T]$ and $\Omega_b = [0,x_0] \times [0,t_{\max}]$, respectively. Also, we assume that

$$\Xi_b = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \tag{14}$$

where

$$\begin{gathered}\Gamma_1 = \{(x_i,t_i) : t_i = 0, \ 0 \leqslant x_i \leqslant x_0, \ i = 1,\dots,n\}, \\ \Gamma_2 = \left\{\left(x_j,t_j\right) : x_j = x_0, \ 0 < t_j \leqslant t_{\max}, \ j = 1,\dots,m\right\}, \\ \Gamma_3 = \{(x_r,t_r) : x_r = x_0, \ 0 < t_r \leqslant t_{\max}, \ r = 1,\dots,s\},\end{gathered} \tag{15}$$

so that $N_B = n + m + s$.

We suppose that the solution of the problem 2 in $\Omega_A$ can be expressed as follows:

$$\tilde{u}(x,t) = \sum_{k=1}^{N_B} \lambda_k \phi_k(x,t) + \sum_{k=N_B+1}^{N} \lambda_k \psi_k(x,t), \tag{16}$$

where $\phi_k(x,t) = k(x - x_k, t - t_k + T)$ and $k(x,t)$ is the fundamental solution of the heat equation which is described in Section 3.2. Also, $\psi_k = \psi(\|(x,t) - (x_k,t_k)\|_2)$ and $\psi$ is a radial function which is defined in Section 3.1. To determine the coefficients in (16), we impose the approximate solution $\tilde{u}$ to satisfy the given partial differential equation with the other conditions at any point $(x,t) \in \Xi_I \cup \Xi_B$, so we achieve the following system of linear equations:

$$\mathbf{A}\Lambda = \mathbf{b}, \tag{17}$$

where $\Lambda = [\lambda_1,\dots,\lambda_N]^T$, $\mathbf{b} = [0, \varphi(x_i), g(t_j), h(t_r)]^T$, and 0 is $N_I \times 1$ zero matrix. Also, the $N \times N$ matrix $\mathbf{A}$ can be subdivided into two submatrices as follows:

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_I \\ \mathbf{A}_B\end{bmatrix}, \tag{18}$$

where $\mathbf{A}_I = [a_{lp}^{(1)}]$ is the $N_I \times N$ obtained submatrix of applying the interior points whose entries are defined as follows:

$$a_{lp}^{(1)} = 0, \quad l = 1,\dots,N_I, \ p = 1,\dots,N_B, \tag{19}$$

because as already mentioned above, the fundamental functions $\phi_k$ satisfy the heat equation and also

$$\begin{gathered}a_{lp}^{(1)} = \left(\frac{\partial}{\partial t} - a^2\frac{\partial^2}{\partial x^2}\right)\psi_k(x,t), \\ l = 1,\dots,N_I, \quad p,k = N_B+1,\dots,N, \quad (x,t) \in \Xi_I.\end{gathered} \tag{20}$$

Using the boundary points of domain $\Omega_b$, namely, $\Xi_b$, the submatrix $\mathbf{A}_B = [a_{lp}^{(2)}]$ results, where

$$\begin{gathered}
a_{lp}^{(2)} = \phi_k(x,t), \\
l = 1,\dots,n, \quad p,k = 1,\dots,N_B, \quad (x,t) \in \Gamma_1, \\
a_{lp}^{(2)} = \psi_k(x,t), \\
l = 1,\dots,n, \quad p,k = N_B+1,\dots,N, \quad (x,t) \in \Gamma_1, \\
a_{lp}^{(2)} = \frac{\partial}{\partial x}\phi_k(x,t), \\
l = n+1,\dots,n+m, \quad p,k = 1,\dots,N_B, \quad (x,t) \in \Gamma_2, \\
a_{lp}^{(2)} = \frac{\partial}{\partial x}\psi_k(x,t), \\
l = n+1,\dots,n+m, \quad p,k = N_B+1,\dots,N, \quad (x,t) \in \Gamma_2, \\
a_{lp}^{(2)} = \phi_k(x,t), \\
l=n+m+1,\dots,n+m+s, \quad p,k=1,\dots,N_B, \quad (x,t)\in\Gamma_3, \\
a_{lp}^{(2)} = \psi_k(x,t), \\
l = n+m+1,\dots,n+m+s, \\
p,k = N_B+1,\dots,N, \\
(x,t) \in \Gamma_3.
\end{gathered} \tag{21}$$

It is essential to keep in mind that inherent errors in measurement data are inevitable. On the other hand, as it was mentioned, a radial basis function depends on the shape parameter $c$, which has an effect on the condition number of $\mathbf{A}$. Therefore, the obtained system of linear equations (17) is ill-conditioned. So, it cannot be solved, directly. Since, small perturbation in initial data may produce a large amount of perturbation in the solution, we use the rand function in matlab in the numerical example presented in Section 6 and we produce noisy data as the follows:

$$\tilde{b}_i = b_i(1 + \delta \cdot \text{rand}(i)), \quad i = 1,\dots,N, \tag{22}$$

where $b_i$ is the exact data, rand($i$) is a random number uniformly distributed in $[-1,1]$, and the magnitude $\delta$ displays the noise level of the measurement data.

## 5. Regularization Method

Solving the system of linear equations (17) usually does not lead to accurate results by most numerical methods, because condition number of matrix $\mathbf{A}$ is large. It means that the ill-conditioning of matrix $\mathbf{A}$ makes the numerical solution unstable. Now, in methods as the proposed method in this study, which is based on RBFs, the condition number of $\mathbf{A}$ depends on some factors such as the shape parameter $c$, as well. On the other hand, for fixed values of the shape parameter $c$, the condition number increases with the number of scattered nodes $N$. In practice, the shape parameter must be adjusted with the number of interpolating points. Also, the accuracy of radial basis functions relies on the shape parameter. So, in case a suitable amount of it is chosen, the accuracy of the approximate solution will be increased. Despite various research works which are done, finding the optimal choice of the shape parameter is still an open problem [32–35]. Accordingly, some regularization methods are presented to solve such ill-conditioned systems. Tikhonov regularization (TR) method is mostly used by researchers [36]. In this method, the regularized solution $\Lambda^\alpha$ for the system of linear equations (17) is explained as the solution of the following minimization problem:

$$\min_{\Lambda}\left\{\left\|\mathbf{A}\Lambda - \tilde{\mathbf{b}}\right\|^2 + \alpha^2\|\Lambda\|^2\right\}, \quad \alpha > 0, \tag{23}$$

where $\|\cdot\|$ denotes the Euclidean norm and $\alpha$ is called the regularization parameter. Some methods such as $L$-curve [37], cross-validation (CV), and generalized cross-validation (GCV) [38] are carried out to determine the regularization parameter $\alpha$ for the TR method. In this work, we apply (GCV) to obtain regularization parameter. In this method, regularization parameter $\alpha$ minimizes the following (GCV) function:

$$G(\alpha) = \frac{\left\|\mathbf{A}\Lambda^\alpha - \tilde{\mathbf{b}}\right\|^2}{\left(\text{trace}\left(\mathbf{I}_N - \mathbf{A}\mathbf{A}^I\right)\right)^2}, \quad \alpha > 0, \tag{24}$$

where $\mathbf{A}^I = (\mathbf{A}^{\text{tr}}\mathbf{A} + \alpha^2\mathbf{I}_N)^{-1}\mathbf{A}^{\text{tr}}$.

The regularized solution (17) is shown by $\Lambda^{\alpha^*} = [\lambda_1^{\alpha^*},\dots,\lambda_N^{\alpha^*}]^T$, in which $\alpha^*$ is a minimizer of $G$. Then the approximate solution for the problem 2 is written as follows:

$$\begin{gathered}
\tilde{u}_\alpha^*(x,t) = \sum_{k=1}^{N_B}\lambda_k^{\alpha^*}\phi_k(x,t) + \sum_{k=N_B+1}^{N}\lambda_k^{\alpha^*}\psi_k(x,t), \\
p(t) = \sum_{k=1}^{N_B}\lambda_k^{\alpha^*}\phi(0-x_k,t-t_k) + \sum_{k=N_B+1}^{N}\lambda_k^{\alpha^*}\psi(0-x_k,t-t_k).
\end{gathered} \tag{25}$$

## 6. Numerical Experiment

In this section, we investigate the performance and the ability of the present method by giving a test problem. Therefore, in order to illustrate the efficiency and the accuracy of the proposed method along with the TR method, initially, we define the root mean square (RMS) error, the relative root mean square (RES) error, and the maximum absolute error $E_\infty(u)$ as follows:

$$\begin{gathered}
\text{RMS}(u) = \sqrt{\frac{1}{N_t}\sum_{k=1}^{N_t}\left(u(x_k,t_k) - \tilde{u}^*(x_k,t_k)\right)^2}, \\
\text{RES}(u) = \frac{\sqrt{\sum_{k=1}^{N_t}\left(u(x_k,t_k) - \tilde{u}^*(x_k,t_k)\right)^2}}{\sqrt{\sum_{k=1}^{N_t}\left(u(x_k,t_k)\right)^2}}, \\
E_\infty(u) = \max_{1\leqslant k\leqslant N_t}\left|u(x_k,t_k) - \tilde{u}^*(x_k,t_k)\right|,
\end{gathered} \tag{26}$$

Table 2: The obtained values of RMS($u$), RES($u$), RMS($p$), and RES($p$) using MFSRBF for various values of $T$, $x_0 = 0.5$, and $\delta = 0.01$ by solving problem $A$.

| $T$ | RMS($u$) | RES($u$) | $E_\infty(u)$ | RMS($p$) | RES($p$) | $E_\infty(p)$ | cond($A$) |
|---|---|---|---|---|---|---|---|
| 1.1 | 0.017391 | 0.012603 | 0.044511 | 0.014968 | 0.012701 | 0.024588 | $1.6111e+09$ |
| 2.5 | 0.005081 | 0.003682 | 0.019141 | 0.007528 | 0.006388 | 0.019141 | $3.5475e+10$ |
| 5 | 0.005061 | 0.003668 | 0.014484 | 0.005682 | 0.004821 | 0.014485 | $5.3298e+11$ |
| 10 | 0.002907 | 0.002107 | 0.008510 | 0.003587 | 0.003043 | 0.008510 | $2.3557e+13$ |
| 20 | 0.003463 | 0.002510 | 0.013148 | 0.004864 | 0.004128 | 0.013147 | $7.9497e+13$ |
| 30 | 0.004472 | 0.003241 | 0.021264 | 0.007954 | 0.006749 | 0.021264 | $1.1148e+11$ |
| 50 | 0.003613 | 0.002618 | 0.012343 | 0.005781 | 0.004905 | 0.012343 | $1.4673e+11$ |

where $N_t$ is the total number of testing points in the domain $\Omega_b$, $u(x_k, t_k)$ and $\tilde{u}^*(x_k, t_k)$ are the exact and the approximated values at these points, respectively. RMS, RES, and $E_\infty$ errors of the functions $p(t)$ and $f(x)$ are similarly defined, as well. In our computation, the Matlab code developed by Hansen [39] is used for solving the discrete ill-conditioned system of linear equations (17).

*Example*. For simplicity, we assume that $x_0 = 0.5$ and $a = l = t_{\max} = 1$. By these assumptions, the exact solution of the problem (1)–(3) is given as follows [40]:

$$u(x,t) = e^{-t}\sin(x) + x^2 + 2t,$$
$$p(t) = 2t, \quad (27)$$
$$f(x) = \sin(x) + x^2.$$

Since, for a large fixed number of scattered points $N$, the matrix $\mathbf{A}$ will be more ill-conditioned and also, smaller values of the shape parameter $c$ generate more accurate approximations, we suppose that $c = 0.1$ and the number of interpolating points is $N = 20$. The obtained values of RMS($u$), RES($u$), $E_\infty(u)$, RMS($p$), RES($p$), $E_\infty(p)$, RMS($f$), RES($f$), and $E_\infty(f)$, as well as condition number of $\mathbf{A}$ for $\delta = 0.01$ and $N_t = 208$ and various values of $T$, are given in Tables 2 and 4. Numerical results indicate that this method is not depended on parameter $T$. By the assumptions $N = 20$ and $T = 4.1$ and with various levels of noise added into the data, Tables 3 and 5 illustrate the relative root mean square error of $p(t)$ and $f(x)$ at three points $x_0 = 0.1$, 0.5, and $x_0 = 0.9$ of the interval $[0, 1]$, respectively. Figures 1 and 3 feature out a comparison between the exact solutions and the approximate solutions for $x_0 = 0.5$ and $T = 3$ and various levels of noise added into the data. It is observed that as the noise level increases, the approximated functions will have acceptable accuracy. Figures 2 and 4 display the relationship between the accuracy and the parameter $T$ with noise of level $\delta = 0.01$ added into the data. They elucidate not only that the numerical results are stable with respect to parameter $T$, but also that they retain at the same level of accuracy for a wide range of values $T$. Therefore, the accuracy of the numerical solutions is not relatively dependent on the parameter $T$. In addition, the study of the presented numerical results, via MFS in [40], indicates that with noise of level $\delta = 0$, namely, with the noiseless data and for $x_0 = 0.5$ of the interval $[0, 1]$, RMS($p$) $= 1.7 \times 10^{-3}$ with $T = 1.9$ and RMS($f$) $= 1.89 \times 10^{-2}$ for $T = 3$. Also, when the data are considered with noise, RMS($p$) $= 2.84 \times 10^{-2}$ and RMS($f$) $= 3.53 \times 10^{-2}$, while by the proposed method in this work and with the same assumptions, we achieve RMS($p$) $= 3.0479 \times 10^{-5}$ and RMS($f$) $= 3.5691 \times 10^{-6}$ for $\delta = 0$. Also, with noise of level $\delta = 10^{-4}$, we obtain RMS($p$) $= 1.1021 \times 10^{-4}$ and RMS($f$) $= 1.9204 \times 10^{-4}$. Accordingly, the MFSRBF, which is based on the fundamental solutions of the heat equation and radial basis functions, is more accurate in comparison to MFS.

Table 3: The resulted values of RES($p$) using MFSRBF for various levels of noise $\delta$, different values of $x_0$, and $T = 4.1$.

| $x_0$ | $\delta = 0.1$ | $\delta = 0.01$ | $\delta = 0.001$ | $\delta = 0.0001$ |
|---|---|---|---|---|
| 0.1 | $3.491e-02$ | $2.949e-03$ | $5.095e-04$ | $4.129e-04$ |
| 0.5 | $2.703e-02$ | $3.306e-03$ | $7.707e-04$ | $3.069e-04$ |
| 0.9 | $6.242e-02$ | $9.011e-03$ | $3.292e-03$ | $1.126e-03$ |

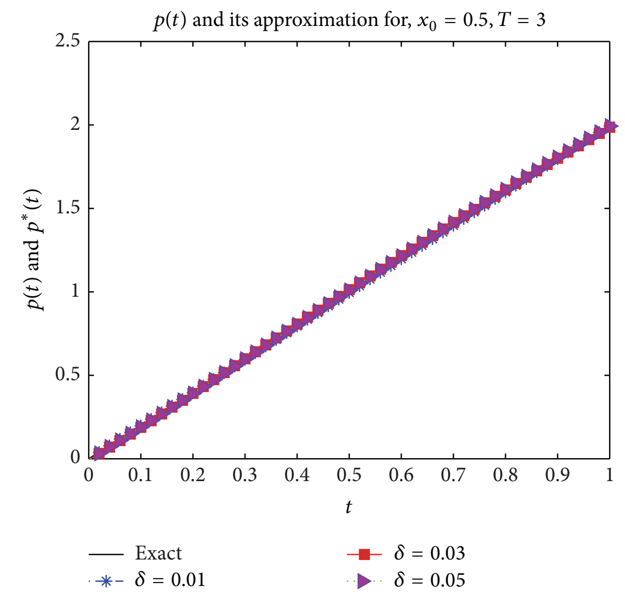


Figure 1: The exact $p$ and its approximation $p^*$ obtained with $N = 20$, $T = 3$, $\delta = 0.01$, and $x_0 = 0.5$ and for various levels of noise added into the data using MQ-RBF with $c = 0.1$.

Table 3 indicates the values of the obtained RES($p$) for various levels of noise, different values of $x_0$ and $T = 4.1$ by MFSRBF.

Table 4: The obtained values of RMS($u$), RES($u$), RMS($f$), and RES($f$) using MFSRBF for various values of $T$, $x_0 = 0.5$, and $\delta = 0.01$ by solving problem $B$.

| $T$ | RMS($u$) | RES($u$) | $E_\infty(u)$ | RMS($f$) | RES($f$) | $E_\infty(f)$ | cond($A$) |
|---|---|---|---|---|---|---|---|
| 1.1 | 0.023701 | 0.011307 | 0.109059 | 0.053926 | 0.041309 | 0.109060 | $2.9272e+08$ |
| 2.5 | 0.003687 | 0.001759 | 0.012035 | 0.005482 | 0.004199 | 0.008311 | $1.1421e+08$ |
| 5 | 0.003769 | 0.001798 | 0.011163 | 0.006226 | 0.004769 | 0.011164 | $3.1868e+10$ |
| 10 | 0.005477 | 0.002613 | 0.019829 | 0.001900 | 0.001455 | 0.003587 | $1.0342e+12$ |
| 20 | 0.005138 | 0.002451 | 0.018525 | 0.003612 | 0.002767 | 0.006490 | $1.4588e+14$ |
| 30 | 0.005022 | 0.002036 | 0.013977 | 0.005012 | 0.002917 | 0.008241 | $5.2325e+13$ |
| 50 | 0.006734 | 0.003212 | 0.012163 | 0.004516 | 0.003459 | 0.006278 | $3.8672e+13$ |

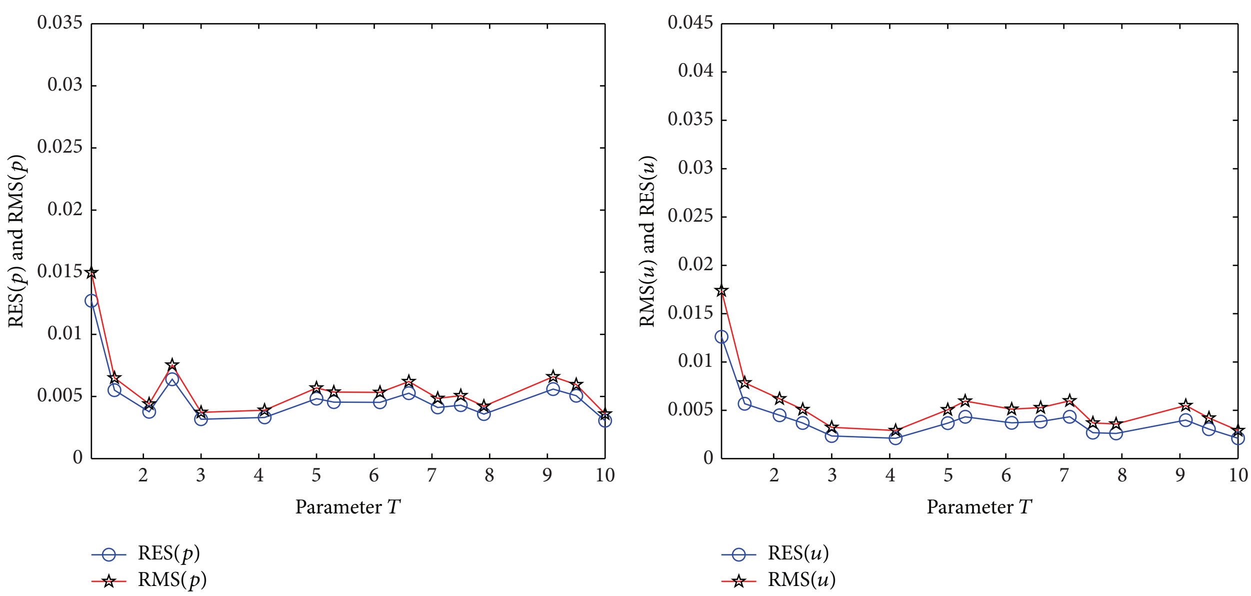


Figure 2: The accuracy of the obtained numerical results by MFSRBF for $\delta = 0.01$ and $x_0 = 0.5$ with respect to the parameter $T$.

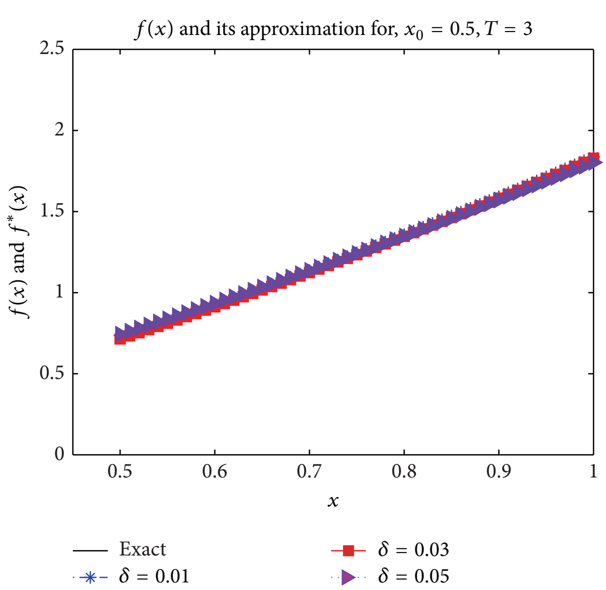


Figure 3: The exact $f$ and its approximation $f^*$ obtained with $N = 20$, $T = 3$, $\delta = 0.01$, and $x_0 = 0.5$ and for various levels of noise added into the data using MQ-RBF with $c = 0.1$.

Table 5: The obtained values of RES($f$) using MFSRBF for various levels of noise $\delta$, different values of $x_0$, and $T = 4.1$.

| $x_0$ | $\delta = 0.1$ | $\delta = 0.01$ | $\delta = 0.001$ | $\delta = 0.0001$ |
|---|---|---|---|---|
| 0.1 | $5.611e-02$ | $1.006e-02$ | $6.731e-03$ | $1.066e-03$ |
| 0.5 | $2.032e-02$ | $2.677e-03$ | $6.219e-04$ | $2.954e-04$ |
| 0.9 | $2.012e-02$ | $1.327e-03$ | $5.852e-04$ | $1.965e-04$ |

## 7. Conclusion

The present study successfully applies a new meshless scheme based on the fundamental solution of the heat equation and the radial basis function with the Tikhonov regularization method in order to solve a backward inverse heat conduction problem. At first, we have divided the inverse problem into two separate problems and then by applying the MFSRBF with the TR method on the obtained problems, two unknown functions in this IHCP are approximated simultaneously. Numerical results clarify that the presented method is an exact and reliable numerical technique to solve a BIHCP and also the accuracy of the numerical solution is not

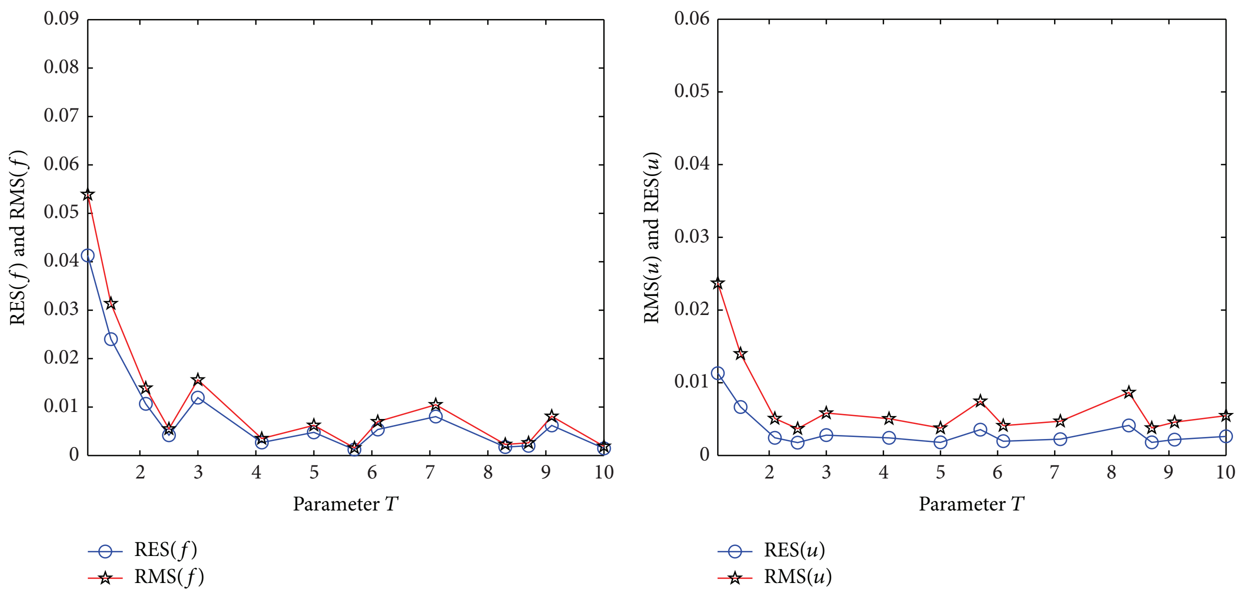


FIGURE 4: The accuracy of the obtained numerical results by MFSRBF for $\delta = 0.01$ and $x_0 = 0.5$ versus the parameter $T$.

relatively depended on the parameter $T$. Accordingly, not only this method can be applied to solve a BIHCP in higher dimensions, but also it is possible to practice it in other inverse problems. Hence, this will be contemplated more in future researches.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## References

[1] J. V. Beck, B. Black-Well, and C. R. St-Clair, *Inverse Heat Conduction Ill-Posed Problems*, John Wiley & Sons, 1985.

[2] J. R. Cannon, *The One-Dimensional Heat Equation*, Addison-Wesley, 1984.

[3] O. M. Alifanov, *Inverse Heat Conduction Problems*, Springer, 1994.

[4] M. N. Ozisik, *Boundary Value Problems of Heat Conduction*, Dover Publications, New York, NY, USA, 1989.

[5] J. R. Cannon and J. A. Vander Hoke, "Implicit finite difference scheme for the diffusion of mass in porous media," in *Numerical Solution of Partial Differential Equations*, vol. 527, pp. 527–539, North-Holland, 1982.

[6] B. T. Johansson and D. Lesnic, "Determination of a spacewise dependent heat source," *Journal of Computational and Applied Mathematics*, vol. 209, no. 1, pp. 66–80, 2007.

[7] A. Farcas and D. Lesnic, "The boundary-element method for the determination of a heat source dependent on one variable," *Journal of Engineering Mathematics*, vol. 54, no. 4, pp. 375–388, 2006.

[8] M. D. Buhmann, *Radial Basis Functions*, Cambridge University Press, Cambridge, UK, 2003.

[9] M. D. J. Powell, "Radial basis functions for multivariable interpolation: a review," in *Numerical Analysis*, D. F. Griffths and G. A. Watson, Eds., pp. 223–241, Longman Scientific & Technical, Harlow, UK, 1987.

[10] Z. M. Wu, "Hermite-Birkhoff interpolation of scattered data by radial basis functions," *Approximation Theory and Its Applications*, vol. 8, no. 2, pp. 1–10, 1992.

[11] M. A. Golberg and C. S. Chen, "The method of fundamental solutions for potential hemholtz and diffusion problems," in *Boundary Integral Methods: Numerical and Mathematical Aspects*, pp. 103–176, Computational Mechanics Publication, Southampton, UK, 1999.

[12] B. Jin and L. Marin, "The method of fundamental solutions for inverse source problems associated with the steady-state heat conduction," *International Journal for Numerical Methods in Engineering*, vol. 69, no. 8, pp. 1570–1589, 2007.

[13] L. Yan, C.-L. Fu, and F.-L. Yang, "The method of fundamental solutions for the inverse heat source problem," *Engineering Analysis with Boundary Elements*, vol. 32, no. 3, pp. 216–222, 2008.

[14] S. Chantasiriwan, "Methods of fundamental solutions for time-dependent heat conduction problems," *International Journal for Numerical Methods in Engineering*, vol. 66, no. 1, pp. 147–165, 2006.

[15] W. Chen and M. Tanaka, "A meshless, integration-free, and boundary-only RBF technique," *Computers & Mathematics with Applications*, vol. 43, no. 3–5, pp. 379–391, 2002.

[16] V. D. Kupradze and M. A. Aleksidze, "The method of functional equations for the approximate solution of certain boundary-value problems," *USSR Computational Mathematics and Mathematical Physics*, vol. 4, pp. 82–126, 1964.

[17] Y. C. Hon and T. Wei, "A fundamental solution method for inverse heat conduction problem," *Engineering Analysis with Boundary Elements*, vol. 28, no. 5, pp. 489–495, 2004.

[18] N. S. Mera, "The method of fundamental solutions for the backward heat conduction problem," *Inverse Problems in Science and Engineering*, vol. 13, no. 1, pp. 65–78, 2005.

[19] L. Marin and D. Lesnic, "The method of fundamental solutions for the Cauchy problem in two-dimensional linear elasticity," *International Journal of Solids and Structures*, vol. 41, no. 13, pp. 3425–3438, 2004.

[20] L. Marin and D. Lesnic, "The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations," *Computers & Structures*, vol. 83, no. 4-5, pp. 267–278, 2005.

[21] E. J. Kansa, "Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics. I. Surface approximations and partial derivative estimates," *Computers & Mathematics with Applications*, vol. 19, no. 8-9, pp. 127–145, 1990.

[22] E. J. Kansa, "Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics—II. Solutions to parabolic, hyperbolic and elliptic partial differential equations," *Computers & Mathematics with Applications*, vol. 19, no. 8-9, pp. 147–161, 1990.

[23] E. J. Kansa, "Exact explicit time integration of hyperbolic partial differential equations with mesh free radial basis functions," *Engineering Analysis with Boundary Elements*, vol. 31, no. 7, pp. 577–585, 2007.

[24] G. E. Fasshauer, "Solving partial differential equations by collocation with radial basis functions," in *Surface Fitting and Multires-Olution Methods*, A. Le Mhaur, C. Rabut, and L. L. Schumaker, Eds., Vanderbilt University Press, Nashville, Tenn, USA, 1997.

[25] M. Tatari and M. Dehghan, "On the solution of the non-local parabolic partial differential equations via radial basis functions," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1729–1738, 2009.

[26] M. Tatari and M. Dehghan, "A method for solving partial differential equations via radial basis functions: application to the heat equation," *Engineering Analysis with Boundary Elements*, vol. 34, no. 3, pp. 206–212, 2010.

[27] M. Dehghan and V. Mohammadi, "The numerical solution of Fokker-Planck equation with radial basis functions (RBFs) based on the meshless technique of Kansa's approach and Galerkin method," *Engineering Analysis with Boundary Elements*, vol. 47, pp. 38–63, 2014.

[28] J. A. Kolodziei and A. UsciLowska, "Application of MFS for determination of effective thermal conductivity of unidirectional composites with linearly temperature dependent conductivity of constituents," *Engineering Analysis with Boundary Elements*, vol. 36, no. 3, pp. 293–302, 2012.

[29] L. Hui and L. Jijun, "Solution of Backward Heat problem by Morozov principle and conditional stability," *Numerical Mathematics*, vol. 14, no. 2, pp. 180–192, 2005.

[30] I. J. Schoenberg, "Metric spaces and completly monotonic functions," *Annals of Mathematics*, vol. 39, pp. 811–841, 1938.

[31] H. Wendland, "Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree," *Advances in Computational Mathematics*, vol. 4, no. 4, pp. 389–396, 1995.

[32] G. E. Fasshauer and J. G. Zhang, "On choosing 'optimal' shape parameters for RBF approximation," *Numerical Algorithms*, vol. 45, no. 1–4, pp. 345–368, 2007.

[33] R. E. Carlson and T. A. Foley, "The parameter $R^2$ in multiquadric interpolation," *Computers & Mathematics with Applications*, vol. 21, no. 9, pp. 29–42, 1991.

[34] A. H. Cheng, "Multiquadric and its shape parameter—a numerical investigation of error estimate, condition number, and round-off error by arbitrary precision computation," *Engineering Analysis with Boundary Elements*, vol. 36, no. 2, pp. 220–239, 2012.

[35] L.-T. Luh, "The shape parameter in the Gaussian function II," *Engineering Analysis with Boundary Elements*, vol. 37, no. 6, pp. 988–993, 2013.

[36] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, vol. 375 of *Mathematics and its Applications*, Kluwer Academic, New York, NY, USA, 1996.

[37] P. C. Hansen, *Rank-Defficient and Discrete Ill-Posed Problems*, SIAM, Philadelphia, Pa, USA, 1998.

[38] G. H. Golub, M. Heath, and G. Wahba, "Generalized cross-validation as a method for choosing a good ridge parameter," *Technometrics*, vol. 21, no. 2, pp. 215–223, 1979.

[39] P. C. Hansen, "Regularization tools: a Matlab package for analysis and solution of discrete ill-posed problems," *Numerical Algorithms*, vol. 6, no. 1-2, pp. 1–35, 1994.

[40] A. Shidfar, Z. Darooghehgimofrad, and M. Garshasbi, "Note on using radial basis functions and Tikhonov regularization method to solve an inverse heat conduction problem," *Engineering Analysis with Boundary Elements*, vol. 33, no. 10, pp. 1236–1238, 2009.