View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by CiteSeerX

Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 179 (2006) 370–377
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Two numerical methods for solving a backward heat conduction problem

Xiang-Tuan Xiong, Chu-Li Fu *, Zhi Qian

Department of Mathematics, Lanzhou University, Lanzhou 730000, People's Republic of China

Abstract

We introduce a central difference method and a quasi-reversibility method for solving a backward heat conduction problem (BHCP) numerically. For these two numerical methods, we give the stability analysis. Meanwhile, we investigate the roles of regularization parameters in these two methods. Numerical results show that our algorithm is effective.
© 2005 Elsevier Inc. All rights reserved.

Keywords: Inverse problem; Backward heat conduction; Central difference; Quasi-reversibility; Regularization

1. Introduction

The backward heat conduction problem (BHCP) is also referred to as final boundary value problem. In general, no solution which satisfies the heat conduction equation with final data and the boundary conditions exists. Even if a solution exists, it will not be continuously dependent on the final data. The BHCP is a typical example of an ill-posed problem which is unstable by numerical methods and requires special regularization methods. In the context of approximation method for this problem, many approaches have been investigated. Such authors as Lattes and Lions [1], Showalter [2], Ames et al. [3], Miller [4] have approximated the BHCP by quasi-reversibility methods. Schröter and Tautenhahn [5] established an optimal error estimate for a special BHCP. Mera and Jourhmane used many numerical methods with regularization techniques to approximate the problem in [6–8], etc. A mollification method has been studied by Hào in [9]. Kirkup and Wadsworth used an operator-splitting method in [10]. A difference approximation method for solving sideways heat equation was provided by Eldén in [11]. So far in the literature (cf. [3,12] and the references therein), most of the authors used the eigenfunctions and eigenvalues to reconstruct the solution of the BHCP by many quasi-reversibility methods numerically, However, the eigenfunctions and eigenvalues are in general not available and the labor needed to compute these and the corresponding Fourier coefficients is very onerous. In this paper, we use two regularization methods to solve the BHCP in one-dimensional setting numerically, but these methods can be generalized to two-dimensional case.

* Corresponding author.
E-mail addresses: xiongxt04@st.lzu.edu.cn (X.-T. Xiong), fuchuli@lzu.edu.cn (C.-L. Fu).

0096-3003/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2005.11.114

The paper is organized as follows. In the forthcoming section, we will present the mathematical problem on a BHCP; in Section 3, we review two regularization methods, one is central difference regularization method, the other is a special quasi-reversibility regularization method; in Section 4, some finite difference schemes are constructed for the inverse problem and the numerical stability analysis is provided; in Section 5, numerical examples are tested to verify the effect of the numerical schemes.

2. Mathematical problem

2.1. The direct problem

We consider the following heat equation:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), && -\pi < x < \pi,\ 0 < t < T,\\ &u(\pi,t) = s(t), && 0 < t < T,\\ &u(-\pi,t) = l(t), && 0 < t < T,\\ &u(x,0) = f(x), && -\pi < x < \pi. \end{aligned} \tag{2.1}$$

Solving the equation with given $s(t)$; $l(t)$ and $f(x)$ is called a direct problem. From the theory of heat equation, we can see that for $s(t)$; $l(t)$ and $f(x)$ in some function space there exists a unique solution [13].

2.2. The inverse problem

Consider the following problem:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), && -\pi < x < \pi,\ 0 < t < T,\\ &u(\pi,t) = s(t), && 0 < t < T,\\ &u(-\pi,t) = l(t), && 0 < t < T,\\ &u(x,T) = g(x), && -\pi < x < \pi. \end{aligned} \tag{2.2}$$

The inverse problem is to determine the value of $u(x, t)$ for $0 \leqslant t < T$ from the data $s(t)$; $l(t)$ and $g(x)$. If the solution exists, then the problem has a unique solution (cf. [14, p. 64]).

The data $g(x)$ are based on (physical) observations and are not known with complete accuracy, due to the ill-posedness of the BHCP, a small error in the data $g(x)$ can cause an arbitrarily large error in the solution $u(x, t)$. Now we want to reconstruct the temperature distribution $u(x, t)$ for $0 \leqslant t < T$ by two different regularization methods.

3. Central difference regularization

By using the central difference with step length h to approximate the second derivative u_{xx}, we can get the following problem:

$$\begin{aligned} &u_t(x,t) = \tfrac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}, && -\pi < x < \pi,\ 0 < t < T,\\ &u(\pi,t) = s(t), && 0 < t < T,\\ &u(-\pi,t) = l(t), && 0 < t < T,\\ &u(x,T) = g(x), && -\pi < x < \pi. \end{aligned} \tag{3.1}$$

Let $\tilde{t} = T - t,\ w(x,\tilde{t}) = u(x,t)$, we have

$$w_{\tilde{t}} = -w_{xx}, \qquad -\pi < x < \pi, \quad 0 < \tilde{t} < T. \tag{3.2}$$

If w_{xx} at x_i is replaced by a second-order central difference, then (3.2) becomes

$$w_{\tilde{t}}(x_i,\tilde{t}) = -\frac{1}{h^2}[w(x_i+h,\tilde{t}) - 2w(x_i,\tilde{t}) + w(x_i-h,\tilde{t})]. \tag{3.3}$$

Let $x_i = -\pi + (i-1)h$, for $i = 1, \ldots, 2n+1$. $h = \frac{\pi}{n}, w_i = w_i(\tilde{t}) = w(-\pi + (i-1)h, \tilde{t})$. Then by the boundary condition of (3.1) there hold $w_1(\tilde{t}) = l(\tilde{t})$ and $w_{2n+1}(\tilde{t}) = s(\tilde{t})$. Meanwhile, Eq. (3.3) with initial boundary conditions can be discretized as

$$\begin{pmatrix} w_2(\tilde{t}) \\ \vdots \\ w_i(\tilde{t}) \\ \vdots \\ w_{2n}(\tilde{t}) \end{pmatrix}_{\tilde{t}} = \underbrace{\begin{pmatrix} \frac{2}{h^2} & -\frac{1}{h^2} & & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & \\ & \ddots & \ddots & \ddots \\ 0 & & -\frac{1}{h^2} & \frac{2}{h^2} \end{pmatrix}}_{A_{(2n-1)\times(2n-1)}} \begin{pmatrix} w_2(\tilde{t}) \\ \vdots \\ w_i(\tilde{t}) \\ \vdots \\ w_{2n}(\tilde{t}) \end{pmatrix} - \frac{1}{h^2} \underbrace{\begin{pmatrix} w_1(\tilde{t}) \\ \vdots \\ 0 \\ \vdots \\ w_{2n+1}(\tilde{t}) \end{pmatrix}}_{2n-1} \tag{3.4}$$

$$\begin{pmatrix} w_2(0) \\ \vdots \\ w_i(0) \\ \vdots \\ w_{2n}(0) \end{pmatrix} = \begin{pmatrix} g(x_2) \\ \vdots \\ g(x_i) \\ \vdots \\ g(x_{2n}) \end{pmatrix}. \tag{3.5}$$

This is an ordinary differential system and there are many numerical methods such as Euler method, Runge–Kutta method for the system. But we find that the eigenvalues of A are $\lambda_k = \frac{4}{h^2}\sin^2\frac{k\pi}{2(2n-1)} \geqslant 0$, $k \in Z$ (the positive integer set). Therefore, the numerical method for (3.4) and (3.5) is unstable. However, via the variable transformation [15]

$$v_i(\tilde{t}) = e^{-a\tilde{t}} w_i(\tilde{t}), \quad i = 1, \ldots, 2n+1, \tag{3.6}$$

where $a > 0$ is a contraction factor to be determined such that (3.3) in terms of $v_i(\tilde{t})$ is changed to

$$\frac{\partial v_i(\tilde{t})}{\partial \tilde{t}} = \left(\frac{2}{h^2} - a\right) v_i(\tilde{t}) - \frac{1}{h^2}[v_{i+1}(\tilde{t}) + v_{i-1}(\tilde{t})], \tag{3.7}$$

similarly, we have

$$\begin{pmatrix} v_2(\tilde{t}) \\ \vdots \\ v_i(\tilde{t}) \\ \vdots \\ v_{2n}(\tilde{t}) \end{pmatrix}_{\tilde{t}} = \underbrace{\begin{pmatrix} \frac{2}{h^2} - a & -\frac{1}{h^2} & & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} - a & -\frac{1}{h^2} & \\ & \ddots & \ddots & \ddots \\ 0 & & -\frac{1}{h^2} & \frac{2}{h^2} - a \end{pmatrix}}_{B_{(2n-1)\times(2n-1)}} \begin{pmatrix} v_2(\tilde{t}) \\ \vdots \\ v_i(\tilde{t}) \\ \vdots \\ v_{2n}(\tilde{t}) \end{pmatrix} - \frac{1}{h^2} \begin{pmatrix} e^{-a\tilde{t}} l(\tilde{t}) \\ \vdots \\ 0 \\ \vdots \\ e^{-a\tilde{t}} s(\tilde{t}) \end{pmatrix} \tag{3.8}$$

$$\begin{pmatrix} v_2(0) \\ \vdots \\ v_i(0) \\ \vdots \\ v_{2n}(0) \end{pmatrix} = \begin{pmatrix} g(x_2) \\ \vdots \\ g(x_i) \\ \vdots \\ g(x_{2n}) \end{pmatrix}. \tag{3.9}$$

We choose $a = b/h^2 \geqslant \frac{2}{h^2}$ such that the eigenvalues of matrix B are negative, thus the numerical methods such as Runge–Kutta method for (3.8) and (3.9) are stable. After $v_i(\tilde{t})$ is obtained numerically, we can obtain $w_i(\tilde{t}) = e^{a\tilde{t}} v_i(\tilde{t})$, furthermore $w(ih, \tilde{t}) \approx u(ih, T - \tilde{t}) = u_i(t)$.

In the central difference regularization method, the space step length h plays the role of regularization parameter. According to the general regularization theory, the h should be neither too small nor too large.

By our experience, the parameter h is selected as $h = 2\left(\ln\left(\frac{1}{\delta}\right)\right)^{-1/2}$, where δ is the noise level of the data $g(x)$. We will see this choice rule is very effective in the subsequent numerical tests.

4. Quasi-reversibility regularization method

The initial boundary value problem (2.2) is replaced by the following problem:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + \epsilon u_{xxt}(x,t), && -\pi < x < \pi,\ 0 < t < T,\\ &u(\pi,t) = s(t), && 0 < t < T,\\ &u(-\pi,t) = l(t), && 0 < t < T,\\ &u(x,T) = g(x), && -\pi < x < \pi, \end{aligned} \tag{4.1}$$

where ϵ is a small positive parameter. For ϵ sufficiently small the solution of (4.1) approximates the solution (if it exists) of (2.1) in some sense. This is one of well-known quasi-reversibility methods [1]. For the above mentioned problem, Ewing [12] has presented a choice rule of the regularization parameter ϵ, i.e., $\epsilon = \left(\ln\left(\frac{1}{\delta}\right)\right)^{-1}$, where δ denotes the noise level of the data $g(x)$, and the error estimate between the approximate solution and the exact solution is given in $L^2(R)$-norm.

Similarly, we take $\tilde{t} = T - t$, then problem (4.1) becomes

$$\begin{aligned} &u_{\tilde{t}}(x,\tilde{t}) + u_{xx}(x,\tilde{t}) - \epsilon u_{xx\tilde{t}}(x,\tilde{t}) = 0, && -\pi < x < \pi, 0 < \tilde{t} < T,\\ &u(\pi,\tilde{t}) = s(\tilde{t}), && 0 < \tilde{t} < T,\\ &u(-\pi,\tilde{t}) = l(\tilde{t}), && 0 < \tilde{t} < T,\\ &u(x,0) = g(x), && -\pi < x < \pi. \end{aligned} \tag{4.2}$$

The problem has a unique solution if a solution exists. Now we prove it for two-dimensional case.

Theorem 1. *There exists a unique solution* (*if it exists*) *for the problem*:

$$\begin{aligned} &u_{\tilde{t}}(x,\tilde{t}) + \Delta u(x,\tilde{t}) - \epsilon \Delta u_{\tilde{t}}(x,\tilde{t}) = 0, && D \times (0,T),\\ &u(x,\tilde{t}) = h(x,\tilde{t}), && \text{on } \partial D \times (0,T),\\ &u(x,0) = g(x), && \text{in } D, \end{aligned} \tag{4.3}$$

where D is a bounded subset in R^2, Δ is the Laplace operator, $\epsilon > 0$.

Proof. We only need to prove the following problem has the zero solution:

$$\begin{aligned} &w_{\tilde{t}}(x,\tilde{t}) + \Delta w(x,\tilde{t}) - \epsilon \Delta w_{\tilde{t}}(x,\tilde{t}) = 0, && D \times (0,T),\\ &w(x,\tilde{t}) = 0, && \text{on } \partial D \times (0,T),\\ &w(x,0) = 0, && \text{in } D. \end{aligned} \tag{4.4}$$

Set

$$\varphi(\tilde{t}) = \int_D w^2(\tilde{t}) + \epsilon |\nabla w(\tilde{t})|^2 \, dx,$$

then

$$\frac{d\varphi(\tilde{t})}{d\tilde{t}} = 2\left(\int_D w w_{\tilde{t}} + \epsilon \nabla w \cdot \nabla w_{\tilde{t}} \, dx\right).$$

Due to the Green's second formula, we have

$$\begin{aligned} \frac{d\varphi(\tilde{t})}{d\tilde{t}} &= 2\left(\int_D w w_{\tilde{t}} - \epsilon w \Delta w_{\tilde{t}} \, dx\right) = 2\left(\int_D w(w_{\tilde{t}} - \epsilon \Delta w_{\tilde{t}}) \, dx\right) = 2\left(\int_D w(-\Delta w) \, dx\right) = 2\left(\int_D |\nabla w|^2 \, dx\right)\\ &\leqslant \frac{2}{\epsilon}\left(\int_D w^2(\tilde{t}) + \epsilon |\nabla w(\tilde{t})|^2 \, dx\right) = \frac{2}{\epsilon}\varphi(\tilde{t}). \end{aligned}$$

Therefore, we have

$$\varphi(\tilde{t}) \leqslant \varphi(0)e^{\frac{2}{\epsilon}\tilde{t}}. \tag{4.5}$$

since $\varphi(0) = 0$, there holds $\varphi(\tilde{t}) \leqslant 0$. Hence $w = 0$. □

Now we construct the finite difference schemes for solving problem (4.2), let $x_i = -\pi + (i-1)h$, $i = 1, \ldots, 2n+1$, $\tilde{t}_j = (j-1)\tau$, $j = 1, \ldots, m+1$, where $h = \frac{\pi}{n}$, $\tau = \frac{T}{m}$. Let $u_i^j = u(x_i, \tilde{t}_j)$ represent the value of the numerical solution of (4.2) at the mesh point $(x_i,\ \tilde{t}_j)$, then Eq. (4.2) is discretized as

$$-ru_{i+1}^{j+1} + \left(2r + \frac{1}{\tau}\right)u_i^{j+1} - ru_{i-1}^{j+1} = -\left(\frac{1}{h^2} + r\right)u_{i+1}^j + \left(\frac{2}{h^2} + 2r + \frac{1}{\tau}\right)u_i^j - \left(\frac{1}{h^2} + r\right)u_{i-1}^j, \tag{4.6}$$

where $r = \frac{\epsilon}{h^2\tau}$, $i = 2, \ldots, 2n$; $j = 1, \ldots, m$.

Now we discuss the stability of difference schemes (4.6) by verifying the *Von Neumann condition*. The propagation factor can be found

$$G(\sigma, \tau) = \frac{\frac{1}{\tau} + 4\left(\frac{1}{h^2} + r\right)\sin^2\frac{\sigma h}{2}}{\frac{1}{\tau} + 4r\sin^2\frac{\sigma h}{2}}.$$

It is easy to verify the fact that the *Von Neumann condition*

$$|G(\sigma, \tau)| \leqslant 1 + c\tau$$

holds with $c = \frac{1}{\epsilon}$. Hence, the numerical algorithm (4.6) is stable.

5. Numerical examples

For convenience, we take $s(t) = l(t) = 0$ in (2.2).

Example 1. We consider the following direct problem:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad -\pi < x < \pi,\ 0 < t < 1,\\ &u(\pi,t) = 0, \qquad 0 < t < 1,\\ &u(-\pi,t) = 0, \qquad 0 < t < 1 \end{aligned} \tag{5.1}$$

with the initial condition:

$$u(x,0) = \begin{cases} \pi + x, & -\pi \leqslant x \leqslant 0,\\ \pi - x, & 0 \leqslant x \leqslant \pi. \end{cases}$$

Let $x_1 = -\pi$, $x_i = -\pi + (i-1)h$, $i = 2, \ldots, 2n$; $x_{2n+1} = 0$, $t_j = (j-1)\tau$, $j = 1, \ldots, m+1$, the space step length $h = \frac{\pi}{n}$ and the time step length $\tau = \frac{1}{m}$, then we solve this problem by an explicit difference scheme in the following form:

$$\begin{aligned} &u_i^{j+1} = ru_{i+1}^j + (1-2r)u_i^j + ru_{i-1}^j, \quad j = 1, \ldots, m,\ i = 2, \ldots, 2n\\ &u(x_{2n+1}, t_j) = u_{2n+1}^j = 0, \qquad j = 1, \ldots, m+1,\\ &u(x_1, t_j) = u_1^j = 0, \qquad j = 1, \ldots, m+1 \end{aligned} \tag{5.2}$$

with

$$u(x_i, t_1 = 0) = \begin{cases} \pi + x_i, & -\pi \leqslant x_i \leqslant 0,\\ \pi - x_i, & 0 \leqslant x_i \leqslant \pi, \end{cases}$$

where $r = \frac{\tau}{h^2}$, and it requires $r < \frac{1}{2}$ for numerical stability reasons.

The numerical result for $g(x) = u(x, T = 1)$ is shown in Fig. 1, where $n = 11$, $m = 50$.

Now we solve the inverse problem by the $g(x)$ generated numerically by the direct problem via the two different regularization methods. We choose to restore the solution $f(x)$ at $t = 0$. We denote the numerical result

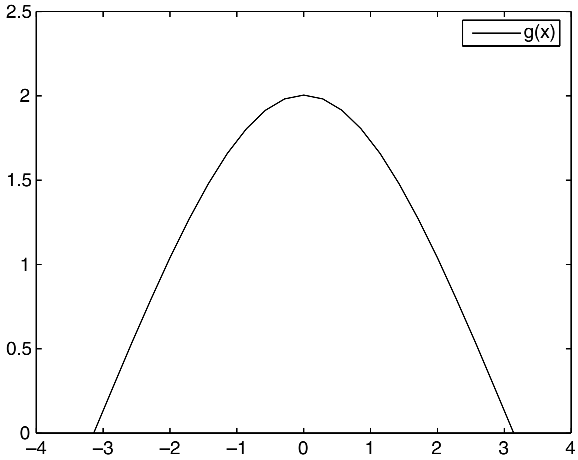

Fig. 1. $g(x)$ computed by (5.2).

of the inverse problem as $f^{\bigstar}(x)$. If we introduce random noises ε to data $g(x)$, i.e., $g_\varepsilon(x_i) = g(x_i) + \varepsilon\,\mathrm{rand}(i)$, where $\mathrm{rand}(i)$ is a random number between $[-1, 1]$, then the total noise δ can be measured in the sense of root mean square (RMS) error according to

$$\delta := \sqrt{\frac{1}{2n-1}\sum_{i=1}^{2n-1}(g_\varepsilon(x_i) - g(x_i))^2}. \tag{5.3}$$

In the computation, we choose the regularization parameters $h = 2\left(\ln\left(\frac{1}{\delta}\right)\right)^{-\frac{1}{2}}$ for the central difference regularization method and $\epsilon = \left(\ln\left(\frac{1}{\delta}\right)\right)^{-1}$ for the quasi-reversibility method, respectively. The numerical results are shown in Fig. 2 (quasi-reversibility), Fig. 3(central difference) with $\delta = 0.001$.

Example 2. Consider the problem

$$\begin{aligned} &u_t = u_{xx}, && -\pi < x < \pi,\ 0 < t < 1,\\ &u(-\pi, t) = u(\pi, t) = 0, && 0 < t < 1,\\ &u(x, 1) = \mathrm{e}^{-1}\sin x, && -\pi < x < \pi. \end{aligned} \tag{5.4}$$

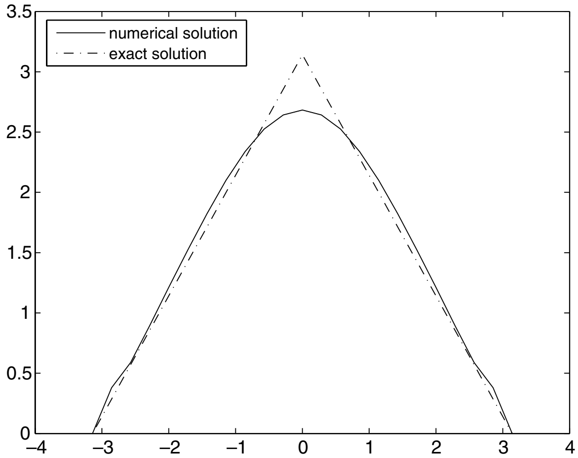

Fig. 2. $f^{\bigstar}$ and f, $m = 50$, $n = 11$, $\epsilon = 0.145$.

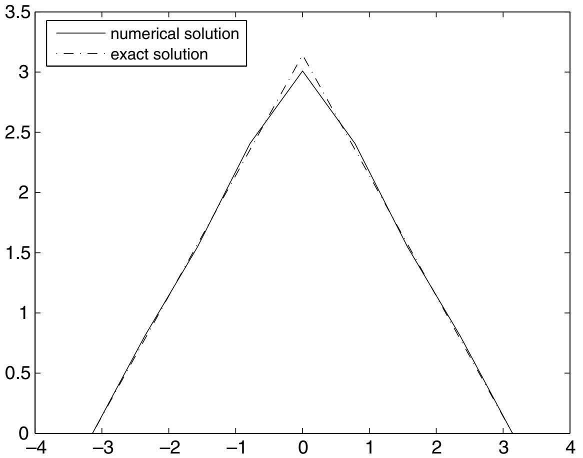

Fig. 3. $f^{\star}$ and f, $b = 2$, $m = 50$, $n = 4$, $h = \pi/4$.

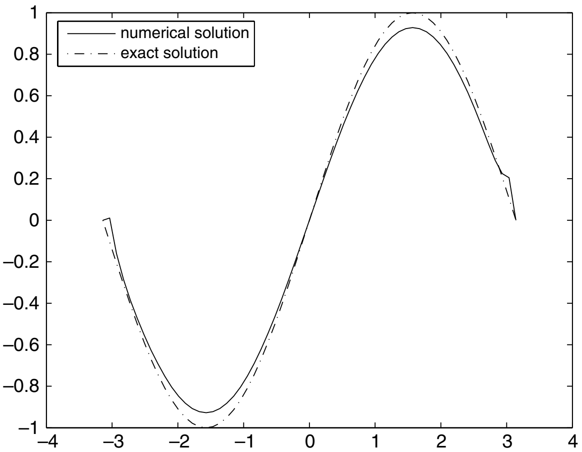

Fig. 4. $f^{\star}$ and f, $m = 21$, $n = 30$, $\epsilon = 0.109$.

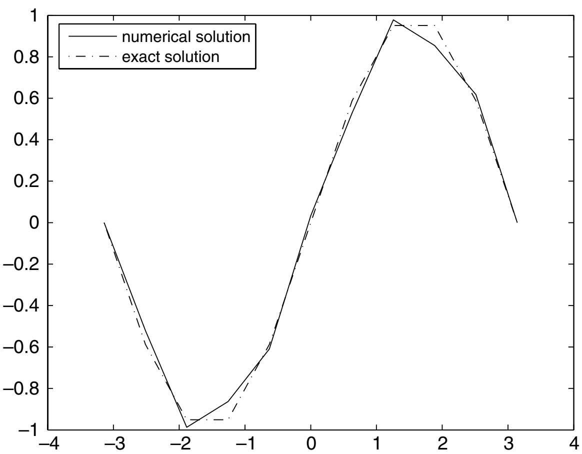

Fig. 5. $f^{\star}$ and f, $b = 3$, $m = 51$, $n = 5$, $h = \pi/5$.

The exact solution is

$$u(x,t) = \mathrm{e}^{-t} \sin x. \tag{5.5}$$

The numerical result for the inverse problem is also denoted as $f^{\bigstar}(x)$. See Fig. 4 (quasi-reversibility) and Fig. 5 (central difference), where $\delta = 0.0001$.

From Examples 1 and 2, we conclude that the choice rules of the regularization parameters h and ϵ are very effective. By our numerical experiments, we can see that the accuracy of the numerical results increases with the decreasing T; at the same time, the numerical solutions of the quasi-reversibility method depend on the parameter ϵ continuously. This accords with the theoretical result (cf. [16]). Here, we will not give the numerical results.

6. Conclusions

In this paper, we discussed two regularization methods for the one-dimensional backward heat conduction problem. We presented two algorithms for the inverse problem. Numerical results show that these method are effective with two appropriately chosen regularization parameters. The algorithm for two-dimensional case is to be considered.

Acknowledgements

The work is supported by the NNSF of China (Nos. 10271050 and 10571079) and the NSF of Gansu Province of China (No. 3ZS051-A25-015).

References

[1] R. Lattes, J.L. Lions, Methode de Quasi-Reversibility et Applications, Dunod, Paris, 1967 (English Transl: R. Bellman, Elsevier, New York, 1969).
[2] R.E. Showalter, The final value problem for evolution equations, J. Math. Anal. Appl. 47 (1974) 563–572.
[3] K.A. Ames, W.C. Gordon, J.F. Epperson, S.F. Oppenhermer, A comparison of regularizations for an ill-posed problem, Math. Comput. 67 (1998) 1451–1471.
[4] K. Miller, Stabilized quasireversibility and other nearly best possible methods for non-well-posed problems, in: Symposium on Non-Well-Posed Problems and Logarithmic Convexity, Lecture Notes in Mathematics, vol. 316, Springer-Verlag, Berlin, 1973, pp. 161–176.
[5] T. Schröter, U. Tautenhahn, On optimal regularization methods for the backward heat equation, Z. Anal. Anw. 15 (1996) 475–493.
[6] N.S. Mera, L. Elliott, D.B. Ingham, D. Lesnic, An iterative boundary element method for solving the one dimensional backward heat conduction problem, Int. J. Heat Mass Transf. 44 (2001) 1973–1946.
[7] M. Jourhmane, N.S. Mera, An iterative algorithm for the backward heat conduction problem based on variable relaxtion factors, Inverse Probl. Eng. 10 (2002) 293–308.
[8] N.S. Mera, The method of fundamental solutions for the backward heat conduction problem, Inverse Probl. Sci. Eng. 13 (1) (2005) 79–98.
[9] D.N. Hào, A mollification method for ill-posed problems, Numer. Math. 68 (1994) 469–506.
[10] S.M. Kirkup, M. Wadsworth, Solution of inverse diffusion problems by operator-splitting methods, Appl. Math. Model. 26 (10) (2002) 1003–1018.
[11] L. Eldén, Numerical solution of the sideways heat equation by difference approximation in time, Inverse Probl. 11 (1995) 913–923.
[12] R.E. Ewing, The approximation of certain parabolic equations backward in time by Sobolev equations, SIAM J. Math. Anal. 6 (1975) 283–294.
[13] J.R. Cannon, The One-Dimensional Heat Equation, Addison-Wesley Publishing Company, California, 1984.
[14] L.C. Evans, Partial Differential Equations, American Mathematical Society, Providence, Rhode Island, 1998.
[15] C.S. Liu, Group preserving scheme for backward heat conduction problems, Int. J. Heat Mass Transf. 47 (2004) 2567–2576.
[16] K.A. Ames, L.E. Payne, Continuous dependence on modeling for some well-posed perturbations of the backward heat equation, J. Inequal. Appl. 3 (1999) 51–64.