Conifold Geometries, Matrix Models and Quantum Solutions

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Abstract. Open topological B-models describing D-branes on 2-cycles of local Calabi–Yau geometries with conical singularities are reviewed. The paper expands in particular on two aspects: the gauge fixing problem in the reduction to two dimensions and the quantum matrix model solutions.

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1 Introduction

Singular Calabi–Yau manifolds represent one of the most interesting developments in string compactifications. For instance, the presence of a conifold point, [1], in a Calabi-Yau opens new prospects: in conjunction with fluxes and branes it may allow for warped compactifications, which in turn may create the conditions for moduli stabilization and for large hierarchies of physical scales. On the other hand singular Calabi-Yau compactifications with conical singularities seem to realize favorable conditions for low energy theory models with realistic cosmological features. A conifold singularity can be smoothed in two different ways, by means of a 2-sphere (resolution) or a 3–sphere (deformation). This leads, from a physical point of view, to a geometric transition that establishes a duality relation between theories defined by the two nonsingular geometries (gauge–gravity or open–closed string duality), [2, 3]. In summary, conifold singularities are at the crossroads of many interesting recent developments in string theory. It is therefore important to study theories defined on conifolds, *i.e.* on singular non compact Calabi-Yau threefolds, as well–defined and (partially) calculable models to approximate more realistic situations.

In [4], building on previous literature, we started to elaborate on an idea that is receiving increasing attention: how data describing the geometry of a local Calabi–Yau can be encoded, *via* a topological field theory, in a (multi–)matrix

model and how they can be efficiently calculated. The framework we considered was IIB string theory with spacetime filling D5–branes wrapped around resolving two–dimensional cycles. This geometry defines a 4D gauge theory, [5–7,20]. On the other hand one can consider the open topological B model describing strings on the conifold. The latter has been shown long ago by Witten to be represented by a six-dimensional holomorphic Chern–Simons theory, [11]. When reduced to a two-dimensional cycle this theory can be shown to reduce to a matrix model. In particular, if one wishes to represent a wide class of deformations of the complex structure satisfying the Calabi–Yau condition, one must resort to very general multi–matrix models, [9]. In [4] we concentrated on the topological string theory part of the story, [12–15, 17, 18], in particular on the formal aspects of the reduction from the six-dimensional holomorphic Chern–Simons theory to a two–dimensional field theory and on the analysis of the matrix model potentials originated from the Calabi-Yau complex structure deformations. Finally we concentrated on the subclass of matrix models represented by two–matrix models with bilinear coupling (for a very recent development see [16]). In this case the functional integral can be explicitly calculated with the method of orthogonal polynomials. Using old results we showed how one can find explicit solutions by solving the *quantum equations of motion* and utilizing the recursiveness granted by integrability. All the data turn out to be encoded in a Riemann surface, which we called *quantum Riemann surface* in order to distinguish it from the Riemann surface of the standard saddle point approach.

In this paper we would like to return to some topics that were only partially developed in [4]. In particular, in section 2, after a concise review of the reduction to from CS theory to matrix models, we explain in detail the gauge fixing problem in this process. Subsequently we return to the problem of solving two–matrix models with bilinear couplings by means of the orthogonal polynomials method *via* the solution of the quantum equation of motion. After a short review of the method in section 3, our main purpose in section 4 is to clarify the similarities and differences with other methods, in particular with the semiclassical saddle point method. We discuss at length the result of [4] that the quantum equations of motion admit in general more vacua than the saddle point method. We interpret these additional solutions as 'quantum' cycles that have no classical analog. Finally in section 5 we give a simple, explicit example of topological open string expansion.

2   Reduction to the Brane

In this section we summarize the reduction of the topological open string field theory (B model) to a holomorphic 2-cycle in a local Calabi-Yau threefold [4]. Let us consider a holomorphic $\mathbb{C}^2-$fibre bundle $X \rightarrow \Sigma$, where Σ is a Riemann surface. The space X is obtained as a deformation of the complex structure of the total space of a rank-2 vector bundle V on Σ. Given an atlas $\{U_\alpha\}$ on Σ, the

transition functions for X can be written

$$
\begin{aligned}
z_{(\alpha)} &= f_{(\alpha\beta)}(z_{(\beta)}) \\
\omega_{(\alpha)}^i &= M_{j(\alpha\beta)}^i\left(z_{(\beta)}\right)\left[\omega_{(\beta)}^j + \Psi_{(\alpha\beta)}^j\left(z_{(\beta)}, \omega_{(\beta)}\right)\right], \quad i,j = 1,2
\end{aligned}
$$

where $f_{(\alpha\beta)}$ are the transition functions on the base, $M_{j(\alpha\beta)}^i$ the transition functions of the vector bundle V and $\Psi_{(\alpha\beta)}^j$ are the deformation terms, holomorphic on the intersections $(U_\alpha \cap U_\beta) \times \mathbb{C}^2$.

The Calabi-Yau condition on the space X, *i.e.* the existence of a nowhere vanishing holomorphic top-form $\Omega = \mathrm{d}z \wedge \mathrm{d}w^1 \wedge \mathrm{d}w^2$, puts conditions on the vector bundle and on the deformation terms. The determinant of the vector bundle has to be equal to the canonical line bundle on Σ and for the transition functions this means $\det M_{(\alpha\beta)} \times f'_{(\alpha\beta)} = 1$. For the deformation terms we have $\det\left(1 + \partial\Psi\right) = 1$, where $(1 + \partial\Psi)_j^i = \delta_j^i + \partial_j\Psi^i$. The solution of this condition can be given in terms of a set of potential functions $X_{(\alpha\beta)}$, the *geometric potential*, which generates the deformation *via*

$$
\epsilon_{ij}w_{(\alpha\beta)}^i\mathrm{d}w_{(\alpha\beta)}^j = \epsilon_{ij}\omega_{(\alpha)}^i\mathrm{d}\omega_{(\alpha)}^j - \mathrm{d}X_{(\alpha\beta)},
$$

where we define the *singular coordinates* $w_{(\alpha\beta)}^i = \omega_{(\alpha)}^i + \Psi_{(\alpha\beta)}^i(z^{(\beta)}, \omega_{(\beta)})$.

The topological open B-model on X can be obtained from open string field theory and reduces [11] to the holomorphic Chern-Simons (hCS) theory on X for a (0,1)-form connection on a $U(N)$ bundle E, where N is the number of space-filling B-branes. We will restrict ourselves to the case in which E is trivial. The action of hCS is

$$
S(\mathcal{A}) = \frac{1}{g_s}\int_X \mathcal{L}, \qquad \mathcal{L} = \Omega \wedge Tr\left(\frac{1}{2}\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{1}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right) \quad (2.1)
$$

where $\mathcal{A} \in T^{(0,1)}(X)$. The dynamics of B-branes on a 2-cycle $\Sigma \subset X$ can be described by reducing the open string field theory from the space X to the B-brane world-volume Σ.

To obtain the reduced action for the *linear geometry* ($\Psi_i \equiv 0$), first we split the form $\mathcal{A}$ into horizontal and vertical components using a reference connection Γ on the vector bundle, then we impose the independence of the fields on the vertical directions and finally we "integrate along the fiber" using a bilinear structure K on the bundle. If the connection Γ is the generalized Chern connection for the bilinear structure, then the result is independent of the particular (Γ, K) chosen.

Let us define $\mathcal{A}_{\bar{z}} = A_{\bar{z}} - A_{\bar{k}}\Gamma_{\bar{z}\bar{j}}^{\bar{k}}\bar{w}^{\bar{j}}$ and $\mathcal{A}_{\bar{i}} = A_{\bar{i}}$, where Γ is a reference connection and impose that the components $(A_{\bar{z}}, A_{\bar{i}})$ are independent on the coordinates along $\mathbb{C}^2$, obtaining for the Lagrangian

$$
L = \frac{1}{2}\Omega \wedge \mathrm{Tr}\left\{A_{\bar{i}}D_{\bar{z}}A_{\bar{j}} + A_{\bar{i}}\Gamma_{\bar{z}\bar{j}}^{\bar{k}}A_{\bar{k}}\right\}\mathrm{d}w^{\bar{i}} \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w^{\bar{j}} \qquad (2.2)
$$

where $D_{\bar{z}}$ is the covariant derivative w.r.t. the gauge structure.

Now let us consider a bilinear structure K in V, *i.e.* a local section $K \in \Gamma(V \otimes \bar{V})$, the components $K^{i\bar{j}}$ being an invertible complex matrix at any point. The "integration along the fiber" is realized contracting the hCS (3,3)-form Lagrangian by the two bi-vector fields $k = \frac{1}{2}\epsilon_{ij}K^{i\bar{l}}K^{j\bar{k}}\frac{\partial}{\partial \bar{w}^l}\frac{\partial}{\partial \bar{w}^k}$ and $\rho = \frac{1}{2}\epsilon^{ij}\frac{\partial}{\partial w^i}\frac{\partial}{\partial w^j}$

$$\mathcal{L}_{red} = i_{\rho \wedge k} L = \frac{1}{2}\mathrm{d}z\mathrm{d}\bar{z}(\det K)\epsilon^{\bar{i}\bar{j}}\mathrm{Tr}\left[A_{\bar{i}}D_{\bar{z}}A_{\bar{j}} + A_{\bar{i}}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}A_{\bar{k}}\right]. \qquad (2.3)$$

Defining the field components $\varphi^i = i_{V^i}A \in V$, where $V^i = K^{i\bar{j}}\frac{\partial}{\partial \bar{w}^j}$, one gets

$$\mathcal{L}_{red} =$$
$$\frac{1}{2}\mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}\left[\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j + (\det K)\varphi^m\varphi^n\epsilon^{\bar{i}\bar{j}}\left(K_{m\bar{i}}\partial_{\bar{z}}K_{n\bar{j}} + K_{m\bar{i}}K_{n\bar{k}}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}\right)\right]$$

where $K_{\bar{i}j}$ are the components of the inverse bilinear structure, that is $K_{\bar{i}j}K^{j\bar{l}} = \delta^{\bar{l}}_{\bar{i}}$. In order to have a result which is independent of the trivialization, just set the reference connection to be the generalized Chern connection of the bilinear structure K, that is $\Gamma^{\bar{k}}_{\bar{z}\bar{j}} = K_{\bar{j}l}\partial_{\bar{z}}K^{l\bar{k}}$. The action for the reduced theory is given by

$$S_{red} = \frac{1}{g_s}\int_{\Sigma}\mathcal{L}_{red} = \frac{1}{2g_s}\int_{\Sigma}\mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}\left[\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j\right].$$

In the *rational case* $\Sigma \simeq \mathbb{P}^1$ with non vanishing deformation terms Ψ_i, X is a deformation of a vector bundle $V \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)$ for some n.

Let us start with the reduction in the *Abelian $U(1)$ case*. In this case the cubic term in the hCS Lagrangian is absent and the reduction is almost straightforward. In the singular coordinates (φ^1, φ^2) we obtain that

$$\mathcal{L}_{red} = \frac{1}{2}\epsilon_{ij}\varphi^i\partial_{\bar{z}}\varphi^j dz\mathrm{d}\bar{z}$$

in both charts of the standard atlas $\{U_N, U_S\}$ of $\mathbb{P}^1$. The potential term X gives the deformation of the action due to the deformation of the complex structure. Passing to the non singular coordinates (ϕ_1, ϕ_2), one obtains

$$S_{red} = \frac{1}{2\,g_s}\left[\int_{\mathbb{P}^1}\mathrm{d}z\mathrm{d}\bar{z}\epsilon_{ij}\phi^i\partial_{\bar{z}}\phi^j + \oint \frac{\mathrm{d}z}{2\pi i}X(z, \phi)\right] \qquad (2.4)$$

where $\oint$ is a contour integral along the equator. Therefore, the reduced theory gives a b–c (β–γ) system on the two hemispheres with a junction interaction along the equator.

The *non–Abelian case* is a bit more complicated than the Abelian one because of the tensoring with the (trivial) gauge bundle. This promotes the vector bundle

sections to matrices and therefore it is not immediate how to unambiguously define the potential function X in the general case. The easiest way to avoid matrix ordering prescriptions is to restrict to the case in which $X(z, \omega)$ does not depend, say, on ω^2. Defining $B := \omega^1 \Psi^2 + X$, one obtains $\Psi^1 = 0$, $\Psi^2 = \partial_{\omega^1} B$ and the reduced action

$$S \equiv S_{red} = \frac{1}{g_s} \left[\int_{\mathbb{P}^1} -\mathrm{Tr}(\phi^2 D_{\bar{z}} \phi^1) \mathrm{d}z \mathrm{d}\bar{z} + \oint \mathrm{Tr} B(z, \phi^1) \mathrm{d}z \right] \qquad (2.5)$$

2.1 Gauge fixing

In order to calculate its partition function, let us now discuss the gauge fixing of the theory. The following discussion is a refinement of the derivation given in [6]. Our starting action is (2.5) and we follow the standard BRST quantization (see for example [8]).

The BRST invariance in the minimal sector is

$$s A_{\bar{z}} = -(Dc)_{\bar{z}}, \quad s\phi^i = [c, \phi^i], \quad sc = \frac{1}{2}[c, c]$$

while we add a further non minimal one to implement the gauge fixing with

$$s\bar{c} = b, \quad sb = 0.$$

The gauge fixed action is obtained by adding to S a gauge fixing term

$$S_{gf} = S + s\Psi, \quad \text{where} \quad \Psi = \frac{1}{g_s} \int_{P^1} \mathrm{Tr} \left[\bar{c} \partial_z A_{\bar{z}} \right]$$

which implements a holomorphic version of the Lorentz gauge. Actually we have

$$s\Psi = \frac{1}{g_s} \int_{P^1} \mathrm{Tr} \left[b \partial_z A_{\bar{z}} - \partial_z \bar{c} (Dc)_{\bar{z}} \right].$$

Our partition function is then the functional integral

$$Z_B = \int \mathcal{D} \left[\phi^i, A_{\bar{z}}, c, \bar{c}, b \right] e^{-S_{gf}}$$

The calculation can proceed as follows. Let us first integrate along the gauge connection $A_{\bar{z}}$ which enters linearly the gauge fixed action and find

$$Z_B = \int \mathcal{D} \left[\phi^i, c, \bar{c}, b \right] \cdot$$

$$\cdot e^{-\frac{1}{g_s} \left[-\int_{P^1} \mathrm{Tr} \phi^2 \partial_{\bar{z}} \phi^1 - \partial_z \bar{c} \partial_{\bar{z}} c + \oint \mathrm{Tr} B(z, \phi^1) \right]} \delta \left(\partial_z b + [\partial_z \bar{c}, c] + [\phi^1, \phi^2] \right)$$

Now we integrate along the field b. By solving the constraint we obtain

$$Z_B = \int \mathcal{D} \left[\phi^i, c, \bar{c} \right] e^{-\frac{1}{g_s} \left[-\int_{P^1} \mathrm{Tr} \phi^2 \partial_{\bar{z}} \phi^1 - \partial_z \bar{c} \partial_{\bar{z}} c + \oint \mathrm{Tr} B(z, \phi^1) \right]} \frac{1}{\det' \partial_z}$$

where $\det'$ is the relevant functional determinant with the exclusion of the zero modes. Then we integrate along the $(c, \bar{c})$ ghosts and get

$$Z_B = \int \mathcal{D}\left[\phi^i\right] e^{-\frac{1}{g_s}\left[-\int_{P^1} \mathrm{Tr}\phi^2 \partial_{\bar{z}}\phi^1 + \oint \mathrm{Tr}B(z,\phi^1)\right]} \frac{\det' \partial_z \partial_{\bar{z}}}{\det' \partial_z}$$

Finally, since the geometric potential B does not depend on ϕ^2, we can also integrate along this variable and obtain

$$Z_B = \int \mathcal{D}\left[\phi^1\right] e^{-\frac{1}{g_s}\left[\oint \mathrm{Tr}B(z,\phi^1)\right]} \delta(\partial_{\bar{z}}\phi^1) \frac{\det' \partial_z \partial_{\bar{z}}}{\det' \partial_z}$$

The delta function constrains the field ϕ^1 to span the $\partial_{\bar{z}}$-zero modes and once it is solved it produces a further $\left(\det' \partial_{\bar{z}}\right)^{-1}$ multiplicative term that cancel the other determinants. Therefore, all in all we get

$$Z_B = \int_{\mathrm{Ker}\partial_{\bar{z}}} d\phi^1 e^{-\frac{1}{g_s}\left[\oint \mathrm{Tr}B(z,\phi^1)\right]}.$$

Lastly we can expand $\phi^1 = \sum_{i=0}^n X_i \xi_i$ along the basis $\xi_i(z) \sim z^i$ of $\mathrm{Ker}\partial_{\bar{z}}$ with $N \times N$ matrix coefficients X_i. Finally we find the multi-matrix integral

$$Z_B = \int \prod_{i=0}^n dX_i e^{-\frac{1}{g_s}\mathcal{W}(X_0,...,X_n)} \tag{2.6}$$

where we defined

$$\mathcal{W}(X_0, \ldots, X_n) = \oint \mathrm{Tr}B(z, \sum_i X_i z^i) \tag{2.7}$$

This is the result of our gauge fixing procedure which covers the details needed to complete the derivation given in [4] and confirms the conjecture in [9].

3 General Properties of Two-Matrix Models

The second part of this paper is devoted to solving some of the matrix models introduced above, eq.(2.6), the two–matrix models with bilinear coupling. The sequel is a short review of [4] on this subject containing some additional remarks and complements. The main point we insist on is that for these models there is the possibility to solve the quantum problem exactly. That is, we perform the path integral exactly and determine all the (quantum) solutions. The exact solvability of these two–matrix model is a well–known fact, but its consequences have not yet been completely explored. As we will see, not all these solutions have a classical analog and, thus, they represent genuine new quantum solutions.

To this purpose the method of orthogonal polynomials turns out to be particularly fit. This method allows one to explicitly perform the path integration, so that one is left with quantum equations of motion and the flow equations of an integrable linear systems. The latter in particular uncover the integrable nature of two–matrix models, which stems from the Toda lattice hierarchy underlying all of them. Our approach for solving two–matrix models consists in solving the quantum equations of motion and, then, using the recursiveness intrinsic to integrability (the flow equations), in finding explicit expressions for the correlators. An alternative method is based on the W constraints on the functional integral. We do not use it here, but one can find definitions, applications and comparisons with the other methods in [21, 22]. For general references on matrix models, see the bibliography in [4, 26].

Let us start with a synthetic review of the approach based on orthogonal polynomials. The model of two Hermitian $N \times N$ matrices M_1 and M_2 with bilinear coupling is defined by the partition function

$$Z_N(t,c) = \int dM_1 dM_2 e^{trW}, \qquad W = V_1 + V_2 + cM_1M_2 \qquad (3.1)$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1,2. \qquad (3.2)$$

where p_α are finite numbers. We denote by $\mathcal{M}_{p_1,p_2}$ the corresponding two–matrix model. With reference to eq.(2.7), this model descends from the geometric potential B defined by

$$B(z,\omega) = \frac{1}{z}\left[V_1(\omega) + V_2\left(\frac{\omega}{z}\right)\right] + \frac{c}{2z^2}\omega^2 \qquad (3.3)$$

We are interested in computing correlation functions (CF's) of the operators

$$\tau_k = trM_1^k, \qquad \sigma_k = trM_2^k, \qquad \forall k,$$

For this reason we complete the above model by replacing (3.2) with the more general potentials

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \qquad \alpha = 1,2 \qquad (3.4)$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \le p_\alpha$. The CF's are defined by

$$\langle \tau_{r_1} \ldots \tau_{r_n} \sigma_{s_1} \ldots \sigma_{s_m}\rangle = \frac{\partial^{n+m}}{\partial t_{1,r_1} \ldots \partial t_{1,r_n} \partial t_{2,s_1} \ldots \partial t_{2,s_m}} \ln Z_N(t,g) \quad (3.5)$$

where, in the RHS, all the $t_{\alpha,r}$ are set equal to $\bar{t}_{\alpha,r}$ for $r \leq p_\alpha$ and the remaining are set to zero. From now on we will not distinguish between $t_{\alpha,r}$ and $\bar{t}_{\alpha,r}$ and use throughout only $t_{\alpha,r}$. We hope the context will always make clear what we are referring to.

We recall that the ordinary procedure to calculate the partition function consists of three steps [23], [24], [25]: (i) one integrates out the angular part so that only the integrations over the eigenvalues are left; (ii) one introduces the orthogonal monic polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_2) = \lambda_2^n + \text{lower powers},$$

for $n = 1, 2, 3, \ldots$. They satisfy the orthogonality relations

$$\int d\lambda_1 d\lambda_2 \xi_n(\lambda_1) e^{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2} \eta_m(\lambda_2) = h_n(t, c)\delta_{nm} \qquad (3.6)$$

(iii), using the orthogonality relation (3.6) and the properties of the Vandermonde determinants, one can easily calculate the partition function

$$Z_N(t, c) = \text{const } N! \prod_{i=0}^{N-1} h_i \qquad (3.7)$$

whereby we see that knowing the partition function amounts to knowing the coefficients $h_n(t, c)$. The crucial point is that the information concerning the latter can be encoded in the flow equations of the Toda lattice hierarchy and the quantum equations of motion. Before coming to this, let us introduce some convenient notations. In the sequel we will meet infinite matrices M_{ij} with $0 \leq i, j < \infty$. For any such matrix M, we define

$$\mathcal{M} = H^{-1}MH, \qquad H_{ij} = h_i\delta_{ij}, \qquad \widetilde{M}_{ij} = M_{ji}$$

We represent such matrices in the lower right quadrant of the (i, j) plane. They all have a band structure, with nonzero elements belonging to a band of lines parallel to the main descending diagonal. We will write $M \in [m, n]$, if all its non–zero lines are between the m–th and the n–th ones, setting $m = 0$ for the main diagonal. We refer to such types of band matrices as Jacobi matrices. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$.

Let us come now to the *quantum equations of motion*. They are written as

$$P^\circ(1) + V_1'(Q(1)) + cQ(2) = 0, \quad cQ(1) + V_2'(Q(2)) + \widetilde{\mathcal{P}}^\circ(2) = 0, (3.8)$$

In these equations $Q(1), Q(2), P^\circ(1), P^\circ(2)$ are infinite Jacobi matrices. They represent the multiplication by λ_1, λ_2 and the derivative by the same parameters, respectively, in the basis of monic polynomials. Eqs.(3.8) can be considered the

quantum analog of the classical equations of motion. The difference with the classical equations of motion of the original matrix model is that, instead of the $N \times N$ matrices M_1 and M_2, here we have infinite $Q(1)$ and $Q(2)$ matrices together with the quantum deformation terms given by $P^\circ(1)$ and $\widetilde{\mathcal{P}}^\circ(2)$, respectively. From the coupling conditions it follows at once that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2$$

where

$$m_1 = p_2 - 1, \qquad m_2 = 1 \qquad n_1 = 1, \qquad n_2 = p_1 - 1$$

where p_α, $\alpha = 1, 2$ is the highest order of the potential V_α (see (3.2)).

The *flow equations* of the Toda lattice hierarchy are

$$\frac{\partial}{\partial t_{\alpha,k}} Q(1) = [Q(1), \ Q^k(\alpha)_-], \qquad \alpha = 1, 2 \tag{3.9j}$$

$$\frac{\partial}{\partial t_{\alpha,k}} Q(2) = [Q^k(\alpha)_+, \ Q(2)] \tag{3.9k}$$

Finally one must use the reconstruction formula for the partition function

$$\frac{\partial}{\partial t_{\alpha,r}} \ln Z_N(t,c) = \sum_{i=0}^{N-1} \left(Q^r(\alpha) \right)_{ii}, \qquad \alpha = 1, 2 \tag{3.10}$$

It is evident that, by using the equations (3.9j,3.9k) above we can express all the derivatives of Z_N in terms of the elements of the Q matrices. For example

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln Z_N(t,c) = \left(Q^r(\alpha) \right)_{N,N-1}, \qquad \alpha = 1, 2 \tag{3.11}$$

and so on. We recall that the derivatives of $F(N,t,c) = \ln Z_N(t,c)$ at $t_{\alpha,r} = \bar{t}_{\alpha,r}$ are nothing but the correlation functions of the model.

To end this section, we collect a few formulas we will need later on. First, we will be using the following parametrization of the Jacobi matrices

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \quad \widetilde{\mathcal{Q}}(2) = I_+ + \sum_i \sum_{l=0}^{m_2} b_l(i) E_{i,i-l} \tag{3.12}$$

where $I_+ = \sum_{i=0} E_{i,i+1}$ and $(E_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l}$. One can immediately see that

$$\left(Q_+(1) \right)_{ij} = \delta_{j,i+1} + a_0(i) \delta_{i,j}, \qquad \left(Q_-(2) \right)_{ij} = R(i) \delta_{j,i-1} \tag{3.13}$$

where $R(i+1) \equiv h_{i+1}/h_i$. As a consequence of this parametrization, eq.(3.11) gives in particular the two important relations

$$\frac{\partial^2}{\partial t_{1,1}^2} F(N,t,c) = a_1(N), \qquad \frac{\partial^2}{\partial t_{1,1} \partial t_{2,1}} F(N,t,c) = R(N) \quad (3.14)$$

To complete this summary, we should mention the W–constraints method. The latter are constraints on the partition function which take the form of a nice algebraic structure, see [21, 22] for instance. They are obtained by putting together quantum equations of motion and flow equations. W–constraints (which are also called loop equations or Schwinger–Dyson equations) can be used to solve matrix models, but such a procedure is less efficient than the one used here.

Finally we must recall that the CF's we compute are genus expanded. The genus expansion is strictly connected with the homogeneity properties of the CF's. The contribution pertinent to any genus is a homogeneous function of the couplings (and N) with respect to appropriate degrees assigned to all the involved quantities, see [4]. In particular we expect the partition function to have, in the large N limit, the expansion

$$F = \sum_{g=0}^{\infty} N^{2-2g} F_g \quad (3.15)$$

where g is the genus. Such expectation, based on a path integral analysis, remains true in our setup due to the fact that the homogeneity properties carry over to the Toda lattice hierarchy. In (3.15) F_g is interpreted as the result of summing the open string partition function at fixed g over all boundaries. Therefore it is a closed string quantity and the correlators obtained from it are closed string correlators, *i.e.* correlators of the geometrically dual closed strings relative to the deformed geometry. From matrix models it is however possible to obtain also genuine open string quantities. To this end we must change expansion. We introduce the string coupling g_s and rescale all couplings in W, so as to extract an overall factor $1/g_s$ out of them. Now the open string expansion is postulated to be

$$F = \sum_{h=0}^{\infty} g_s^{2g-2+h} N^h F_{g,h} \quad (3.16)$$

$F_{g,h}$ refers to the contribution from a world–sheet of genus g with h boundaries.

4 Solving Two-Matrix Models

Our procedure to solve two–matrix models consists in solving the quantum equations of motion. This allows us to determine the 'lattice fields' $a_i(n), b_i(n)$ and

$R(n)$. Once these are known we can compute all the correlation functions starting from (3.10) by repeated use of eqs.(3.9j,3.9k), which form the flows of the Toda lattice hierarchy. As for the free energy $F(t, N, c)$ itself, see for instance ([21]). In the following we describe some explicit examples of this method. In reality we will concentrate on solving the equations of motion, since the calculus of correlators is of algorithmic nature and, therefore, not particularly interesting; in any case, it has already been illustrated in a number of examples, [4, 21, 22]. The equations of motion are definitely more interesting, because some aspects of them have not been stressed enough or ignored in the existing literature. Therefore our purpose is to use some (simple) examples first of all in order to exemplify our method and compare it with others, in particular with the saddle point method; second, in order to single out the novelties with respect to the existing literature. Our exposition will be very close to [4], but with more examples and more details. For other approaches to matrix models see, for instance, [26] and references therein.

4.1 The Gaussian model

The bi-Gaussian model $\mathcal{M}_{2,2}$ was fully solved in [22]. It is a very simple model, but we further simplify it by considering the decoupling limit $c \to 0$ and keeping only half of couplings, say $t_k = t_{1,k}$. This operation makes sense and leads to the Gaussian one–matrix model $\mathcal{M}_2$. It allows us to make an explicit comparison of our method and the traditional one based on eigenvalue density and resolvent.

The quantum equations of motion of the simplified model are, see [22],

$$t_1 + 2t_2 a_0(n) = 0, \qquad n + 2t_2 a_1(n) = 0 \qquad (4.1)$$

We can set $t_1 = 0$, because the linear term in the potential can always be gotten rid of with a matrix redefinition. Therefore we have $a_0 = 0$ and $a_1 = -n/(2t_2) = g_s n/N$. In the last equality we have introduced the string coupling $g_s = -N/(2t_2)$, thus making contact with the notation of [13] where $1/g_s$ is factored out in front of the potential in the path integral. Since the latter reference contains a compact review of the Gaussian one–matrix model with the saddle point method, we refer to it for a comparison. In the large N limit, n/N becomes the continuous variable x, and so, in this limit, $a_1(x) = g_s x$. Now we can easily compute all the correlators. For a comparison with [13] we need the one–point functions in genus 0. Following [21] and section 5.5 of [22], we can compute

$$\langle \tau_{2k+1} \rangle = 0$$
$$\langle \tau_{2k} \rangle = \int_0^x L_{(0)}^{2k}(y)\, dy = \int_0^x \binom{2k}{k} g_s^k \frac{y^{k+1}}{k+1} dy \qquad (4.2)$$

where $L = \zeta + a_1(x)\zeta^{-1}$ is the genus 0 continuous version of $Q(1)$, and the

label $_{(0)}$ denotes the coefficient of ζ^0. In [13] the resolvent is defined as

$$\omega = \frac{1}{N} \sum_{k=0}^{\infty} \frac{\tau_k}{p^{k+1}} \tag{4.3}$$

Therefore in the genus 0 case we have[*]

$$\omega_0 = g_s \sum_{k=0}^{\infty} \frac{\langle \tau_k \rangle_0}{p^{k+1}} = g_s \sum_{k=0}^{\infty} \frac{(2k)!}{k!(k+1)!} g_s^{2k} \frac{x^{2k+1}}{p^{k+1}}$$

$$= \frac{1}{u} \left(1 - \sqrt{1 - u^2} \right) = \frac{1}{2t} \left(p - \sqrt{p^2 - 4t} \right) \tag{4.4}$$

which is the result of [13] for the resolvent, provided we set $t = xg_s$ and $u = \frac{2t}{p}$. From this we can reconstruct the eigenvalue density

$$\rho(\lambda) = \frac{1}{2\pi t} \sqrt{4t - \lambda^2}$$

In the saddle point method this leads to introducing an auxiliary hyperelliptic Riemann surface

$$y^2 = p^2 - 4t \tag{4.5}$$

where all data of the model are encoded.

As we see from this elementary example, starting from our formulas above we can reconstruct all the formulas of the saddle point method. In this case there is no difference between the two methods.

4.2 The cubic model

The full $\mathcal{M}_{3,2}$ model has been discussed at length in [22] and, in particular, in [4]. Here we would like to consider its decoupling limit $c = 0$ and single out the cubic potential part, which amounts to considering the cubic one–matrix model $\mathcal{M}_3$. In the genus 0 limit this model is described by the discrete algebraic equations

$$a_0^3 + \frac{t_2}{t_3} a_0^2 + \frac{2}{9} \left(\frac{t_2}{t_3} \right)^2 a_0 - \frac{n}{3t_3} = 0 \tag{4.6}$$

$$a_1 = -\frac{1}{2} a_0^2 - \frac{1}{3} \frac{t_2}{t_3} a_0 \tag{4.7}$$

where, for simplicity and without loss of generality, we have set $t_1 = 0$. One can extract from these equations a_0 and a_1 and calculate all the correlators with

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[*]In the continuous limit the couplings get renormalized, $t_k \rightarrow \tilde{t}_k = t_k/N$ and, due to the expansion (3.15), we define $\langle \tau_k \rangle_0 = \frac{\partial F_0}{\partial \tilde{t}_k}$.

the algorithm described in the previous section. Here we are not interested in this, but rather in analyzing eq.(4.6) and its solutions.

In the large N limit we shift to $x = \frac{n}{N}$ and, in order to make contact with section 4 of [23] for a comparison, we simplify a bit further our notation setting $t_2 = -\frac{N}{2}$ and $t_3 = -Ng$, where g is the cubic coupling constant there. Moreover we denote $z = 3ga_0$. Then eq.(4.6) becomes

$$18g^2 x + z(1 + z)(1 + 2z) = 0 \tag{4.8}$$

This can be solved exactly for z and gives the three solutions

$$z_1 = -\frac{1}{2} + \frac{1}{2I(x)} + \frac{I(x)}{6} \tag{4.9}$$

$$z_2 = -\frac{1}{2} + \frac{1 + i\sqrt{3}}{4I(x)} + \frac{1 - i\sqrt{3}I(x)}{12} \tag{4.10}$$

$$z_3 = -\frac{1}{2} + \frac{1 - i\sqrt{3}}{4I(x)} + \frac{1 + i\sqrt{3}I(x)}{12} \tag{4.11}$$

where

$$I(x) = 3^{1/3}\left(-324g^2 x + \sqrt{3}\sqrt{-1 + 34992g^4 x^2}\right)^{1/3} \tag{4.12}$$

From these we can extract three solutions for a_0 and, consequently, for a_1. For small x the three solutions can be expanded as follows

$$z_1 = -18g^2 x - 972g^4 x^2 - 93312g^6 x^3 - 11022480g^8 x^4 + O(x^5)$$
$$z_2 = -1 - 18g^2 x + 972g^4 x^2 - 93312g^6 x^3 + 11022480g^8 x^4 + O(x^5)$$
$$z_3 = -\frac{1}{2} + 36g^2 x + 186624g^6 x^3 + O(x^5). \tag{4.13}$$

The best way to analyze these solutions is to notice that they represent a plane curve in the complex z, x plane. It is a genus 0 Riemann surface with punctures at $x = 0$ and $x = \infty$, made of three sheets joined through cuts running from $z = -1/(\sqrt{3}108g^2)$ to $z = 1/(108\sqrt{3}g^2)$. The three solutions (4.13) correspond to the values z takes near $x = 0$, away from the cuts. In order to pass from one solution to another we have to cross the cuts. We call the Riemann surface so constructed the *quantum Riemann surface associated to the model**. This Riemann surface picture is the clue to understanding the solutions with multiple brane configurations, as was explained in [4].

Let us analyze the meaning of these three solutions. To this end it is useful to make a comparison with [23]: we see that the first solution corresponds to the unique solution found there, which corresponds to the minimum of the classical

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*In the example of the previous subsection the quantum Riemann surface was just one point.

potential (see below). In fact the correspondence with [23] can be made more precise: one can easily verify that eqs.(46) there are nothing but eqs.(4.6,4.7), provided we make the identifications: $a + b = a_0$ and $(b - a)^2 = 4a_1$ and the rescalings $a_0 \to \sqrt{x}a_0, a_1 \to xa_1$ and $g \to g/\sqrt{x}$. In [23] the interval $(2a, 2b)$ represents a cut in the eigenvalue λ plane. Also in [23], as in the previous subsection, we have therefore an auxiliary Riemann surface. The latter is not to be confused with the quantum Riemann surface defined above, although the two are related.

Let us now discuss the correspondence between our three solutions and the classical extrema of the potential. The classical potential for the continuous eigenvalue function $\lambda(x)$ (which is $\lambda_n/\sqrt{N}$ in the large N limit), is $V_{cl} = \frac{1}{2}\lambda^2 + g\lambda^3$. It has extrema at $\lambda = 0$ and $\lambda = -1/3g$. To find the classical limit in our quantum approach instead, we rescale t_k, $k = 2, 3$ as: $t_k \to t_k/\hbar$, and take $\hbar \to 0$. This amounts to dropping the last term in eq.(4.6). The extrema are three, $z = 0, -1, -1/2$, which corresponds to $a_0 = 0, -1/3g, -1/6g$, not two as in the classical case. $z = 0$ corresponds to the minimum of the potential, $z = -1$ corresponds to the maximum, while $z = -1/2$ to the flex. The latter solution does not have a classical analog.

Discussion. Before taking seriously this new quantum solution, we have to check whether our method of solving the quantum problem has any flaw. Our method consists in solving the quantum equations of motion (3.8). One might wonder whether there are other independent conditions beside (3.8) that one must impose on the solutions. Eqs.(3.8) are obtained from the identity

$$\int d\lambda_1 d\lambda_2 \, \frac{\partial}{\partial \lambda_1} \left(\xi_n(\lambda_1) e^{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2} \eta_m(\lambda_2) \right) = 0 \quad (4.14)$$

and a similar equation for the derivative with respect to λ_2. Of course one has also

$$\int d\lambda_1 d\lambda_2 \, \frac{\partial}{\partial \lambda_1} \left(\lambda_1^k \xi_n(\lambda_1) e^{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2} \eta_m(\lambda_2) \right) = 0 \quad (4.15)$$

for any k. The question is whether these equations imply additional constraints on the solutions of the quantum equations of motion. However one can easily see that eq.(4.15) can be written as

$$Q(1)^k \left(P^\circ(1) + V_1'(Q(1)) + cQ(2) \right) = 0 \quad (4.16)$$

Therefore a solution to eqs.(3.8) is automatically a solution to (4.16). Thus these additional equations (as well as all those that can be obtained by inserting, instead of λ_1^k, generic monomials of λ_1 and λ_2 in (4.15)), cannot further constrain the solutions to (3.8). By the way, the W–constraints are obtained precisely by taking the trace of all the expressions like (4.16) and expressing them in terms of derivative of $\ln Z$. Therefore the three quantum solutions we have found above are also solutions to the W–constraints

(or loop equations). Finally, if we consider higher order derivatives with respect to λ_1 in (4.14) instead of the first order one, it is easy to see that we do not get any additional constraints either.

This result is somewhat puzzling, but we should remember that the saddle point method is semiclassical: one cannot exclude that the quantum problem admits solution without classical analog. This is precisely what happens in the present case. One can phrase it also by saying that, in general, the large N limit and the $\hbar \to 0$ limit do not commute.

The next question is: what is the meaning of the third solution, $z = -1/2$? Let us recall what the other two solutions at $z = 0$ and $z = -1$ mean. On the basis of the discussion in section 2, we know that they represent two Riemann spheres located at the minimum and the maximum of the potential. They replace the continuous family of $\mathbb{P}^1$ which characterizes the conifold geometry before the deformation W is introduced. This is the interpretation on the basis of classical geometry. What we learn now is that solving the quantum problem we obtain a third solution, which we can interpret as a *quantum* $\mathbb{P}^1$, located at the flex of the potential. This is a pure quantum geometry effect.

Before we end this section we would like to make a few remarks. First we notice that the classical extrema are characterized by the fact that $a_1 = 0$, while the pure quantum solution corresponds to a non-vanishing a_1. Moreover, after setting $a_1 = 0$ we get for a_0 an equation that coincides with the classical eigenvalue equation. From this simple example we learn three important pieces of informations.

- The number of solutions of the quantum problem (*i.e.* the number of solutions to eq.(4.8)) is in general larger than the number of the extrema of the classical potential.

- The field a_0 can be regarded as the quantum version of the classical eigenvalue function.

- The classical extrema are obtained by neglecting the n term in eq.(4.6) and setting $a_1 = 0$.

These conclusion are valid in general, except for the fact that, the condition $a_1 = 0$ in the last remark must be replaced by the fields $a_1, a_2, ..., b_1, b_2, ...,$ being set to zero in the general case.

4.3 The $\mathcal{M}_{3,3}$ model

We study the model in the case $t_1 = s_1 = 0$ and limit ourselves to writing down the genus 0 quantum equations of motion:

$$3t_3ca_0^2 + 2t_2ca_0 - 36s_3t_3b_0R + c^2b_0 - 12s_2t_3R = 0 \qquad (4.17)$$

$$3s_3cb_0^2 + 2s_2cb_0 - 36s_3t_3a_0R - 12s_3t_2R + a_0c^2 = 0$$

$$nc + Rc^2 - 18s_3t_3R^2 - 36s_3t_3a_0b_0R - 12s_2t_3a_0R$$
$$\qquad - 12t_2s_3b_0R - 4s_2t_2R = 0$$

$$a_1 = -\frac{6s_3}{c}b_0R - \frac{2s_2}{c}R, \qquad a_2 = -\frac{3s_3}{c}R^2$$

$$b_1 = -\frac{6t_3}{c}a_0R - \frac{2t_2}{c}R, \qquad b_2 = -\frac{3t_3}{c}R^2$$

From the previous section we see that we are indeed interested in finding all the solutions that have an analytic expansion in $x = n/N$ around $x = 0$. In order to compute all possible solutions of this type, that is the quantum vacua, we therefore drop the first term in the LHS of the third eq. (4.17) and solve the resulting system. The third equation, in particular, admits the solution $R = 0$.

$$R = 0 \qquad (4.18)$$

$$3t_3a_0^2 + 2t_2a_0 + cb_0 = 0 \qquad (4.19)$$

$$3s_3b_0^2 + 2s_2b_0 + ca_0 = 0 \qquad (4.20)$$

which give rise to four (in general) distinct solutions. The alternative $R \neq 0$ leads to

$$3\frac{s_3}{s_2}(c^2 - 16s_2t_2) + c^2 - 4s_2t_2 = \frac{9}{2}c\frac{s_3t_3}{s_2}a_0^2 \qquad (4.21)$$

$$+3(\frac{s_3t_2}{s_2}c + 8s_2t_3)a_0 + 108\,s_3t_3a_0b_0 + 54\frac{s_362t_2}{s_2}b_0^2 + 162\frac{s_3^2t_3}{s_2}a_0b_0^2,$$

$$3\frac{t_3}{t_2}(c^2 - 16s_2t_2) + c^2 - 4s_2t_2 = \frac{9}{2}c\frac{s_3t_3}{t_2}b_0^2, \qquad (4.22)$$

$$+3(\frac{t_3s_2}{t_2}c + 8s_3t_2)a_0 + 108\,s_3t_3a_0b_0 + 54\frac{s_2t_3^2}{t_2}a_0^2 + 162\frac{s_3t_3^2}{t_2}a_0^2b_0.$$

This leads to an algebraic equation of order 10 for a_0, for instance. Therefore, generically, we have 10 (possibly complex) solutions for a_0, each of which gives rise to two different values for b_0. Altogether we are going to find 24 different quantum vacua. Once again it is interesting to compare these solutions with the classical ones. To this end in the above equations we set $a_2 = b_2 = a_1 = b_1 = 0$ as well as $R = 0$, from which we get eqs. (4.19, 4.20).

From the first we can get $b_0 = -\frac{1}{c}(3t_3a_0^2 + 2t_2a_0)$, whence we get either $a_0 = 0$ or the cubic equation

$$27s_3t_3^2a_0^3 + 36t_2s_3t_3a_0^2 + (12s_3t_2^2 - 6cs_2t_3)a_0 + c(c^2 - 4s_2t_2) = 0$$

Therefore in general we have four classical extrema. In ref. [4] it is shown how to find an explicit series expansion in x about each of these solutions.

Before we end this section it is interesting to discuss the geometric meaning of the first three equations in (4.17). We can think of the third equation as a definition of the complex $x = n/N$ plane. The two remaining ones are quadratic equations in a_0, b_0, R. Introducing homogeneous coordinates they can be seen to represent two hypersurfaces in $\mathbb{P}^3$. The intersection is a genus 1 Riemann surface.

4.4 The $\mathcal{M}_{p_1,p_2}$ model

In the general case the matrix rank for $Q(1)$ and $Q(2)$ was given above and the quantum equations of motion become of course very complicated. It is however simple to write down the equations that identify the extrema with classical analog. They are

$$V_1'(a_0) + cb_0 = 0, \qquad V_2'(b_0) + ca_0 = 0 \tag{4.23}$$

while all the other fields are set to zero. We have in general $(p_1 - 1)(p_2 - 1)$ solutions of this type in perfect correspondence with the classical analysis. The simplest solution is $a_0 = b_0 = 0$. Other solutions may be hard or even impossible to determine explicitly. Anyhow, once one such solution is known it is possible to find explicit expressions for the fields around it in terms of $x = n/N$. The quantum Riemann surface corresponding to this model has the same general structure as the one discussed at the end of the previous subsection. It is an intersection curve of two hypersurfaces in $\mathbb{P}^3$. Its genus can be calculated with standard methods in intersection theory. For instance, the quantum Riemann surface for the $\mathcal{M}_{4,3}$ model is in general genus 2.

5 Topological Open String Expansions

As we have pointed out in section 3 the matrix formalism allows us to compute both closed and open string amplitudes. In this final section we would like to show how to obtain the latter. To avoid clogging the text with cumbersome formulas we work in the $\mathcal{M}_{2,2}$ model. The most explicit formulas for the latter can be found in [22]. We adapt the results to the present situation by rescaling all the coupling, such as $t_k \rightarrow t_k/g_s$, etc. For instance the exact one point function is

$$\langle \tau_r \rangle = \sum_{2l=0}^{r} \sum_{k=0}^{l} \frac{r!2^{-k}}{(r-2l)!k!(l-k)!} \binom{N}{l-k+1} g_s^l A^l B^{r-2l} \tag{5.1}$$

where

$$A = \frac{2s_2}{c^2 - 4s_2 t_2}, \qquad B = \frac{2s_2 t_1 - cs_1}{c^2 - 4s_2 t_2} \tag{5.2}$$

Now, for large N it makes sense to expand the binomial coefficient in (5.1) in powers of N:

$$\binom{N}{s+1} = \sum_{p=0}^{s} \frac{\beta_{s-p}(s)}{(s+1)!} (-1)^{p+s} N^p \qquad (5.3)$$

where

$$\beta_k(s) = \sum_{1 \le s_1 < s_2 \ldots < s_k \le s} s_1 s_2 \ldots s_k, \qquad 1 \le k \le s,$$

$$\beta_0(s) = 1, \qquad \beta_k(s) = 0 \quad \text{otherwise}$$

As a consequence we get

$$\langle \tau_r \rangle = \sum_{2l=0}^{r} \sum_{k=0}^{l} \sum_{p=0}^{l-k+1} \frac{r! \, 2^{-k} \, (-1)^{p+l-k+1}}{(r-2l)! k! (l-k)! (l-k+1)!} \cdot$$

$$\cdot \, \beta_{l-k-p+1}(l-k) \, A^l \, B^{r-2l} \, g_s^l \, N^p \qquad (5.4)$$

Setting $p = h$ and $l = 2g - 2 + h$, we can easily extract the contribution to the correlator from world–sheets of genus g with h boundaries. This of course requires $l - p$ to be even. But in (5.4) we have also contributions with odd $l - p$. We do not have an interpretation for these additional contributions. They may be perhaps related to the possibility of describing the effect of punctures.

Acknowledgments

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