Proceedings of the European Control Conference 2007
Kos, Greece, July 2-5, 2007

An exponential observer for discrete-time systems with bilinear drift and rational output functions

Alfredo Germani Costanzo Manes

Abstract— **This paper presents an observer with exponential error decay for systems described by bilinear input-state dynamics and output functions that are ratios of polynomials in the state. It is shown that such kind of systems can be immersed into systems of higher dimension, with time-varying linear state dynamics and linear output map. The observer here presented is derived exploiting the structure of the extended system. Conditions of global exponential convergence are given and discussed. Computer simulations demonstrate good behavior of the observer, even in the presence of disturbances on the state and output equations.**

Index Terms— **State observers, Kronecker algebra, bilinear systems.**

I. INTRODUCTION

When designing a state observers for discrete time nonlinear systems, many authors adopt the approach to find a nonlinear change of coordinates and an output transformation such to put the system into a canonical form suitable for the observer design using linear techniques. In [10],[15],[16], where autonomous systems are considered, conditions are given for the existence of a coordinate transformation that allows the design of an observer with linear error dynamics. Other papers concerned with the problem of finding conditions for the existence of such a change of coordinates are [20] and [21], for autonomous systems, and [3], [7] for nonautonomous systems. Note that, in general, the computation of such a coordinate transformation is a very difficult task. Another approach consists in designing observers in the original coordinates, finding iterative algorithms that asymptotically solve some suitable defined extensions of the state-output map, (see e.g. [11] and [17]). Sufficient conditions of local convergence are provided, in general, under the assumption of Lipschitz nonlinearities. The use of the Extended Kalman Filter as a local observer for noise-free systems has been investigated in [19], [4], [5] and [18]. Nonlinear systems with linear measurements are considered in [1], [6] and [22]. In [22] autonomous systems with the Lipschitz nonlinearities are considered and an observer with constant gain is proposed. Sufficient convergence conditions are given in terms of LMI's.

This paper presents an asymptotic observer for the class of systems whose state evolves according to bilinear difference equations, and whose outputs are ratios of polynomials of the state. Such kind of output structure is useful to approximate generic nonlinear output functions. It is shown that the dynamics of systems in this class can be immersed into the dynamics of a linear time-varying system of larger dimension, that can be used for the observer construction. Conditions of global exponential convergence are given, in terms of observability and reachability Gramians of the extended system. The observer here presented can be implemented without computing any nonlinear change of coordinates or Jacobian of the output function.

The paper is organized as follows. In section II the class of systems considered is introduced and an useful representation for them is derived. The extended system is defined in Section III and the observation algorithm is presented in Section IV. Section V reports simulations results. Conclusions follow. A short Appendix reports the definitions of Kronecker products and powers and some of their properties, used in sections II and III.

II. SYSTEMS WITH BILINEAR DRIFT AND RATIONAL OUTPUT

The class of systems considered in this paper is characterized by discrete-time bilinear input-state dynamics and by output functions that are ratios of polynomials (BDRO Systems: Bilinear Drift-Rational Output). Such systems are described, for all integers $t \geq t_0$, by equations of the form

$$x(t+1) = A(t)x(t) + B_0(t)u(t) + \sum_{i=1}^{p} u_i(t)B_i(t)x(t),$$
$$y_k(t) = \frac{n_k\big(t, x(t)\big)}{d_k\big(t, x(t)\big)}, \quad k = 1, \dots q, \tag{1}$$

where $x(t) \in I\!R^n$ is the system state, $y(t) \in I\!R^q$ is the measured output, $u(t) \in I\!R^p$ is a known input, and $n_k(t, x)$ and $d_k(t, x)$ are polynomials of x, with possibly time-varying coefficients. In this paper the Kronecker formalism is used for the representation of polynomials of vectors. The definition of Kronecker products and powers and some of their properties are reported in the Appendix. Using the Kronecker powers of the state, the output of the system (1) can be written as

$$y_k(t) = \frac{n_{k,0}(t) + \sum_{j=1}^{r_k} n_{k,j}^T(t)x^{[j]}(t)}{d_{k,0}(t) + \sum_{j=1}^{s_k} d_{k,j}^T(t)x^{[j]}(t)} \quad k = 1, \dots q, \tag{2}$$

where r_k and s_k are the degrees of the numerator and denominator polynomials of each scalar output, and $n_{k,j}(t) \in I\!R^{n^j}$ and $d_{k,j}(t) \in I\!R^{n^j}$ are the vector coefficients of the powers $x^{[j]}(t) \in I\!R^{n^j}$. Denoting with m the maximal degree of the

This work is supported by *MIUR* (Italian Ministry for University and Research) under grant PRIN 2005097521, and by *CNR* (National Research Council of Italy)

A. Germani and C. Manes are with the Department of Electrical and Information Engineering, University of L'Aquila, Poggio di Roio, 67040 L'Aquila, Italy. E-mail: germani@ing.univaq.it, manes@ing.univaq.it. They are also with Istituto di Analisi dei Sistemi ed Informatica del CNR "A. Ruberti", Viale Manzoni 30, 00185 Roma, Italy.

ISBN: 978-960-89028-5-5

polynomials, i.e. $m = \max_{k=1,\dots,q}(r_k, s_k)$, and defining a *polynomial extended* state vector $X_m(t)$ as follows

$$X_m(t) = \begin{pmatrix} x(t) \\ x^{[2]}(t) \\ \vdots \\ x^{[m]}(t) \end{pmatrix} \in I\!R^{b(n,m)}, \tag{3}$$

where $b(n,m) = \sum_{i=1}^{m} n^i$, the components of the system output can be written in the following compact form

$$y_k(t) = \frac{n_{k,0}(t) + N_k^T(t) X_m(t)}{d_{k,0}(t) + D_k^T(t) X_m(t)} \tag{4}$$

where $N_k(t) \in I\!R^{b(n,m)}$ and $D_k(t) \in I\!R^{b(n,m)}$ are defined as follows

$$\begin{aligned} N_k^T(t) &= [n_{k,1}^T(t) \ \cdots \ n_{k,m}^T(t)], \\ D_k^T(t) &= [d_{1,m}^T(t) \ \cdots \ d_{k,m}^T(t)]. \end{aligned} \tag{5}$$

Moreover, defining the matrix function

$$B\big(t, u\big) = \sum_{i=1}^{p} B_i(t) u_i, \tag{6}$$

the BDRO system (1) can be written in the form

$$\begin{aligned} x(t+1) &= A(t)x(t) + B\big(t, u(t)\big)x(t) + B_0(t)u(t), \\ y_k(t) &= \frac{n_{k,0}(t) + N_k^T(t) X_m(t)}{d_{k,0}(t) + D_k^T(t) X_m(t),} \quad k = 1, \dots q, \end{aligned} \tag{7}$$

Each scalar output in system (7) can put in the form

$$y_k(t) d_{k,0}(t) - n_{k,0}(t) = N_k^T(t) X_m(t) - y_k(t) D_k^T(t) X_m(t), \tag{8}$$

so that, defining transformed outputs $\tilde{y}_k(t)$ as

$$\tilde{y}_k(t) = y_k(t) d_{k,0}(t) - n_{k,0}(t), \quad k = 1, \dots, q, \tag{9}$$

the following q output equations can be written

$$\tilde{y}_k(t) = \Big(N_k^T(t) - y_k(t) D_k^T(t)\Big) X_m(t), \tag{10}$$

where the new outputs $\tilde{y}_k(t)$ appears as a linear function of the extended state. The equations (10) can be put in matrix form as follows

$$\tilde{y}(t) = \mathcal{C}_y(t) X_m(t) \tag{11}$$

where

$$\tilde{y}(t) = \begin{pmatrix} y_1(t) d_{1,0}(t) - n_{1,0}(t) \\ \vdots \\ y_q(t) d_{q,0}(t) - n_{q,0}(t) \end{pmatrix}, \tag{12}$$

and

$$\mathcal{C}_y(t) = \begin{pmatrix} N_1^T(t) - y_1(t) D_1^T(t) \\ \vdots \\ N_q^T(t) - y_q(t) D_q^T(t). \end{pmatrix} \tag{13}$$

Defining the matrices

$$A_u(t) = A(t) + B\big(t, u(t)\big), \quad B_u(t) = B_0(t) u(t). \tag{14}$$

The BDRO system (7) with the transformed output (12) can be written in the following compact form

$$\begin{aligned} x(t+1) &= A_u(t)x(t) + B_u(t), \\ \tilde{y}(t) &= \mathcal{C}_y(t) X_m(t). \end{aligned} \tag{15}$$

Thanks to the definition (14) of matrix $A_u(t)$ the input-state equation in (15) appears as a linear time-varying system, and thanks to the definition of the transformed output $\tilde{y}(t)$, given in (12), the output function appears as a polynomial of degree m of the state.

III. THE EXTENDED SYSTEM

This section shows that a BDRO system can be *immersed* into a system of larger dimension, characterized by a linear time-varying dynamics. The main result consists in showing that the dynamics of the polynomial extended state $X_m(t)$ defined in (3) obeys the time varying-linear equation presented by the following lemma:

Lemma III.1. *Consider the bilinear input-state equation of system* (1) *and its representation given by the first of* (15)*, with* $A_u(t)$ *and* $B_u(t)$ *defined in* (14)*. The dynamics of the polynomial extended state* $X_m(t)$ *defined in* (3) *is governed by the following recursive equation:*

$$X_m(t+1) = \mathcal{A}_u(t) X_m(t) + \mathcal{B}_u(t), \tag{16}$$

where matrix $\mathcal{A}_u$ *has the following block-triangular structure*

$$\mathcal{A}_u(t) = \begin{bmatrix} \mathcal{A}_{1,1} & 0 & \cdots & 0 & 0 \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \cdots & 0 & 0 \\ \mathcal{A}_{3,1} & \mathcal{A}_{3,2} & \cdot & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{A}_{m,1} & 0 & \cdots & \mathcal{A}_{m,m-1} & \mathcal{A}_{m,m} \end{bmatrix}, \tag{17}$$

where the matrices $\mathcal{A}_{h,k}$*, defined for* $h = 1, \dots, m$ *and* $k = 1, \dots, h$*, are recursively defined by the following equations, defined for* $h = 1, \cdots, m-1$ *and* $k = 2, \cdots, h$

$$\begin{aligned} \mathcal{A}_{1,1} &= A_u, \\ \mathcal{A}_{h+1,1} &= B_u^{[h]} \otimes A_u + \mathcal{A}_{h,1} \otimes B_u, \\ \mathcal{A}_{h+1,k} &= \mathcal{A}_{h,k-1} \otimes A_u + \mathcal{A}_{h,k} \otimes B_u, \\ \mathcal{A}_{h+1,h+1} &= \mathcal{A}_{h,h} \otimes A_u. \end{aligned} \tag{18}$$

and

$$\mathcal{B}_u = \begin{bmatrix} B_u \\ B_u^{[2]} \\ \vdots \\ B_u^{[m]} \end{bmatrix}. \tag{19}$$

(in equations (17)*,* (18) *and* (19) *time-dependence is omitted for brevity).*

Proof: Taking into account the definitions (17) and (19), the state dynamics (16) is equivalent to the following equations

$$x^{[h]}(t+1) = \sum_{i=1}^{h} \mathcal{A}_{h,i} x^{[i]}(t) + B_u^{[h]}(t), \quad h = 1, \dots, m. \tag{20}$$

The equation with $h = 1$ is readily proved by observing that, by definition, $\mathcal{A}_{1,1} = A_u(t)$.

Now, proceed by induction: assume that (20) is true for a given $h > 1$ and prove that it is also true for $h + 1$. Indeed

$$x^{[h+1]}(t+1) = x^{[h]}(t+1) \otimes x(t+1) \\ = \left(\sum_{i=1}^{h} \mathcal{A}_{h,i} x^{[i]}(t) + B_u^{[h]}(t)\right) \big(A_u(t)x(t) + B_u(t)\big) \tag{21}$$

From this

$$x^{[h+1]}(t+1) = \sum_{i=1}^{h} \Big(\mathcal{A}_{h,i} x^{[i]} \otimes (A_u x) + \mathcal{A}_{h,i} x^{[i]} \otimes B_u\Big) \\ + B_u^{[h]} \otimes (A_u x) + B_u^{[h+1]} \tag{22}$$

By means of property (69)

$$x^{[h+1]}(t+1) = \sum_{i=1}^{h} \Big((\mathcal{A}_{h,i} \otimes A_u) x^{[i+1]} + (\mathcal{A}_{h,i} \otimes B_u) x^{[i]}\Big) \\ + (B_u^{[h]} \otimes A_u) x + B_u^{[h+1]} \tag{23}$$

Reorganizing the summation this can be put in the form

$$x^{[h+1]}(t+1) = (\mathcal{A}_{h,h} \otimes A_u) x^{[h+1]} + \\ + \sum_{k=2}^{h} \Big(\mathcal{A}_{h,k-1} \otimes A_u + \mathcal{A}_{h,k} \otimes B_u\Big) x^{[k]} \\ + \Big(B_u^{[h]} \otimes A_u + \mathcal{A}_{h,1} \otimes B_u\Big) x + B_u^{[h+1]} \tag{24}$$

Thus, from definitions (18), it follows that also $x^{[h+1]}(t)$ obeys the recursive equation (20) and the induction is proved.

Remark III.2. The state dynamics of system (1) is said to be *immersed* into the dynamics

$$X(t+1) = \mathcal{A}_u(t) X(t) + \mathcal{B}_u(t), \tag{25}$$

whose state $X(t) \in I\!R^{b(m,n)}$ is of larger dimension than $x(t)$. Such an immersion is to be intended as follows: if $X(t_0) = X_m(t_0)$, where

$$X_m(t_0) = \begin{pmatrix} x(t_0) \\ x^{[2]}(t_0) \\ \vdots \\ x^{[m]}(t_0) \end{pmatrix} \in I\!R^{b(n,m)}, \tag{26}$$

then the following identity relates the state $x(t)$ of (1) and the state $X(t)$ of (25) for $t \geq t_0$

$$x(t) = S_m X(t), \tag{27}$$

with

$$S_m = \begin{bmatrix} I_n & O_{n \times (n^2 + \cdots + n^m)} \end{bmatrix}. \tag{28}$$

This happens because, thanks to Lemma III.1, the initialization (26) implies that $X(t) = X_m(t)$ for all $t \geq t_0$, so that the product $S_m X(t)$ in (27) simply selects the first n components of $X_m(t)$, that is the state $x(t)$ of system (1).

From what discussed, the following equations can be used to represent system (15) for $t \geq t_0$

$$X(t+1) = \mathcal{A}_u(t) X(t) + \mathcal{B}_u(t), \quad X(t_0) = X_m(t_0) \tag{29}$$

$$\tilde{y}(t) = \mathcal{C}_y(t) X(t) \tag{30}$$

It must be stressed that the Kronecker powers of vectors contain redundant terms. It follows that redundant components are present in the extended state vector X_m, so that the extended state space results to be output-indistinguishable. Such redundancy can be eliminated by considering suitably defined reduction matrices. First of all note that $x^{[i]}$, the i-th Kronecker power of $x \in I\!R^n$, has n^i components, but only $\binom{n+i-1}{i}$ are distinct terms (the number of ways to choose i elements from a set of n, with repetitions allowed). Defining the following functions of pairs of integers

$$b(n,m) = n\frac{1-n^m}{1-n} = \sum_{i=1}^{m} n^i, \tag{31}$$

$$c(n,m) = \binom{n+m}{m} - 1 = \sum_{i=1}^{m} \binom{n+i-1}{i}, \tag{32}$$

it is easy to see that the vector X_m has $b(n,m)$ components, but only $c(n,m)$ are distinct (obviously $c(n,m) < b(n,m)$).

A block-diagonal reduction matrix $\overline{T}_{n,m} \in I\!R^{c(n,m) \times b(n,m)}$ can be suitably defined, as described in detail in [8], for the selection of a nonredundant subvector $\overline{X}_m \in I\!R^{c(n,m)}$ from $X_m \in I\!R^{b(n,m)}$. A block-diagonal matrix $T_{n,m} \in I\!R^{b(n,m) \times c(n,m)}$ allows to reconstruct the redundant vector X_m from $\overline{X}_m$. In formulas

$$\overline{X}_m(k) = \overline{T}_{n,m} X_m, \quad X_m = T_{n,m} \overline{X}_m(k). \tag{33}$$

Using Lemma III.1 and the reduction matrices (33), system (III) is obviously equivalent to the following system

$$\mathcal{X}(t+1) = \bar{\mathcal{A}}_u(t) \mathcal{X}(t) + \overline{\mathcal{B}}_u(t) \\ \tilde{y}(t) = \bar{\mathcal{C}}_y \mathcal{X}(t), \tag{34}$$

where $\mathcal{X}(t) \in I\!R^{c(n,m)}$ and

$$\bar{\mathcal{A}}_u(t) = T_{n,m} \mathcal{A}_u(t) \overline{T}_{n,m}, \qquad \overline{\mathcal{B}}_u(t) = T_{n,m} \mathcal{B}_u(t), \\ \bar{\mathcal{C}}_y(t) = \mathcal{C}_y(t) \overline{T}_{n,m}. \tag{35}$$

The equivalence of the system (15) with the reduced system (34) should be intended as follows: if the initial value of the extended state of (34) is set to

$$\mathcal{X}(t_0) = T_{n,m} \begin{pmatrix} x(t_0) \\ x^{[2]}(t_0) \\ \vdots \\ x^{[m]}(t_0) \end{pmatrix} = T_{n,m} X_m(t_0), \tag{36}$$

then the outputs of the two systems is the same for any input, and the state $x(t)$ of system (15) is recovered by selecting the first n components of the extended state $\mathcal{X}(t)$:

$$x(t) = \Sigma_m \mathcal{X}(t), \\ \text{where } \Sigma_m = \begin{bmatrix} I_n & 0_{n \times (c(n,m) - n)} \end{bmatrix}. \tag{37}$$

IV. AN EXPONENTIAL STATE OBSERVER FOR *BDRO* SYSTEMS

The construction of an observer for system (1) is achieved in this section through the construction of an observer for the extended system (34) that provides an estimate $\widehat{\mathcal{X}}(t)$ (observation) of the true extended state $\mathcal{X}(t)$. The observation of the original state $x(t)$ is achieved by selecting the first n components of $\widehat{\mathcal{X}}(t)$ through the selection matrix Σ_m defined in (37). The observer presented in this section is sought among dynamic systems of the form

$$Z(\tau+1) = \bar{\mathcal{A}}_u(\tau)Z(\tau) + \overline{\mathcal{B}}_u(\tau) + v(\tau), \tag{38}$$

$$Z(t_0) = Z_0, \tag{39}$$

where $v(\tau)$ is some *driving* sequence and Z_0 is some initial observer state. Let $\widehat{\mathcal{X}}_0$ be an *a priori* estimate of the true state $\mathcal{X}(t_0)$, and let $w(t)$ be the output error for the observer (38)-(39), defined as

$$w(\tau) = \tilde{y}(\tau) - \bar{\mathcal{C}}_y(t)Z(\tau). \tag{40}$$

Let $J_{[t_0,t]}$ be an *energy cost functional* defined over the interval $[t_0, t]$ as follows

$$\begin{aligned} J_{[0,t]}(Z_0, v_{[t_0,t]}, w_{[t_0,t]}) &= \frac{1}{2}(Z_0 - \widehat{\mathcal{X}}_0)^T S_0 (Z_0 - \widehat{\mathcal{X}}_0) \\ &+ \frac{1}{2}\sum_{\tau=t_0}^{t}\left(v^T(\tau)Q_\tau v(\tau) + w^T(\tau)R_\tau w(\tau)\right), \end{aligned} \tag{41}$$

where S_0, R_τ and Q_τ, $\tau \in [0,t]$, are positive definite symmetric weight matrices that can be defined by the user. The *minimum energy triple* $(Z_0^*, v^*_{[t_0,t]}, w^*_{[t_0,t]})$, given all observations in $[t_0, t]$, is the one that provides the minimum value for $J_{[t_0,t]}$ under constraints (38)–(40). Let $Z^*(\tau|t)$, $\tau \in [t_0, t]$, denote the trajectory provided by equations (38) and (39) driven by the minimum energy input $v^*_{[t_0,t]}$ and with initial state Z_0^*. For $\tau < t$ the solution $Z^*(\tau|t)$ is the optimal smoothing (minimum energy non-causal estimate), while the endpoint $Z^*(t|t)$ of the trajectory is the optimal *causal* estimate. Let $\widehat{\mathcal{X}}(t) = Z^*(t|t)$. It is known that $\widehat{\mathcal{X}}(t)$ can be computed through the Kalman Filter equations (forward recursion), while a backward recursion is needed for the computation of $Z^*(\tau|t)$, $\tau < t$ (see e.g. [2], [14]). When a new observation $y(t+1)$ is available, the solution of the constrained minimization problem $\min J_{[t_0,t+1]}(Z_0, v_{[t_0,t+1]}, w_{[t_0,t+1]})$ subject to (38)–(40) provides a different trajectory $Z^*(\tau|t+1)$, whose endpoint is the optimal estimate $\widehat{\mathcal{X}}(t+1)$. The sequence of minimum energy estimates can be computed through the following recursion

$$K_t = \widetilde{P}_t\bar{\mathcal{C}}_y^T(t)\Big(\bar{\mathcal{C}}_y(t)\widetilde{P}_t\bar{\mathcal{C}}_y^T(t) + R_t^{-1}\Big)^{-1}, \tag{42}$$

$$\widehat{\mathcal{X}}(t) = \widehat{\mathcal{X}}_p(t) + K_t\Big(\tilde{y}(t) - \bar{\mathcal{C}}_y(t)\widehat{\mathcal{X}}_p(t)\Big), \tag{43}$$

$$P_t = \Big(I - K_t\bar{\mathcal{C}}_y(t)\Big)\widetilde{P}_t, \tag{44}$$

$$\widehat{\mathcal{X}}_p(t+1) = \bar{\mathcal{A}}_u(t)\widehat{\mathcal{X}}(t) + \overline{\mathcal{B}}_u(t), \tag{45}$$

$$\widetilde{P}_{t+1} = \bar{\mathcal{A}}_u(t)P_t\bar{\mathcal{A}}_u^T(t) + Q_t^{-1}. \tag{46}$$

The initialization of the algorithm is

$$\widehat{\mathcal{X}}_p(t_0) = \widehat{\mathcal{X}}_0, \qquad \widetilde{P}_{t_0} = S_0^{-1}. \tag{47}$$

The state observation of the original system (1) is computed as

$$\hat{x}(t) = \Sigma_m\widehat{\mathcal{X}}(t) \tag{48}$$

where the selection matrix Σ_m is defined in (37) (see the discussion at the beginning of the section).

In order to present conditions that ensure convergence, let $\Phi(t,\tau)$ denote the transition matrix associated to system (34), recursively defined as

$$\begin{aligned} \Phi(t,t) &= I_{c(n,m)}, \quad \forall t \\ \Phi(t+1,\tau) &= \bar{\mathcal{A}}_u(t)\Phi(t,\tau), \quad t > \tau. \end{aligned} \tag{49}$$

Theorem IV.1. *Consider the BDRO system* (1) *and the construction that has led to the extended system* (34) *and to the observer* (42)–(48)*, where the sequences Q_t and R_t are chosen uniformly upper and lower bounded. Assume that for a given pair $(x(t_0), u_{[t_0,\infty)})$ the matrix $\bar{\mathcal{A}}_y(t)$ is nonsingular and bounded for all $t \geq t_0$. Let*

$$\gamma = \sup_{t\in[t_0,\infty)} \|\bar{\mathcal{A}}_y(t)\|. \tag{50}$$

Assume that the pair $(\bar{\mathcal{A}}_u(t), \bar{\mathcal{C}}_y(t))$ is such that there exist an integer N and positive scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that for all $t \geq t_0 + N$

$$\alpha_1 I \geq \sum_{\tau=t-N}^{t}\Phi(t,\tau)Q_\tau^{-1}\Phi^T(t,\tau) \geq \alpha_2 I, \tag{51}$$

and

$$\beta_1 I \leq \sum_{\tau=t-N}^{t}\Phi^T(\tau,t-N)\bar{\mathcal{C}}_y^T(\tau)R_\tau\bar{\mathcal{C}}_y(\tau)\Phi(\tau,t-N) \leq \beta_2 I, \tag{52}$$

where I is the identity in $\mathbb{R}^{c(n,m)\times c(n,m)}$ (obviously $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$).

Then, the observation error exponentially converges to 0 with the following bound

$$\|x(t) - \hat{x}(t)\| \leq a^{t-t_0}b\|\widehat{\mathcal{X}}(t_0) - X_m(t_0)\|. \tag{53}$$

where $a \in (0,1)$ and $b > 0$ are as follows:

$$a = \frac{1}{\sqrt{1 + (\gamma^2 q_M)^{-1}p_m}}, \qquad b = \sqrt{\frac{p_M}{p_m}}, \tag{54}$$

where q_M is the upper bound of the sequence Q_t, i.e. $q_M I \geq Q_t$, $\forall t \geq t_0$, and

$$p_m = \frac{\beta_1}{1+\alpha_1\beta_1}, \qquad p_M = \frac{1+\alpha_2\beta_2}{\alpha_2}. \tag{55}$$

Remark IV.2. Note that (52) is an assumption of *uniform observability* of the linear time-varying system (34), while the assumption (51) is an assumption of *uniform reachability* of the pair $(\bar{\mathcal{A}}_u(t), Q_t)$, and is satisfied provided that both $\bar{\mathcal{A}}_u(t)$ and Q_t are uniformly lower and upper bounded.

The proof of Theorem IV.1, omitted due to lack of space, is based on the fact that inequalities (51) and (52) imply that the

Riccati sequence P_t satisfies the following lower and upper bounds

$$\frac{1}{p_M} I \leq P_t \leq \frac{1}{p_m} I, \tag{56}$$

with p_M and p_M defined in (55) (see [12]). Then, defining the extended state observation error $\varepsilon_t = \mathcal{X}(t) - \widehat{\mathcal{X}}(t)$, by showing the asymptotic convergence to zero of the following positive definite function of ε_t

$$V(t, \varepsilon_t) = \varepsilon_t^T P_t^{-1} \varepsilon_t, \tag{57}$$

suitable computations lead to inequality (53).

Remark IV.3. Since the matrices $\bar{\mathcal{C}}_u(t)$ depend on the measurements $y(t)$, the condition (52) can only be checked *on-line*. On the other hand, the condition (51) can be verified *off-line* only when the input $u(t)$ is *a priory* known. From a practical point of view, it is easier to check the observability on-line by monitoring the minimum and maximum eigenvalues of P_t during its evolution. Note that if the uniform bounds p_m and p_M of inequality (56) exist, then

$$p_m^{-1} \geq \lambda_{\mathrm{Max}}(P_t), \qquad p_M^{-1} \leq \lambda_{\min}(P_t). \tag{58}$$

V. SIMULATION RESULTS

Simulations results are here reported in order to show the effectiveness of the proposed observer. Consider the following system, with $x \in I\!R^3$, $u \in I\!R$ and $y \in I\!R$,

$$x(t+1) = Ax(t) + B_0 u(t) + u(t)Bx(t), \tag{59}$$

$$y(t) = \frac{x_1 + 0.2x_2^3}{1 + 0.1x_1^2 + 0.1x_2^2}, \tag{60}$$

where:

$$A = \begin{bmatrix} 0 & -0.8 & 0 \\ 0.4 & 0 & 0.4 \\ 0 & -0.4 & 0.4 \end{bmatrix} \tag{61}$$

$$B = \begin{bmatrix} 0 & 0 & 0.2 \\ -0.1 & 0.4 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}. \tag{62}$$

The output in (60) can be put in the form (2) with:

$$\begin{aligned} & n_0 = 0, \quad n_1^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ & n_2^T = 0_{1\times 9}, \quad [n_3]_{1,14} = 0.2, \quad [n_3]_{1,j} = 0, \ j \neq 14, \\ & d_0 = 1, \quad d_1^T = 0_{1\times 3}, \quad d_3^T = 0_{1\times 27}, \\ & [d_2]_{1,1} = 0.1, \quad [d_2]_{1,5} = 0.1, \quad [d_2]_{1,j} = 0, \ \begin{matrix} j \neq 1 \\ j \neq 5 \end{matrix}. \end{aligned} \tag{63}$$

Figures 1–3 report the true and the observed states over the interval $[0, 70]$ when the input is

$$u(t) = 1.5 + 0.5\mathrm{sign}(\cos(4\pi t/70)) \tag{64}$$

($u(t)$ switches five times between 1 and 2 in the interval $[0, 70]$). The weight matrices S_0, Q_t and R_t in this simulation are constant and equal to the identity matrices.

The performance of the observer in the presence of unknown disturbances, both in the state equation and in the measurements, has been also evaluated through simulations.

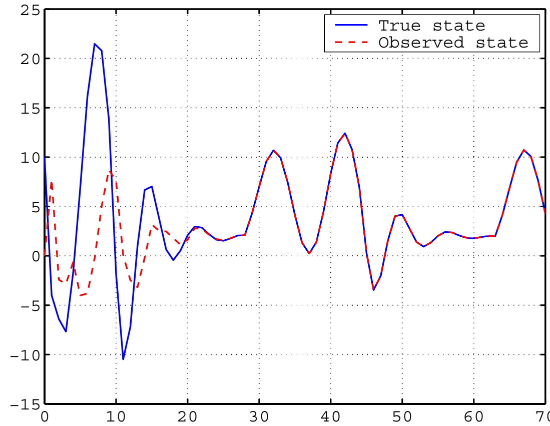

Fig. 1. **The first component of the true and observed state without noise.**

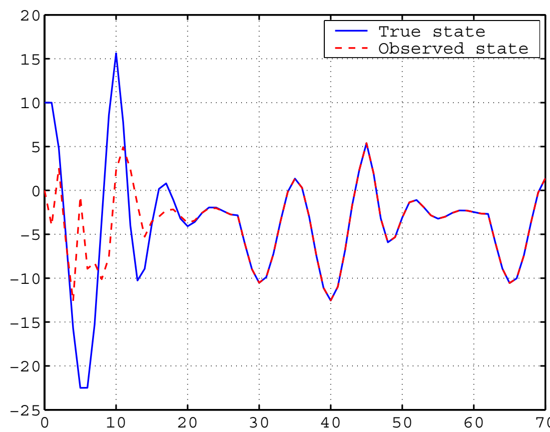

Fig. 2. **The second component of the true and observed state without noise.**

Figure 4 reports simulation results (due to lack of space, only the third component is reported) on the same system, with the same input, when additive standard Gaussian noises (unitary variance) are present as forcing terms and as measurement noise in the equations (1). Good performances are obtained.

VI. CONCLUSIONS

This paper presents an exponential observer for the class of systems with bilinear input-state dynamics and state-output functions that are ratios of polynomials (Bilinear Drift, Rational Output). The dimension of the state space of the observer depends on the state space of the system and on the degree of the maximal polynomial present in the output functions. The observer gain is time varying and is obtained on-line as the solution of discrete-time Riccati equations. The Jacobian of the output function does not need to be computed. Global convergence conditions are provided. The observer behavior has been numerically tested on some examples, even in the presence of disturbances, and has always given good results.

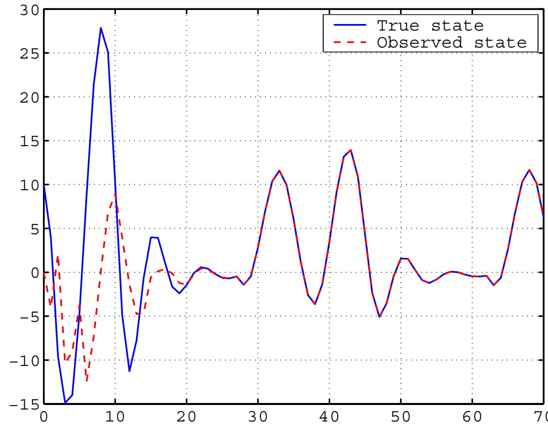

Fig. 3. **The third component of the true and observed state without noise.**

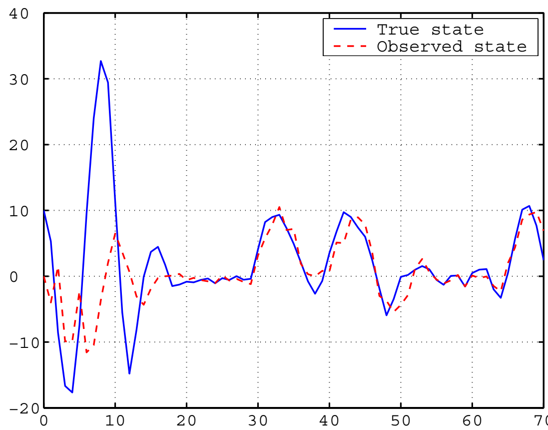

Fig. 4. **The third component of the true and observed state with, state and output noise.**

APPENDIX

The Kronecker product of two matrices M and N of dimensions $r \times s$ and $p \times q$ respectively, is the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \vdots & \ddots & \vdots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix}, \qquad (65)$$

where the m_{ij} are the entries of M. The Kronecker power of a matrix M is recursively defined as

$$M^{[0]} = 1, \quad M^{[i]} = M \otimes M^{[i-1]}, \quad i \geq 1. \qquad (66)$$

Note that if $M \in I\!R^{a \times b}$, then $M^{[i]} \in I\!R^{a^i \times b^i}$. See the Appendix in [9] for a quick survey on the Kronecker algebra. Some properties of the Kronecker product used throughout the paper are the following:

$$(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D \qquad (67)$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \qquad (68)$$

$$(A \cdot C) \otimes (B \cdot D) = (A \otimes B) \cdot (C \otimes D) \qquad (69)$$

In particular, repeated application of properties (68) and (69) provides the identity

$$(Ax)^{[i]} = A^{[i]} x^{[i]}, \qquad (70)$$

intensively used throughout the paper. See [13] for more properties.

References

[1] V.C. Aitken and H.M. Schwartz, "On the exponential stability of discrete-time systems with applications in observer design." *IEEE Trans. A. C.*, Vol. 39, No. 9, pp. 1959–1962, 1994.

[2] B.D.O. Anderson and J.B. Moore, *Optimal filtering,* Prentice-Hall, Englewood Cliffs, New Jersey, 1979.

[3] G. Besançon, H. Hammouri, S. Benamor, " State equivalence of discrete-time nonlinear control systems to state affine form up to input/output injection", *Syst. Contr. Lett.*, Vol. 33, No. 1, pp. 1–10, 1998.

[4] M. Boutayeb, H. Rafaralahy, and M. Darouach, "Convergence analysis of the Extended Kalman Filter used as an observer for nonlinear deterministic discrete-time Systems," *IEEE Trans. A. C.*, Vol. 42, No. 4, pp. 581–586, 1997.

[5] M. Boutayeb and D. Aubry, "A strong tracking extended Kalman observer for nonlinear discrete-time systems." *IEEE Trans. A. C.*, Vol. 44, No. 8, pp. 1550 - 1556, 1999.

[6] M. Boutayeb and M. Darouach, "A reduced-order observer for nonlinear discrete-time systems," *Syst. Contr. Lett.*, Vol. 39, No. 2, pp. 141–151, 2000.

[7] C. Califano, S. Monaco, and D. Normand-Cyrot, "On the observer design in discrete-time," *Syst. Contr. Lett.*, Vol. 49, No. 4, pp. 255–265, 2003.

[8] F. Carravetta, A. Germani, M. Raimondi, "Polynomial filtering for linear discrete-time non-Gaussian systems," *SIAM J. Contr. Optim.*, Vol. 34, No. 5, pp. 1666–1690, 1996.

[9] F. Carravetta, A. Germani, M. Raimondi, "Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise," *IEEE Trans. A. C.*, Vol. 42, No. 8, pp. 1106–1126, 1997.

[10] S.T. Chung and J.W. Grizzle, "Sampled-data observer error linearization," *Automatica*, Vol. 26, pp. 997–1007, 1990.

[11] G. Ciccarella, M. Dalla Mora, and A. Germani, "Observers for discrete-time nonlinear systems," *Syst. Contr. Lett.*, Vol. 20, No. 5, pp. 373–382, 1993.

[12] J.J. Deyst, Jr., and C.F. Price, "Conditions for asymptotic stability of the discrete minimum-variance linear estimator," *IEEE Trans. A. C.*, Vol. 13, No. 6, pp. 702–705, 1968.

[13] A. Graham, *Kronecker Products and Matrix Calculus with Applications,* Ellis Horwood ltd. Publishers, Chichester, 1981.

[14] A.H. Jazwinski, *Stochastic Processes and Filtering Theory,* Academic Press, New York, 1970.

[15] W. Lee and K. Nam, "Observer design for autonomous discrete-time nonlinear systems," *Syst. Contr. Lett.*, Vol. 17, No. 1, pp. 49–58, 1991.

[16] W. Lin and C.I. Byrnes, "Remarks on linearization of discrete-time autonomous systems and nonlinear observer design," *Syst. Contr. Lett.*, Vol. 25, No. 1, pp. 31–40, 1995.

[17] P.E. Moraal, J.E. Grizzle, "Observer design for nonlinear systems with discrete-time measurements," *IEEE Trans. A. C.*, Vol. 40, No. 3, pp. 395–404, 1995.

[18] K. Reif, R. Unbehauen, "The extended Kalman filter as an exponential observer for nonlinear systems," *IEEE Trans. Signal Process.*, Vol. 47, pp. 2324–2328, 1999.

[19] Y. Song, J.W. Grizzle, "The extended Kalman filter as a local asymptotic observer for discrete-time nonlinear systems," *J. Math. Systems Estimate and Control*, Vol. 5, pp. 59–78, 1995.

[20] M. Xiao, N. Kazantzis, C. Kravaris, and A.J. Krener, "Nonlinear discrete-time observer design with linearizable error dynamics," *IEEE Trans. A. C.*, vol. 48, No. 4, pp. 622–626, 2003.

[21] M. Xiao, "A direct method for the construction of nonlinear discrete-time observer with linearizable error dynamics," *IEEE Trans. A. C.*, Vol. 51, No. 1, pp. 128–135, 2006.

[22] A. Zemouche and M. Boutayeb, "Observer design for lipschitz nonlinear systems: the discrete-time case," *IEEE Trans. on Circ. and Systems II: Express Briefs,* Vol. 53, No. 8, pp. 777–781, 2006.