**Higher-Order Iterative Methods for Solving Nonlinear Equations**

I.A. Al-Subaihi [1], A. J. Al-qarni[2]

[1]. Department Department of Mathematics, Faculty of Science, Taibah University, KSA.
[2]. Department of Mathematics, Faculty of Science, King Khaled University, KSA.
aw-aw910@hotmail.com

**Abstract:** In this report, we presented three high-order iterative methods for solving nonlinear equations of the form $f(x)=0$. These proposed iterative methods are obtained by combining a fourth-order iterative method with the classical Newton's method and approximating the first derivative in the third step by three different approaches of combinations of previously evaluated function values. The convergence analyses of the new methods are discussed, and several examples are given to illustrate the methods' efficiency.

[Al-Subaihi IA., Al-qarni AJ. **Higher-Order Iterative Methods for Solving Nonlinear Equations.** *Life Sci J* 2014;11(12):85-91]. (ISSN:1097-8135). http://www.lifesciencesite.com. 15

**Keywords:** Newton's method; Iterative methods; Efficiency index; Order of Convergence; Optimal Iterative methods.

**AMS** subject classification: 65H05, 49M99.

## 1. Introduction

One of the most important and challenging problems in science and engineering applications is to solve nonlinear equations of the form $f(x)=0$ .The Newton method is most likely the best-known iterative method for solving nonlinear equations and is given as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Recently, several modifications of the Newton method have been proposed and analyzed, which have either an equal or better performance than the Newton method; for examples, refer to to (Kou et al., 2010 ) and (Neta and Petkovic, 2010 ). Researchers have devoted a significant amount of attention to developing three-step iterative methods with an eighth-order of convergence for solving nonlinear equations of the form $f(x)=0$ ; for examples, refer to (Kou et al., 2010 ), (Neta and Petkovic, 2010 ),( Sharma and Sharma , 2010), (Siyyam et al., 2011), (Siyyam et al., 2011), (Soleymani et al., 2012).

Combining two methods of third order modifications of Newton's methods (Chun, 2006), (Chun, 2005) ; for solving nonlinear equations of the form $f(x)=0$ by R. Ezzati and F. Saleki (Ezzati and Saleki, 2011); gives the iterative method as follows:

$$x_{n+1} = A[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)}] + B[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)f(y_n)}{f'(x_n)(f(x_n)-f(y_n))}],$$

When $A=-1$ and $B=2$, the fourth-order iterative method can be determined as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)\left(f(x_n)-f(y_n)\right)}. \tag{2}$$

The error equation corresponding to the above method is given as follows:

$$e_{n+1} = (-c_2c_3 + 3c_2^3)e_n^4 + O(e_n^5),$$

where $e_n = x_n - \alpha$ ; and $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$ for $k = 2,3,...$ .

Combining the iterative method (2) with Newton's method, the iterative method can be obtained as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)\left(f(x_n)-f(y_n)\right)}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{3}$$

According to the following theorem, the iterative method (3) has an eighth-order of convergence.

**Theorem 2.1**: (Traub, 1982) Let $\phi_1(x)$ and $\phi_2(x)$ be two iterative methods with an order of convergence $p$ and $q$, respectively; then, the order of convergence of the iterative method $\phi(x)=\phi_2(\phi_1(x))$ is $pq$.

Furthermore, per one cycle, the iterative method (3) requires three evaluations of the functions and two evaluations of its first derivatives, $s=5$. Therefore, the efficiency index is given as follows:

$$EI=\rho^{\frac{1}{s}}$$
$$=8^{\frac{1}{5}}\approx 1.5157.$$

An iterative method is an optimal method if it's order of convergence is $2^{s-1}$ and efficiency index is $2^{\frac{(s-1)}{s}}$ (Kung and Traub, 1974). Thus, method (3) is not optimal and does not meet the Kung and Traub conjecture (Kung and Traub, 1974). However, (1) and (2) are optimal iterative methods.

The goal of this study is to simultaneously increase the order of convergence and efficiency index as high as possible . Therefore, we have to replace the first derivative in the last step of (3), i.e., $f'(z_n)$, by a combination of previously evaluated function values, $f(x_n), f'(x_n)$ and $f(y_n)$.

This report is organized as follows: In Section 2, we describe the concept of the composition of the iterative methods and, consequently, prove that the iterative method defined in (3) has an eighth-order of convergence. Three different estimations for the first derivative in the last step of (3) are presented in Section 2. Furthermore, Section two, proves that the order of convergence of the last two new resulting iterative methods is eight with an efficiency index of $8^{1/4}\approx 1.6818$. Therefore, these last two new iterative methods defined in (18) and (27) are optimal and satisfy the conjecture of Kung and Traub (1974). In Section 3 various numerical examples are presented, to illustrate and, confirm the performance and accuracy of our proposed iterative methods as well as compare methods of the same order of convergence.

## 2. Construction of Higher-Order Iterative Methods

To improve the efficiency index of method (3), several estimations for the first derivative in the last step $f'(z_n)$ of (3) are proposed by using a combination of previously evaluated function values. In Section (2.1), we presented the first method, which is the estimation based on Cordero et al. (2010); and proved in Theorem (2.1.1) that the order of convergence of the resulting iterative method is seven. More, two estimations of $f'(z_n)$ are described based on Neta and Petkovic (2010), and Siyyam et al. (2011). The last two methods will be proved to have an eighth-order of convergence.

### 2.1 Method One

A three-step family of iterative methods based on the modified Kou's method (Kou , Li and Wang, 2007); is considered by Cordero et al. (2010). A second degree Taylor polynomial of $f(z_n)$ is used to approximate $f'(y_n)$, and substituting this with an appropriate approximation of $f''(y_n)$ in $f'(z_n)$ is given as follows:

$$f'(z_n)\approx f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n). \quad (4)$$

By substituting the estimation obtained in (4) into (3), the following new iterative method can be obtained as follows:

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$
$$z_n=y_n+\frac{f(y_n)}{f'(x_n)}-2\frac{f(x_n)f(y_n)}{f'(x_n)\left(f(x_n)-f(y_n)\right)}$$
$$x_{n+1}=z_n-\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}. \quad (5)$$

Per one full iteration, the method requires three evaluations of the function and one evaluation of its first derivative; therefore, its efficiency index is $\rho=7^{1/4}\approx 1.6266$. Furthermore, as $\rho\neq 2^{\frac{3}{4}}$, the method is not optimal.

**Theorem 2.1.1:** Let $\alpha\in I$ be a simple zero of a sufficiently differentiable function $f:I\rightarrow R$ for an open interval $I$. Then, the method that is defined by equation (5) has a seventh-order of convergence and satisfies the error equation as follows:

$$e_{n+1}=(2c_3^2c_2^2-6c_3c_2^4)e_n^7+O(e_n^8), \quad (6)$$

where $e_n=x_n-\alpha$; and $c_k=\frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$, for $k=2,3,...$.

**Proof :** Let $\alpha$ be a simple zero of the nonlinear equation $f(x)=0$, and $x_n=\alpha+e_n$

By the Taylor expansion, we have the equation as follows:

$$f(x_n)=f'(\alpha)[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6+c_7e_n^7+c_8e_n^8+O(e_n^9)], \quad (7)$$

and

$$f'(x_n)=f'(\alpha)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+8c_8e_n^7+O(e_n^8)]. \quad (8)$$

Dividing (7) by (8), the equation becomes as follows:

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (-2c_3 + 2c_2^2)e_n^3 + \ldots$$
$$(31c_4c_5 - 64c_2c_4^2 - 75c_4c_3^2 - 176c_4c_2^4 + 92c_5c_2^3 + 27c_6c_3$$
$$-44c_6c_2^2 + 135c_2c_3^3 - 408c_3^2c_2^3 + 304c_3c_2^5 - 64c_2^7 + 19c_2c_7$$
$$-7c_8 + 348c_4c_3c_2^2 - 118c_5c_2c_3)e_n^8 + O(e_n^9). \quad (9)$$

Substituting (7), (8) and (9) into $y_n$ in (5) gives the equation as follows:

$$y_n = \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + \ldots$$
$$+(-31c_4c_5 + 64c_2c_4^2 + 75c_4c_3^2 + 176c_4c_2^4 - 92c_5c_2^3 - 27c_6c_3$$
$$+44c_6c_2^2 - 135c_2c_3^3 + 408c_3^2c_2^3 - 304c_3c_2^5 + 64c_2^7 - 19c_2c_7$$
$$+7c_8 - 348c_4c_3c_2^2 + 118c_5c_2c_3)e_n^8 + O(e_n^9). \quad (10)$$

By expanding $f(y_n)$ with respect to $\alpha$, the expression is given as follows:

$$f(y_n) = f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + \ldots$$
$$+(-31c_4c_5 + 73c_2c_4^2 + 75c_4c_3^2 + 297c_4c_2^4 - 134c_5c_2^3$$
$$+54c_6c_2^2 - 147c_2c_3^3 + 582c_3^2c_2^3 - 552c_3c_2^5 + 144c_2^7 - 19c_2c_7$$
$$+7c_8 - 455c_4c_3c_2^2 + 134c_5c_2c_3)e_n^8 + O(e_n^9)]. \quad (11)$$

Substituting (7), (8), (10), and (11) into $z_n$ in (5), gives the equation as follows:

$$z_n = \alpha + (-c_2c_3 + 3c_2^3)e_n^4 + \ldots + (164c_5c_2c_3 - 965c_4c_3c_2^2$$
$$-5c_2c_7 - 13c_6c_3 + 50c_6c_2^2 - 17c_5c_4 - 239c_5c_2^3 + 122c_4c_3^2$$
$$+91c_2c_4^2 + 799c_4c_2^4 - 395c_2c_3^3 + 1862c_3^2c_2^3 - 2076c_3c_2^5$$
$$+624c_2^7)e_n^8 + O(e_n^9). \quad (12)$$

By expanding $f(z_n)$ with respect to $\alpha$, the expression is given as follows:

$$f(z_n) = f'(\alpha)[(-c_2c_3 + 3c_2^3)e_n^4 + \ldots + (164c_5c_2c_3$$
$$-965c_4c_3c_2^2 - 5c_2c_7 - 13c_6c_3 + 50c_6c_2^2 - 17c_5c_4 - 239c_5c_2^3$$
$$+122c_4c_3^2 + 91c_2c_4^2 + 799c_4c_2^4 - 395c_2c_3^3 + 1863c_3^2c_2^3$$
$$-2082c_3c_2^5 + 633c_2^7)e_n^8 + O(e_n^9). \quad (13)$$

By using (10), (11), (12) and (13), the expression is obtained as follows:

$$f[y_n, z_n] = f'(\alpha)[1 + c_2^2e_n^2 + (2c_2c_3 - 2c_2^3)e_n^3 + \ldots$$
$$+(-18c_3c_2c_4 + 86c_2^6 + 59c_4c_2^3 - 16c_5c_2^2 + 5c_2c_6 + 4c_3^3$$
$$+52c_3^2c_2^2 - 167c_3c_2^4)e_n^6 + O(e_n^7)] \quad (14)$$

By using (6), (7), (12), and (13), the equation is obtained as follows:

$$f[z_n, x_n] = f'(\alpha)[1 + c_2e_n + c_3e_n^2 + c_4e^3 + \ldots$$
$$+(-12c_5c_3^2 - 257c_5c_2^4 + 53c_6c_2^3 - 1279c_3c_4c_2^3 - 22c_4c_2c_5$$
$$+288c_4c_2c_3^2 + 224c_3c_5c_2^2 - 18c_3c_2c_6 - 13c_3c_4^2 - 738c_3^3c_2^2$$
$$+624c_2^8 + 28c_3^4 + c_9)e_n^8 + O(e_n^9)]. \quad (15)$$

By using (6), (8), (12), and (15), the expression $f[z_n, x_n, x_n]$ can be expressed in terms of $e_n$ as follows:

$$f[z_n, x_n, x_n] = f'(\alpha)[c_2 + 2c_3e_n + 3c_4e_n^2 + 4c_5e_n^3 + \ldots$$
$$-2c_3^3)e_n^5 + O(e_n^6)]. \quad (16)$$

Substituting (10), (12), (13), (14) and (16) into $x_{n+1}$ in equation (5), the equation can be expressed as follows:

$$x_{n+1} = \alpha + (2c_3^2c_2^2 - 6c_3c_2^4)e_n^7 + O(e_n^8).$$

Therefore, the equation is given as follows:

$$e_{n+1} = (2c_3^2c_2^2 - 6c_3c_2^4)e_n^7 + O(e_n^8). \quad (17)$$

Equation (17) establishes the seventh-order convergence of the method that is defined by equation (5). □

### 2.2 Method Two

A general technique is given by Neta and Petkovic (2010); to construct such methods using inverse interpolation and any optimal two point methods and present an approximation of the third step, which can be placed into the last step in the method (3). Thus, the new iterative method to solve the nonlinear equation of the form $f(x) = 0$ is given as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = y_n + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)(f(x_n) - f(y_n))}$$
$$x_{n+1} = y_n + c[f(x_n)]^2 - d[f(x_n)]^3. \quad (18)$$

where

$$c = \frac{1}{(f(y_n) - f(x_n))f[y_n, x_n]} - \frac{1}{f'(x_n)(f(y_n) - f(x_n))} - d(f(y_n) - f(x_n)), \quad (19)$$

and

$$d = \frac{1}{(f(y_n) - f(x_n))(f(y_n) - f(z_n))f[y_n, x_n]} - \frac{1}{(f(z_n) - f(x_n))(f(y_n) - f(z_n))f[z_n, x_n]} + \frac{1}{f'(x_n)(f(z_n) - f(x_n))(f(y_n) - f(z_n))} - \frac{1}{f'(x_n)(f(y_n) - f(x_n))(f(y_n) - f(z_n))}. \quad (20)$$

Per one full iteration, the method requires three evaluations of the function and one evaluation of its first derivative; therefore, its efficiency index is $8^{1/4} \approx 1.6818$, which implies that the method is an optimal eighth -order method according to Kung and Traub's conjecture (1974). In the next theorem, we will show that the order of convergence of the iterative method (18) is eight.

**Theorem 2.2.1:** Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \to R$ for an open interval $I$. Then, the method that is defined by equation (18) has an eighth-order of convergence and satisfies the error equation as follows:

$$e_{n+1} = (-c_4c_3c_2^2 + 15c_2^7 + 3c_4c_2^4 + 5c_3^2c_2^3 - 20c_3c_2^5)e_n^8 + O(e_n^9),$$

**Proof**: Let $\alpha$ be a simple zero of $f$ and $x_n = \alpha + e_n$. Using (6), (7), (10) and (11), we obtain the expression as follows:

$$f[y_n, x_n] = f'(\alpha)[1 + c_2e_n + (c_3 + c_2^2)e_n^2 + \dots(-8c_5c_3^2 - 93c_5c_2^4 + 8c_2c_8 + 45c_6c_2^3 - 313c_3c_4c_2^3 - 40c_4c_2c_5 + 56c_4c_2c_3^2 + 116c_3c_5c_2^2 - 36c_3c_2c_6 + 8c_3c_7 - 8c_3c_4^2 - 42c_3^3c_2^2 + 264c_3^2c_2^4 - 256c_3c_2^6 + 69c_4^2c_2^2 + 8c_4c_6 + 184c_4c_2^5 - 20c_7c_2^2 + 64c_2^8 + 4c_5^2 - 6c_3^4 + c_9)e_n^8 + O(e_n^9)]. \quad (21)$$

The parameters $c$ and $d$ can be expressed in terms of $e_n$ as follows:

$$c = \frac{1}{f'(\alpha)^2}(-c_2 + (6c_2^2 - 3c_3)e_n + \dots(350c_2^3c_5 - 166c_2^2c_6 - 189c_2c_4^2 - 530c_2^4c_4 - 1034c_2^3c_3^2 + 447c_2^5c_3 + 382c_2c_3^3 - 13c_8 + 58c_2c_7 + 54c_2^7 - 209c_4c_3^2 + 72c_6c_3 - 376c_2c_5c_3 + 76c_5c_4 + 1084c_2^2c_4c_3)e_n^6 + O(e_n^7). \quad (22)$$

and

$$d = \frac{1}{f'(\alpha)^3}(-c_3 + 2c_2^2 + (-10c_2^3 - 2c_4 + 10c_2c_3)e_n + (27c_2^4 - 3c_5 + 9c_3^2 + 15c_2c_4 - 48c_2^2c_3)e_n^2 \dots + (-6c_8 + 268c_2^7 + 186c_2^3c_5 - 92c_2^2c_6 - 90c_2c_4^2 - 176c_2^4c_4 - 216c_2^3c_3^2 - 316c_2^5c_3 + 158c_2c_3^3 + 30c_2c_7 - 100c_4c_3^2 + 36c_6c_3 + 36c_5c_4 - 192c_2c_5c_3 + 480c_2^2c_4c_3)e_n^5 + O(e_n^6), \quad (23)$$

By substituting (7), (10), (22) and (23) into $x_{n+1}$ in (18), the equation is obtained as follows:

$$x_{n+1} = \alpha + (-c_4c_3c_2^2 + 15c_2^7 + 3c_4c_2^4 + 5c_2^2c_3^3 - 20c_3c_2^5)e_n^8 + O(e_n^9).$$

Therefore, the expression becomes as follows:

$$e_{n+1} = (-c_4c_3c_2^2 + 15c_2^7 + 3c_4c_2^4 + 5c_3^2c_2^3 - 20c_3c_2^5)e_n^8 + O(e_n^9). \quad (24)$$

Equation (24) establishes the eighth-order convergence of the method that is defined by equation (18). □

### 2.3 Method Three

Siyyam et al. (2011); considers composing the two-step family of a fourth-order equation based on Chun (2007), with the classical Newton's method to obtain a three-step iterative method. A new estimation for $f'(z_n)$, also proposed in (Siyyam et al., 2011), is provided by the third-degree polynomial as follows:

$$P(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)^2 + a_3(x - x_n)^2(x - y_n) \quad (25)$$

where $f$ is interpolated at $x_n, y_n$ and $z_n$; and $f'(x_n) = P'(x_n)$ is satisfied. Therefore, the expression becomes as follows:

$$f'(z_n) \approx P'(z_n) = f'(x_n) + (f[x_n, y_n, z_n] + f[x_n, x_n, y_n])(z_n - x_n)$$

$$f'(z_n) \approx P'(z_n) = f'(x_n) + (f[x_n, y_n, z_n] + f[x_n, x_n, y_n])(z_n - x_n) + 2(f[x_n, y_n, z_n] - f[x_n, x_n, y_n])(z_n - y_n). \quad (26)$$

Substituting (26) into the last step of equation (3), we obtain the new iterative method as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)(f(x_n) - f(y_n))}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{P_3'(z_n)}. \quad (27)$$

The next theorem shows that the order of convergence of the iterative method defined in (27) is eight. Moreover, per one full iteration, the method requires three evaluations of the function and one evaluation of its first derivative; therefore, its efficiency index is $8^{1/4} \approx 1.6818$, which implies that the method is an optimal eighth order method according to Kung and Traub's conjecture (1974).

**Theorem 2.3.1:** Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \to R$ for an open interval $I$. Then, the method that is defined by equation (27) has an eighth-order of convergence.

**Proof:** Use equations (6), (12), (14) and (21), (23) to write the expression $f[x_n, y_n, z_n]$ in terms of $e_n$ as follows:

$$f[x_n, y_n, z_n] = f'(\alpha)[c_2 + c_3e_n + (c_2c_3 + c_4)e_n^2 + \dots + (-16c_3c_2c_4 - 8c_3^3 + 3c_4^2 + c_7 + 3c_4c_2^3 + 6c_3c_5 + 40c_3^2c_2^2 - 26c_3c_2^4 - c_5c_2^2 + c_2c_6)e_n^5] + O(e_n^6), \quad (28)$$

Using (6), (8), (10) and (21), the expression $f[x_n, x_n, y_n]$ can be expressed in terms of $e_n$ as follows:

$$f[x_n, x_n, y_n] = f'(\alpha)[c_2 + 2c_3e_n + (3c_4 + c_3c_2)e_n^2 + \dots$$

$$+(-36c_5c_2c_4+58c_5c_3c_2^2-32c_6c_2c_3+118c_4c_2c_3^2$$
$$-92c_4c_3c_2^3+8c_9+12c_5^2+18c_3^4-6c_7c_2^2+6c_6c_2^3$$
$$+6c_2c_8-32c_5c_3^2-6c_5c_2^4+16c_7c_3+22c_6c_4-34c_4^2c_3$$
$$+24c_4^2c_2^2-126c_3^3c_2^2+128c_3^2c_2^4-32c_3c_2^6)e_n^7]$$
$$+O(e_n^8). \quad (29)$$

Thus, using equations (8), (10) , (12) , (28) and (29), we can write the estimation of Siyyam et al.in $f'(z_n)\approx P_{3'}(z_n)$ as described in the last step of (27) in terms of $e_n$ as follows:

$$f'(z_n)\approx P_{3'}(z_n)$$
$$=f'(\alpha)[1+(-2c_3c_2^2+c_2c_4+6c_2^4)e_n^4+...$$
$$+(-20c_3c_2c_4+3c_4^2+140c_2^6+61c_4c_2^3+4c_3c_5$$
$$+84c_3^2c_2^2-260c_3c_2^4-9c_5c_2^2+3c_2c_6)e_n^6]+O(e_n^7). \quad (30)$$

By substituting equations (12), (13) and (30) into $x_{n+1}$ in (27), the equation becomes as follows:

$$x_{n+1}=\alpha+(-c_4c_3c_2^2+3c_4c_2^4+c_3^2c_2^3-6c_3c_2^5+9c_2^7)e_n^8$$
$$+O(e_n^9). \quad (31)$$

Therefore, the expression can be written as follows:

$$e_{n+1}=(-c_4c_3c_2^2+3c_4c_2^4+c_3^2c_2^3-6c_3c_2^5+9c_2^7)e_n^8$$
$$+O(e_n^9). \quad (32)$$

Equation (32) establishes the eighth-order convergence of the method that is defined by equation (27).□

## 3. Numerical Examples

Three high-order iterative methods have been derived for solving nonlinear equations of the form $f(x)=0$, namely the seventh-order iterative method defined in equation (5), which we will refer to as (SQM1), the eighth-order iterative method defined in (18), which we will refer to as (SQM2) and the eighth-order iterative methods defined in (27), which we will refer to as (SQM3). To confirm our theoretical results and illustrate the efficiency and accuracy of our developed methods, we tested these iterative methods using several numerical examples and compared them with other existing eighth-order iterative methods.

We can compare our iterative methods, (SQM1), (SQM2) and (SQM3), with the eighth-order iterative methods of Kou et al., (KM1), (KM2) in (Kou et al., 2010), which are defined as follows:

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$z_n=y_n-H_2(x_n,y_n)(x_n-y_n),$$
$$x_{n+1}=z_n-[(1+H_2(x_n,y_n))^2+(1+4H_2(x_n,y_n))$$
$$\times H_\beta(y_n,z_n)]\frac{f(z_n)}{f'(x_n)}, \quad (33)$$

where

$$H_2(x_n,y_n)=\frac{f(y_n)}{f(x_n)-2f(y_n)},$$
$$H_\beta(y_n,z_n)=\frac{f(z_n)}{f(y_n)-\beta f(z_n)}, \quad (34)$$

with $\beta$ =0 ,

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$z_n=y_n-H_2(x_n,y_n)(x_n-y_n),$$
$$u_n=z_n-(1+H_2(x_n,y_n))^2\frac{f(z_n)}{f'(x_n)},$$
$$x_{n+1}=u_n-(1+4H_2(x_n,y_n))$$
$$\times\frac{(z_n-u_n)}{(y_n-u_n-\beta(z_n-u_n))}\frac{f(z_n)}{f'(x_n)}, \quad (35)$$

With $\beta$ =0

and the eighth-order iterative method of Nazir et al. (NAM) in (Mir and Akram, 2009), the equation is given as follows:

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$z_n=y_n-\frac{h(y_n)}{1-(h(y_n))^2},$$
$$x_{n+1}=z_n-(y_n-z_n)\frac{f(z_n)}{f(y_n)-2f(z_n)}, \quad (36)$$

where

$$h(y_n)=\frac{2f(y_n)}{f'(y_n)+\sqrt{(f'(y_n))^2+4(f(y_n))^2}}.$$

The particular eighth-order iterative method of Sharma et al. (SM1) in (Sharma and Sharma, 2010), is defined as follows:

$$w_i=x_i-\frac{f(x_i)}{f'(x_i)},$$
$$z_i=w_i-\frac{f(x_i)}{f(x_i)-2f(w_i)}\frac{f(w_i)}{f'(x_i)}, \quad (37)$$
$$x_{i+1}=z_i-W(\mu_i)\frac{f[x_i,w_i]f(z_i)}{f[x_i,z_i]f[w_i,z_i]},$$

where

$$\mu_i=\frac{f(z_i)}{f(x_i)},$$

| |
|---|
| $f_1(x) = xe^x + \log(1+x+x^4)$, $\alpha \approx 0.22045246\ 21592135$. |
| $f_2(x) = \sqrt{x} - \frac{1}{x} - 3$, $\alpha \approx 9.63359556\ 28326952$. |
| $f_3(x) = \log(x) + \sqrt{x} - 5$, $\alpha \approx 8.30943269\ 42315718$. |
| $f_4(x) = \arcsin(x^2-1) - x/2 + 1$, $\alpha \approx 0.59481096\ 83983692$. |
| $f_5(x) = e^{-x} + \cos(x)$, $\alpha \approx 1.74613953\ 04080124$. |
| $f_6(x) = \cos(x) - x$, $\alpha \approx 0.73908513\ 32151606$. |

$$f[x_i, z_i] = \frac{f(z_i) - f(x_i)}{z_i - x_i},$$

$$f[x_i, w_i] = \frac{f(w_i) - f(x_i)}{w_i - x_i},$$

$$f[w_i, z_i] = \frac{f(z_i) - f(w_i)}{z_i - w_i},$$

$W$ can be expressed as follows:

$$W(t) = 1 + t + t^2,$$

The same method was used by Sharma et al.(SM2) in (Sharma et al., 2011), to produce another eighth-order method as follows:

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$z_i = w_i - \frac{f(w_i)}{f'(x_i)} \frac{f(x_i)}{f(x_i) - 2f(w_i)},$$

$$x_{i+1} = z_i - (\phi)^{-1} \frac{f(z_i)}{f'(x_i)}, \tag{38}$$

where

$$\phi = [f(w_i)(f(w_i) - f(x_i))^3 - f(x_i)f(z_i)(f(w_i) - f(x_i))(f(x_i) - 2f(w_i)) - 2f^2(w_i)(f(z_i) - f(x_i))(f(x_i) - 2f(w_i))][f(x_i)f(w_i)(f(w_i) - f(x_i))(f(x_i) - 2f(w_i))]^{-1}. \tag{39}$$

All computations were performed using MATLAB 7.11 with 1000 significant digits. Table(1) shows the absolute value of the function at $x_n$, $|x_{n+1} - x_n|$ and $|x_n - \alpha|$ for the test functions for all the iterative methods mentioned in this section, where $n$ represents the number of iterations and is taken in the table to be 3, $\alpha$ is the zero of the function and $x_0$ is the initial estimation of $\alpha$. Furthermore, the computational order of convergence for all the iterative methods mentioned is displayed, where the computational order of convergence ( $COC$ ) can be approximated by using the formula as follows (Weerakoon and Fernando, 2000):

$$COC \approx \frac{ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

The following test problems are used to demonstrate the performance of the new developed iterative methods and compare with other existing eighth- order iterative methods.

Table 1: Comparison of various iterative methods with $n = 3$

| Method | $\|f(x_n)\|$ | $\|x_n - \alpha\|$ | $COC$ |
|---|---|---|---|
| $f_1(x)$, $x_0 = 0.25$ | | | |
| $SQM1$ | 0.214e-338 | 0.107e-338 | 7.000 |
| $SQM2$ | 0.335e-522 | 0.167e-522 | 8.000 |
| $SQM3$ | 0.670e-515 | 0.335e-515 | 8.000 |
| $KM1$ | 0.107e-461 | 0.535e-462 | 7.972 |
| $KM2$ | 0.426e-451 | 0.213e-451 | 8.029 |
| $NAM$ | 0.412e-497 | 0.206e-497 | 8.000 |
| $SM1$ | 0.933e-497 | 0.467e-497 | 8.000 |
| $SM2$ | 0.479e-441 | 0.240e-441 | 8.000 |
| $f_2(x)$, $x_0 = 15.5$ | | | |
| $SQM1$ | 0.132e-246 | 0.769e-246 | 7.000 |
| $SQM2$ | 0.321e-548 | 0.187e-547 | 8.000 |
| $SQM3$ | 0.546e-365 | 0.318e-364 | 8.000 |
| $KM1$ | 0.120e-491 | 0.696e-491 | 7.930 |
| $KM2$ | 0.100e-414 | 0.584e-414 | 7.834 |
| $NAM$ | 0.437e-407 | 0.254e-406 | 8.000 |
| $SM1$ | 0.359e-518 | 0.209e-517 | 8.000 |
| $SM2$ | 0.644e-386 | 0.375e-385 | 8.000 |
| $f_3(x)$, $x_0 = 11.9$ | | | |
| $SQM1$ | 0.244e-258 | 0.832e-258 | 7.000 |
| $SQM2$ | 0.150e-511 | 0.510e-511 | 8.000 |
| $SQM3$ | 0.625e-385 | 0.213e-384 | 8.000 |
| $KM1$ | 0.699e-482 | 0.238e-481 | 7.915 |
| $KM2$ | 0.486e-501 | 0.165e-500 | 7.877 |
| $NAM$ | 0.332e-447 | 0.113e-446 | 8.000 |
| $SM1$ | 0.392e-491 | 0.134e-490 | 8.000 |
| $SM2$ | 0.472e-433 | 0.161e-432 | 8.000 |
| $f_4(x)$, $x_0 = 0.5$ | | | |
| $SQM1$ | 0.286e-515 | 0.270e-515 | 7.000 |
| $SQM2$ | 0.234e-796 | 0.221e-796 | 8.000 |

| | | | |
|---|---|---|---|
| $SQM3$ | 0.186e-774 | 0.175e-774 | 8.000 |
| $KM1$ | 0.298e-777 | 0.282e-777 | 7.998 |
| $KM2$ | 0.103e-724 | 0.974e-725 | 7.986 |
| $NAM$ | 0.468e-803 | 0.442e-803 | 8.000 |
| $SM1$ | 0.839e-775 | 0.793e-775 | 8.000 |
| $SM2$ | 0.488e-745 | 0.461e-745 | 8.000 |
| $f_5(x)$, $x_0 = 1.6$ | | | |
| $SQM1$ | 0.369e-474 | 0.319e-474 | 7.000 |
| $SQM2$ | 0.660e-700 | 0.570e-700 | 8.000 |
| $SQM3$ | 0.294e-738 | 0.254e-738 | 8.000 |
| $KM1$ | 0.281e-674 | 0.242e-674 | 7.970 |
| $KM2$ | 0.445e-660 | 0.384e-660 | 7.973 |
| $NAM$ | 0.135e-822 | 0.117e-822 | 8.000 |
| $SM1$ | 0.109e-712 | 0.939e-713 | 8.000 |
| $SM2$ | 0.105e-749 | 0.908e-750 | 8.000 |
| $f_6(x)$, $x_0 = 0.6$ | | | |
| $SQM1$ | 0.185e-448 | 0.110e-448 | 7.000 |
| $SQM2$ | 0.181e-635 | 0.108e-635 | 8.000 |
| $SQM3$ | 0.413e-675 | 0.247e-675 | 8.000 |
| $KM1$ | 0.369e-650 | 0.221e-650 | 8.018 |
| $KM2$ | 0.213e-645 | 0.127e-645 | 8.015 |
| $NAM$ | 0.401e-730 | 0.240e-730 | 8.000 |
| $SM1$ | 0.950e-673 | 0.568e-673 | 8.000 |
| $SM2$ | 0.309e-747 | 0.184e-747 | 8.000 |

**Acknowledgements:**
We gratefully thank Dr. Siyyam, Hani I. for the positive comments suggested on the first version of this report.

**Corresponding Author:**
Dr. I.A. Al-Subaihi
Department of Mahematics,
College of Science,
Taibah University, KSA.
Mobile: +966-555327731
E-mail:alsubaihi@hotmail.com

**References**

1. Chun C. A family of composite fourth-order iterative methods for solving nonlinear equations. Appl. Math. Comput 2007;187:951-956.
2. Chun C. Construction of Newton-like iteration methods for solving nonlinear equations. Numer. Math. 2006;104:297-315.
3. Chun C. Iterative method improving Newton's method by the decomposition method. Computers Math. Applic. 2005;50:1559-1568.
4. Cordero A, Hueso JL, Martinez E, Torregrosa JR. A Family of Iterative Methods with Sixth and Seventh Order Convergence for Nonlinear Equations. Math. Comput. Model. 2010;52:1490-1496.
5. Ezzati R, Saleki F. On the Construction of New Iterative Methods with Fourth-Order Convergence by Combining Previous Methods. Int. Math. Forum. 2011;6:1319-1326.
6. Sharma JR, Guha KR, Sharma R. Improved Ostrowski-Like methods based on cubic curve interpolation. Appl. Math. 2011;2:618-823.
7. Sharma JR, Sharma R. A New Family of Modified Ostrowski's Methods with Accelerated Eighth Order Convergence. Numer. Algor. 2010;54:445-458.
8. Siyyam HI, Al-Subaihi IA, Shatnawi M. A Note on Some Higher-Order Iterative Methods Free from Second Derivative for Solving Nonlinear Equations. Int. Journal of Math. Analysis. 2011;5:2337-2347.
9. Siyyam HI, Shatnawi M, Al-Subaihi IA. A New One Parameter Family of Iterative Methods with Eighth-Order of Convergence for Solving Nonlinear Equations. Accepted for publication in International Journal of Pure and Appl. Math. 2011.
10. Soleymani F,Vanani SK, Siyyam HI, Al-Subaihi IA. Numerical solution of nonlinear equations by an optimal eighth-order class of iterative methods. ANN UNIV FERRARA . 2012;59:159-171.
11. Traub JF. Iteraive Methods for the Solution of Equations. Chelsea publishing company. 1982 (in New York) .
12. Weerakoon S, Fernando TGI. A Variant of Newton's Method with Accelerated Third-Order Convergence. Appl. Math. Lett. 2000;13:87-93.

7/19/2014