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Complete caps in projective spaces $\mathrm{PG}(n, q)$

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Abstract. A computer search in the finite projective spaces $\mathrm{PG}(n, q)$ for the spectrum of possible sizes k of complete k-caps is done. Randomized greedy algorithms are applied. New upper bounds on the smallest size of a complete cap are given for many values of n and q. Many new sizes of complete caps are obtained.

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1. Introduction

Let $\mathrm{PG}(n, q)$ be the projective space of dimension n over the Galois field $\mathrm{GF}(q)$. A k-cap in $\mathrm{PG}(n, q)$ is a set of k points, no three of which are collinear. A k-cap in $\mathrm{PG}(n, q)$ is called complete if it is not contained in a $(k + 1)$-cap of $\mathrm{PG}(n, q)$. For an introduction to these geometric objects, see [10]–[12].

A complete cap in a geometry $\mathrm{PG}(n, q)$, points of which are treated as $(n+1)$-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension $n+1$, Hamming distance 4, and covering radius 2 [12]. For an introduction to coverings of vector spaces over finite fields and the concept of code covering radius, see [3].

We use the following notation for constants of the projective space $\mathrm{PG}(n, q)$: as usual, $m_2(n, q)$ is the size of the largest complete cap, $m'_2(n, q)$ is the size of the second largest complete cap, and $t_2(n, q)$ is the size of the smallest complete cap. The corresponding *best known* values are denoted by $\bar{m}_2(n, q)$, $\bar{m}'_2(n, q)$, and $\bar{t}_2(n, q)$.

In this work, by computer search, we obtain a number of new values of $\bar{m}_2(n, q)$, $\bar{m}'_2(n, q)$, and $\bar{t}_2(n, q)$. Also, many new sizes k for which a complete k-cap in $\mathrm{PG}(n, q)$ exists are obtained.

This work uses results of the survey [8]. The reference to the paper [8] means "see [8] and the references therein", and similarly for [12].

An approach to computer search is considered in Section 2. The sizes of the known complete k-caps in $\mathrm{PG}(n, q)$ with $n \geq 3$, $q \geq 2$, are given in Section 3. New small complete k-caps with $k = \bar{t}_2(3, q)$, $q < 30$, are listed in the Appendix.

q	$t_2(3,q)$	Sizes k of the known complete caps with $t_2(3,q) \le k \le m'_2$	$\frac{q^2+q}{2}+2$	m'_2	m_2	References
7	≥ 12	$17 \le k \le 30$ and $k = 32$	30	32	50	[8], [12], [14]
8	≥ 14	$20 \le k \le 41$	38	≤ 60	65	[8], [14]
9	≥ 15	$24 \le k \le 48$	47	≤ 78	82	[8], [14], [15]
11	≥ 18	$30 \le k \le 69$	68	≤ 116	122	[8], [14]
13	≥ 21	$37 \le k \le 89$ and $k = 93$	93	≤ 162	170	[8], [14]
16	≥ 25	$41 \le k \le 138$	138	≤ 245	257	[8], [12], [16]
17	≥ 26	$51 \le k \le 153$ and $k = 155$	155	≤ 278	290	[8]
19	≥ 29	$59 \le k \le 187$ and $k = 189, 192$	192	≤ 348	362	[8]

Table 1 The sizes of the known complete k-caps in $\mathrm{PG}(3,q)$, $7 \le q \le 19$.

2. An approach to computer search

For the computer search we use a randomized greedy algorithm. At every step the algorithm minimizes or maximizes an objective function f, but some steps are executed in a random manner. The number of these steps and their ordinal numbers have been taken intuitively. If the same extremum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete cap using a starting set S_0 of points. At every step one point is added to the set. As the value of the objective function f we consider the number of points in the projective space that lie on bisecants of the set obtained. As S_0 we use a subset of points of a cap obtained in previous stages of the computer search. A random number generator is used for a random choice. To get caps with distinct sizes, starting conditions of the generator are changed for the same set S_0.

3. On the spectrum of sizes of complete caps in PG(n,q)

In the beginning we consider non-binary caps with $q \ge 3$. We use bounds of [8, Tables 3.2, 4.3], [12, Section 4], and the following bounds [8, Theorems 3.3, 3.4, 4.4].

In $\mathrm{PG}(3,q)$ if K is a complete k-cap, then

$$k(k-1)(q+1)/2 - k(k-2) \ge |\mathrm{PG}(3,q)|; \tag{1}$$

$$m_2'(3, q) \le q^2 - q + 6 \text{ if } q \ge 7 \text{ is odd;} \tag{2}$$

$$t_2(n, q) > \sqrt{2q^{n-1}}. \tag{3}$$

For known constructions of k-caps in PG(n, q) see [8],[11], [12].

Table 1 gives sizes of the known complete caps in PG$(3, q)$. We used the values of the cardinality of complete caps from [8, Table 3.2], [14]–[16]. New sizes are obtained by computer. In the tables $m_2' = m_2'(n, q)$, $m_2 = m_2(n, q)$.

Table 2 gives the sizes of the known complete caps in PG(n, q), $n \ge 4$, $q \ge 3$. We used the values of the cardinality of complete caps from [1], [5], [2], [8, Table 4.3], and [14]. The new sizes of caps in this table are obtained by computer.

n	q	$t_2(n, q)$	Sizes k of the known complete caps with $t_2(n, q) \le k \le m_2'$	m_2'	m_2	References
4	4	$16 \le$	$k = 20$ and $22 \le k \le 40$	40	41	[2], [8]
4	5	$21 \le$	$31 \le k \le 61$		≤ 96	[8],[12], [14]
4	7	$29 \le$	$57 \le k \le 113$		≤ 285	[8],[12], [14]
5	3	$20 \le$	$k = 22$, $26 \le k \le 44$, and $k = 48$	48	56	[1],[8]
5	4	$31 \le$	$50 \le k \le 108$ and $k = 112, 126$		≤ 159	[8],[12], [14]
6	3	$34 \le$	$22 \cdot 2 = 44 \le k \le 94$, $k \ne 45, 47, 49, 51$, and $k = 56 \cdot 2 = 112$		≤ 137	[1], [5], [8],[12]
6	4	$61 \le$	$117 \le k \le 254$		≤ 631	[8],[12]
7	3	$58 \le$	$44 \cdot 2 = 88 \le k \le 188$ and $k = 112 \cdot 2 = 224$		≤ 407	[5],[8], [12]
8	3	$100 \le$	$88 \cdot 2 = 176 \le k \le 380$, and $k = 224 \cdot 2 = 448$		≤ 1217	[5], [8], [12]
9	3	$172 \le$	$176 \cdot 2 = 352 \le k \le 784$ and $k = 448 \cdot 2 = 896$		≤ 3647	[5], [8], [12]

Table 2 The sizes of the known complete k-caps in PG(n, q), $n \ge 4$, $q \ge 3$.

Tables 3 and 4 give the sizes of the known small complete caps in PG$(3, q)$ and PG(n, q). The new sizes of caps obtained in this work are marked by the asterisk $\star$.
From Table 3 and [8, Table 3.1] the following is deduced.

q	$t_2(3,q)$	$\bar{t}_2(3,q)$	Refs.	q	$t_2(3,q)$	$\bar{t}_2(3,q)$	Refs.
7	≥ 12	$3q-4=17$	[14]	43	≥ 63	$3q+26=154$	⋆
8	≥ 14	$3q-4=20$	[14]	47	≥ 69	$3q+33=173$	⋆
9	≥ 15	$3q-3=24$	[14]	49	≥ 72	$3q+38=184$	⋆
11	≥ 18	$3q-3=30$	[14]	53	≥ 77	$3q+40=199$	⋆
13	≥ 21	$3q-2=37$	[14]	59	≥ 86	$3q+45=222$	⋆
16	≥ 25	$3q-7=41$	[16]	61	≥ 89	$3q+50=233$	⋆
17	≥ 26	$3q=51$	⋆	64	≥ 93	$3q+2=194$	[8]
19	≥ 29	$3q+2=59$	⋆	67	≥ 97	$3q+58=259$	⋆
23	≥ 35	$3q+4=73$	⋆	71	≥ 103	$3q+63=276$	⋆
25	≥ 38	$3q+7=82$	⋆	73	≥ 106	$3q+69=288$	⋆
27	≥ 41	$3q+9=90$	⋆	79	≥ 114	$4q=316$	⋆
29	≥ 43	$3q+10=97$	⋆	81	≥ 117	$4q-1=323$	⋆
31	≥ 46	$3q+13=106$	⋆	83	≥ 120	4q=332	⋆
32	≥ 48	$3q+2=98$	[8]	89	≥ 128	$4q=356$	⋆
37	≥ 55	$3q+20=131$	⋆	97	≥ 140	$4q+8=396$	⋆
41	≥ 60	$3q+24=147$	⋆				

Table 3 The sizes $\bar{t}_2(3,q)$ of the known small complete caps in PG$(3,q)$.

THEOREM 1. *In* PG$(3,q)$,

$$t_2(3,q) \leq 4q \quad \text{for } 2 \leq q \leq 89. \tag{4}$$

Now we consider binary caps with $q = 2$. We use the obvious relation

$$t_2(n,2) \geq \sqrt{2^{n+1}} \tag{5}$$

and the bound $t_2(6,2) \geq 19$ based on the corresponding bound for linear covering codes [3]. We consider only k-caps with $k \leq 2^{n-1}$ since all possible parameters of binary complete caps of size $k > 2^{n-1}$ are known [7].

In [9] binary k-caps in PG$(n,2)$ with $k = f(n)$ are constructed. Here

$$f(7) = 28,\ f(2m) = 23 \times 2^{m-3} - 3, m \geq 4,\ f(2m-1) = 15 \times 2^{m-3} - 3, m \geq 5. \tag{6}$$

From Table 5 and (6) the following result is deduced.

THEOREM 2. *In spaces PG$(n,2)$, $7 \leq n \leq 12$, there exist k-caps of all sizes with*

$$f(n) + D(n) \leq k \leq 2^{n-1} - 1, \quad 0 \leq D(n) < 1.5n. \tag{7}$$

n	q	$t_2(n,q)$	$\bar{t}_2(n,q)$	References	n	q	$t_2(n,q)$	$\bar{t}_2(n,q)$	References
4	4	≥ 16	20	[16]	5	4	≥ 31	50	[14]
4	5	≥ 21	31	[14]	5	5	≥ 36	83	[14]
4	7	≥ 29	57	[14]	5	7	≥ 70	176	$\star$
4	8	≥ 33	72	[14]	5	8	≥ 91	218	[16]
4	9	≥ 39	87	$\star$	5	9	≥ 115	304	$\star$
4	11	≥ 52	124	$\star$	6	3	≥ 34	44	[5]
4	13	≥ 67	163	$\star$	6	4	≥ 61	117	$\star$
4	16	≥ 91	233	$\star$	6	5	≥ 80	131	[5]
4	17	≥ 100	257	$\star$	6	7	≥ 121	349	[5]
5	3	≥ 20	22	[16]					

Table 4 The sizes $\bar{t}_2(n, q)$ of the known small complete caps in PG(n, q).

n	$t_2(n,2)$	Sizes k of the known complete caps with $t_2(n,q) \leq k \leq 2^{n-1}$	References
6	≥ 19	$21 \leq k \leq 31,\ k \neq 23, 30$	[3],[4],[9]
7	≥ 16	$28 \leq k \leq 63$	[9]
8	≥ 23	$43 \leq k \leq 127$	[9]
9	≥ 32	$60 \leq k \leq 255,\ k = 57$	[9]
10	≥ 46	$92 \leq k \leq 511,\ k = 89$	[9]
11	≥ 64	$133 \leq k \leq 1023,\ k = 117, 125, 126, 129, 130$	[9]
12	≥ 91	$196 \leq k \leq 2047,\ k = 181, 189, 190, 193, 194$	[9]

Table 5 The sizes of the known complete k-caps in PG$(n, 2)$, $k \leq 2^{n-1}$.

In fact, from Table 5 and (6), we have $f(8) = 43,\ f(9) = 57,\ f(10) = 89,\ f(11) = 117,$ $f(12) = 181$, and $D(7) = D(8) = 0,\ D(9) = D(10) = 3,\ D(11) = 16, D(12) = 15.$

We conjecture that the relation (7) holds for all $n \geq 7$ and moreover that $D(n) = 0$ for all $n \geq 7$.

Appendix

We give new small complete k-caps with $k = \bar{t}_2(3, q),\ q < 30$. Similarly to [6] and [14], we represent the elements of a Galois field GF(q) as follows:

$\{0, 1, \ldots, q - 1\}$ if q is prime and we operate on these modulo q; $\{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, q - 1 = \alpha^{q-2}\}$, where α is a primitive element, if $q = p^n$, p prime.

For addition we use a primitive polynomial generating the field. In this work the primitive polynomials are x^2+x+1 for $q=4$, x^3+x+1 for $q=8$, x^2+2x+2 for $q=9$, x^4+x^3+1 for $q=16$, x^2+x+2 for $q=25$, x^3+2x^2+x+1 for $q=27$, x^5+x^3+1 for $q=32$, x^2+x+3 for $q=49$, [13]. We write a cap as a set of points.

$\bar{t}_2(3, 17) = 51$:
(1,0,0,0), (0,1,7,15), (1,9,5,2), (0,1,2,9), (1,2,12,12), (1,6,15,14), (1,4,15,8), (1,8,16,1), (1,12,6,1), (1,5,11,11), (1,16,14,11), (1,11,8,9), (1,3,7,7), (1,9,6,12), (0,1,9,12), (1,6,5,2), (1,2,2,16), (1,4,15,16), (1,2,2,13), (1,5,3,14), (1,4,4,6), (1,12,8,5), (1,3,16,11), (0,1,3,2), (1,12,9,15), (1,0,5,13), (1,3,6,9), (0,1,4,7), (1,14,10,16), (1,12,12,9), (1,12,1,13), (1,8,16,10), (1,2,10,10), (1,3,3,2), (1,13,16,7), (1,4,7,12), (1,2,9,16), (1,4,1,13), (1,15,8,3), (1,13,16,3), (1,9,7,12), (1,11,16,14), (1,1,4,10), (1,13,1,14), (1,4,14,15), (0,1,6,4), (1,8,14,9), (1,8,0,12), (1,2,10,1), (1,10,3,9), (1,2,5,12)

$\bar{t}_2(3, 19) = 59$:
(1,0,0,0), (0,1,3,4), (1,10,5,13), (1,9,14,4), (1,12,11,10), (1,17,3,8), (1,6,0,4), (1,4,9,3), (1,13,10,10), (1,13,1,6), (0,1,15,14), (1,13,4,11), (1,15,18,17), (1,5,11,10), (1,9,13,4), (1,4,8,15), (1,9,2,8), (1,16,18,3), (1,9,16,5), (1,7,17,16), (1,1,5,9), (1,11,2,13), (1,11,10,3), (1,1,14,2), (1,3,9,7), (1,16,10,16), (1,5,18,0), (1,1,14,10), (1,18,9,15), (1,8,15,13), (0,1,8,2), (1,5,14,14), (1,7,12,13), (1,5,6,6), (1,17,4,7), (1,2,3,7), (0,1,8,3), (1,3,14,13), (1,4,13,13), (1,2,5,7), (1,16,14,16), (1,9,9,15), (1,4,11,2), (1,8,3,8), (1,11,0,15), (1,7,11,8), (1,8,6,16), (1,13,16,16), (1,3,13,7), (1,5,5,10), (1,6,1,6), (1,0,10,16), (1,7,1,2), (1,18,4,10), (1,2,2,2), (1,12,11,12), (1,16,9,3), (1,14,18,0), (1,4,10,3)

$\bar{t}_2(3, 23) = 73$:
(1,0,0,0), (0,1,1,7), (0,0,1,20), (1,6,22,14), (1,7,18,20), (1,8,0,12), (1,12,20,5), (1,3,1,16), (1,12,9,8), (1,19,14,9), (1,2,8,22), (1,15,18,21), (1,3,9,11), (1,21,12,9), (1,1,18,21), (1,2,1,11), (1,11,4,10), (1,21,1,1), (1,20,8,13), (1,9,15,10), (1,12,9,12), (0,1,16,7), (1,17,11,12), (1,8,19,13), (1,4,13,4), (1,18,4,16), (1,2,20,21), (1,7,11,4), (1,22,19,19), (1,14,18,10), (1,11,2,20), (1,10,8,20), (1,8,8,22), (1,12,20,6), (1,4,9,1), (1,0,5,7), (1,0,16,7), (1,2,10,10), (1,22,20,13), (1,21,7,10), (1,10,11,22), (1,22,21,15), (0,1,21,20), (1,11,6,14), (1,3,10,14), (1,2,8,19), (1,1,5,13), (1,9,4,9), (1,22,16,21), (1,21,7,9), (0,1,11,19), (1,12,19,15), (1,15,1,11), (1,21,5,0), (1,11,11,0), (1,19,1,15), (1,13,0,1), (1,14,0,20), (1,18,20,9), (1,8,16,19), (1,21,10,10), (1,14,14,5), (1,3,19,19), (1,22,18,4), (1,0,13,12), (1,21,17,13), (1,8,0,11), (1,17,16,4), (1,0,13,15), (1,0,22,21), (1,15,0,19), (1,4,19,18), (1,13,1,5)

$\bar{t}_2(3, 25) = 82$:
(1,0,0,0), (0,1,1,23), (1,3,22,1), (1,0,1,8), (1,0,15,10), (1,8,8,2), (1,6,16,7), (1,5,20,8), (1,2,16,2), (1,4,6,19), (1,14,24,21), (1,17,24,21), (1,11,22,0), (1,18,13,2), (1,4,24,15), (1,16,3,13), (1,23,2,9), (1,10,12,10), (1,14,23,20), (1,14,4,19), (1,23,5,9), (1,14,13,5), (1,19,23,11), (1,3,19,13), (1,10,13,16), (1,3,5,11), (1,0,16,9), (1,7,11,3), (1,2,16,22), (1,1,7,4), (1,10,1,3), (1,10,24,9), (1,1,4,23), (1,23,1,8), (1,15,0,8), (1,15,4,0), (1,14,16,20), (0,1,21,18), (1,8,9,13), (1,5,13,11), (1,7,3,20), (1,18,10,22), (1,13,19,10), (1,1,15,0), (1,12,19,4), (1,24,19,1), (1,1,14,14), (1,16,3,5), (1,17,23,9), (1,21,9,17), (1,1,20,14), (1,9,10,7), (1,20,0,19), (1,13,20,8), (1,1,5,19), (1,0,14,19), (1,0,6,24), (1,23,1,16), (1,6,3,7), (1,8,18,10), (0,1,3,5), (1,18,7,23), (1,8,19,13), (1,19,16,23), (1,3,9,8), (1,11,14,15), (1,21,19,9), (1,15,11,7), (1,7,18,1), (1,9,14,21), (1,21,0,19), (1,12,5,4), (1,19,14,14), (1,2,3,20), (1,20,18,7), (1,18,19,9), (1,0,12,24), (1,21,19,4), (1,5,7,17), (1,4,15,21), (0,1,6,10), (1,11,24,9)

$\bar{t}_2(3, 27) = 90$:
(1,0,0,0), (0,1,0,13), (1,1,22,6), (1,2,4,1), (1,2,0,25), (1,4,12,23), (1,9,15,16), (1,19,26,24), (1,16,22,1), (1,26,21,17), (1,5,13,12), (1,8,15,7), (1,0,24,25), (1,14,16,15), (1,2,4,2), (1,21,1,21), (1,3,7,20), (1,0,7,1), (1,3,24,12), (1,14,13,12), (1,19,10,24), (1,15,26,0), (1,5,5,2), (1,11,17,4), (1,4,13,11), (1,2,17,5), (1,11,14,6), (1,25,22,26), (1,24,3,22), (1,5,21,9), (1,16,8,1), (1,8,7,0), (1,6,26,18), (1,2,14,0), (1,7,11,1), (1,23,14,6), (1,21,3,16), (1,11,5,13), (1,26,9,18), (1,7,1,24), (1,5,24,18), (1,20,5,3), (1,0,18,22), (1,19,9,13), (1,21,2,13), (1,7,26,13), (1,15,20,14), (1,24,7,3), (1,24,13,24), (1,1,21,21), (1,21,22,8), (1,13,10,4), (1,2,18,7), (1,1,14,15), (1,10,17,23), (1,24,4,15), (1,21,1,8), (1,11,9,12), (1,11,5,22), (1,9,25,15), (1,0,24,16), (1,22,3,4), (1,26,21,6), (1,23,0,25), (1,20,24,4), (1,20,22,25), (1,13,26,5), (1,20,0,5), (1,22,21,12), (1,6,17,18), (1,16,11,14), (1,17,11,14), (1,0,14,3), (1,3,0,20), (1,21,18,21), (1,5,13,19), (1,3,16,0), (1,17,8,5), (1,15,17,10), (1,10,13,19), (1,0,26,21), (1,23,8,14), (1,25,18,8), (1,13,10,26), (1,0,16,22), (1,4,15,19), (1,20,11,19), (1,14,2,14), (1,2,20,9), (1,25,6,16)

$\bar{t}_2(3, 29) = 97$:
(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,3,19), (1,6,21,25), (1,3,13,20), (1,7,19,23), (1,14,25,14), (1,23,24,9), (1,16,7,27), (1,11,26,21), (1,13,14,10), (1,28,4,17), (1,20,27,5), (1,19,14,4), (1,20,3,23), (1,15,23,7), (1,14,21,2), (1,26,15,2), (1,3,5,8), (1,24,8,0), (1,5,19,3), (1,17,6,13), (1,6,0,22), (1,25,11,22), (1,19,19,15), (1,5,25,24), (1,6,20,11), (1,16,6,1), (1,28,22,2), (1,23,27,19), (1,7,6,4), (1,26,22,13), (1,26,26,26), (1,23,16,11), (1,15,24,23), (1,3,0,6), (1,4,15,15), (1,4,0,18), (1,14,28,24), (1,11,16,14), (1,2,16,21), (1,20,6,17), (1,5,27,6), (0,1,9,21), (1,21,1,11), (1,27,3,3), (1,12,18,22), (1,13,1,4), (1,27,16,6), (1,10,26,12), (1,2,13,6), (1,28,7,14), (1,11,13,0), (1,4,5,3), (1,24,13,8), (1,16,1,17), (1,1,0,26), (1,16,17,9), (1,12,13,26), (1,25,21,10), (1,4,1,5), (1,3,22,4), (1,9,28,19), (1,6,9,12), (1,25,5,11), (1,4,28,28), (1,25,12,12), (1,28,6,7), (1,9,9,10), (1,18,24,6), (1,22,21,12), (1,1,23,16), (1,19,20,12), (1,11,11,3), (1,17,21,14), (1,10,12,5), (1,1,12,2), (1,12,6,19), (1,17,24,28), (1,28,11,28), (1,11,23,2), (1,8,25,8), (1,14,12,25), (1,17,17,26), (1,0,20,9), (1,18,14,25), (1,5,8,4), (1,17,7,5), (1,10,3,11), (1,3,26,27), (1,9,0,25), (1,22,12,4), (1,8,16,5), (1,16,25,20), (1,14,27,3)

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