

# Electrohydrodynamic Plumes in Point-plane Geometry

P. Atten, B. Malraison and M. Zahn<sup>1</sup>

LEMD, CNRS and Joseph Fourier University, Grenoble, France

## ABSTRACT

We examine the flow organization induced by charge injection from a needle of small radius of curvature into an insulating dielectric liquid. Experimental results are presented for the electrical current as a function of the point-plane distance and of the applied voltage. Schlieren visualization reveals the existence of very thin plumes which are slightly vacillating. An approximate analysis of the laminar plume is developed which takes into account the finite but very thin axial region where charge is confined. A classical type of asymptotic treatment results in a set of ordinary differential equations, provided the current and the field distribution are known. A new expression for the order of magnitude of the liquid velocity is proposed which leads to estimates of the radius  $a$  of the charge core and the typical radial scale  $\delta$  of the velocity profile. For a gap spacing  $d$ , these estimates are of the order of  $10^{-3}d$  and  $10^{-2}d$  which confirm the thinness of such charge plumes.

## 1. INTRODUCTION

WHEN applying a high enough potential difference between a so-called point electrode and an opposite plate in an insulating liquid, ions are injected into the liquid by the needle electrode, more precisely by the small area in the immediate vicinity of the needle tip. The Coulomb force  $q\vec{E}$  exerted by the electric field  $\vec{E}$  on the space charge  $q$  induces a motion with characteristic velocities much higher than the drift velocity of ions with respect to the liquid. The problem is drastically different from that relative to gases. In air at atmospheric pressure for instance, the ions created by corona effect close to the tip of a needle electrode have a drift velocity  $K\vec{E}$  (with  $K$  the ion mobility) much higher than the air velocity (electric wind) induced by the force  $q\vec{E}$ . In that case the motion has but a negligible effect and the ion trajectories are the field lines. This greatly simplifies the EHD (electrohydrodynamic) problem.

To quantify the difference between gas and liquid in EHD problems it is convenient to consider the parameter  $M = (1/K)\sqrt{\varepsilon/\rho}$  [1] which is the ratio of a hydrodynamic turbulent mobility  $K_H = \sqrt{\varepsilon/\rho}$  (obtained by balancing electrostatic energy density  $\frac{1}{2}\varepsilon E^2$  and kinetic energy density  $\frac{1}{2}\rho w^2$ ) and the true mobility  $K$  of charge carriers. In gases  $M \ll 1$  (for instance in air at atmospheric pressure  $K_H = 2.6 \times 10^{-2} \text{ cm}^2/(\text{V.s})$  and  $K \simeq 2 \text{ cm}^2/(\text{V.s})$ ) whereas in liquid  $M$  is larger or much larger than 3 [2]. Therefore in gases the problem splits into two rather simple problems: the electric one which is completely decoupled from the aerodynamic motion of the gas (Figure 1(a)), and the aerodynamic one to be solved with a known force distribution  $q\vec{E}$

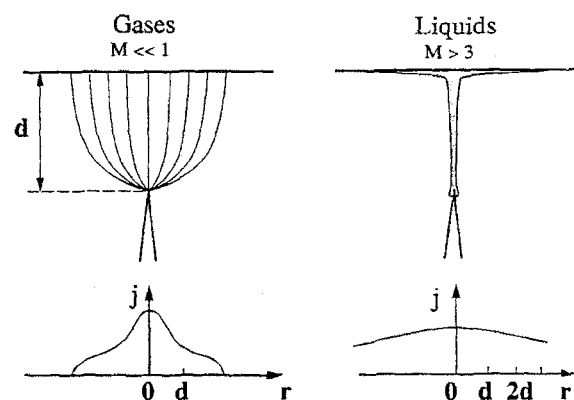


Figure 1. Effect of fluid motion on charge transport in point-plane geometry. Schematic view of ion trajectories (top part) and distribution of current density  $j$  on the facing plate (bottom part). Case of (a) gases, the ions trajectories are practically the field lines and  $j$  is confined to a finite part of the plate. (b) liquids, the ions are entrained by the liquid flow which takes the form of a narrow plume around the axis of symmetry. When arriving on the plate the charged plume spreads out.

given by the solution of the electric problem. Even the simpler electrostatic problem has received complete analytical solutions only in symmetrical geometries. Analytical solutions have not yet been obtained in asymmetric geometries like a point facing a plate, despite the attention which have been paid to it (e.g. [3] for a discussion of a well-posed problem that refutes the assumptions made in some models such as the Deutsch approximation [4] assuming that Laplacian field lines are not distorted by space charge). In [3], there is proposed a general method of obtaining numerical solutions and in [5] there is a discussion of some

Table 1. Table of Symbols

Symbol	Definition	Unit
$a$	charge core radius	m
$A$	constant	
$c$	constant	
$d$	point to plane spacing	m
$\vec{E}$	electric field vector	V/m
$\bar{E}$	mean electric field amplitude	V/m
$E_0$	electric field on axis	V/m
$E_{th}$	critical electric field	V/m
$F_1$	Coulomb force density on axis	N/m <sup>3</sup>
$I$	electric current	A
$I_t$	electric current collected by plane	A
$I_p$	electric current emitted by needle	A
$\vec{j}$	current density vector	A/m <sup>2</sup>
$j$	current density value	A/m <sup>2</sup>
$K$	charge carrier mobility	m <sup>2</sup> /Vs
$K_H$	electrohydrodynamic mobility	m <sup>2</sup> /Vs
$K_{eq}$	equivalent mobility	m <sup>2</sup> /Vs
$L$	length scale	m
$M$	mobility parameter	
$q$	volume charge density	C/m <sup>3</sup>
$q_0$	charge density on the injector	C/m <sup>3</sup>
$Q$	mean value of $q$ in horizontal plane	C/m
$r$	radial coordinate	m
$r_0$	radius of curvature of the point	m
$t$	time	s
$T$	stability parameter	
$u$	radial component of plume velocity	m/s
$\vec{V}$	fluid velocity vector	m/s
$V$	applied voltage	V
$w$	$z$ component of plume velocity	m/s
$w_0$	velocity on plume axis	m/s
$z$	coordinate along the axis	m
$z_1$	thickness of needle boundary layer	m
$\alpha$	non dimensional exponent	
$\beta$	non dimensional exponent or constant	
$\gamma$	acceleration	m/s <sup>2</sup>
$\delta$	plume radius	m
$\epsilon$	fluid permittivity	F/m
$\epsilon_r$	relative permittivity	F/m
$\xi$	coordinate	m
$\eta$	dynamic viscosity	Pas
$\nu$	kinematic viscosity	m <sup>2</sup> /s
$\rho$	mass density	kg/m <sup>3</sup>
$\tau_0$	time constant	s

analytical solutions. At the present time, despite some crude assumptions on approximate current density profile on the angle of the cone of the charged zone, and a final formula obtained only for a zero electric field on the tip, which is not realistic, the solution of  $I(V)$  proposed by Coelho and Debeau [6] for SCLC (space-charged limited current) seems to be the better one and the mobility values  $K$  obtained experimentally

[7] for low electric fields between plane parallel plates can be recovered [8] in gases like air or nitrogen over a large range of pressure values, from the measurements of the current in point-plane geometry when using the relation

$$I = 4\epsilon K \frac{V^2}{d} \quad (1)$$

Applying this relation to the case of liquids leads to apparent mobility values much higher than  $K$  and which characterize the overall effect of liquid motion. The analysis by Coelho and Debeau [6] is of no help here because these authors only retained the mobility term in the expression of current density and assumed that the space charge occupies a large solid angle around the axis. We show below that the induced liquid motion entrains the charge carriers which are confined in a very thin tube around the axis (Figures 1(b) and 2). We then deal with a problem analogous to the thermal plume where buoyancy is the driving force. Adapting the analysis of thermal plumes [9, 10] to the case of charged plumes leads to asymptotic laws for the characteristic velocity and thickness of the plume, provided the current is known [11–15].

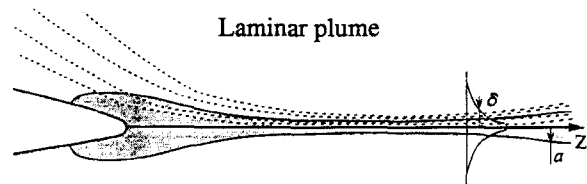


Figure 2. Schematic representation of charged plume in the vicinity of the needle tip. The dashed lines are the streamlines of the plume. The grey part is the core where the charge is confined (injection is assumed to occur only in the region of high enough field). Note that due to acceleration of the flow in the region near the needle, there is first a shrinking of both the flow and the charged core. At some distance the plume begins to expand at a small rate.

A full self-consistent treatment of charge plumes should lead to predicting both the current and the plume characteristics. Before examining this problem theoretically, we turn towards an experimental study confirming previous observations upon the injection phenomena (and laws) as well as revealing a dependence of the current on the point-plane distance  $d$  which, to our knowledge, has never been mentioned.

## 2. EXPERIMENTS

Experiments were performed in point-plane geometry using a transformer oil of relative permittivity  $\epsilon_r = 2.15$ , kinematic viscosity  $\nu = 26.6 \times 10^{-6} \text{ m}^2/\text{s}$ , mass per unit volume  $\rho = 897 \text{ kg}/\text{m}^3$ , and thus a dynamic viscosity  $\eta = \rho\nu = 2.3810^{-2} \text{ kg}/(\text{m}\cdot\text{s})$  at room temperature and a typical value of mobility  $K \approx 10^{-9} \text{ m}^2/(\text{Vs})$ . The oil is contained in a cubic box whose vertical transparent PVC walls allow easy visualization. The bottom metallic plate ( $22 \times 22 \text{ cm}^2$ ) has been replaced in some experiments by a metallic grid. In order to measure the electrical current injected by the needle only, the needle was insulated from its supporting rod and both were connected to a negative HV power supply (Figure 3). The needle is an iron cylinder ending with a conical part and a tip of small radius of curvature ( $r_0 \approx 3 \mu\text{m}$ ). This needle sticks out  $\sim 3 \text{ mm}$  from the rod (diameter  $\phi = 2 \text{ cm}$ ), the outer part of which is threaded which facilitates the control of the distance  $d$  (varied from 5 to 60 mm) between tip and plane. The rod plays sort of a shielding role

by capturing the major part of conduction current. Therefore to first approximation,  $I_p$  is the 'injected' current. Both the electrical current  $I_p$  passing through the needle and the total current  $I_t$  collected by the plane were measured. The current-voltage characteristics  $I_p(V)$  plotted on doubly logarithmic scales (Figure 4) show first a residual ohmic part followed by a steep increase of the current due to injection and motion in the plume, and a third part at HV with an approximate quadratic variation. In order to characterize the voltage dependence of  $I_p$  and to determine the best variation law, by analogy with the behavior in gases [7], we plotted  $I_p/V$  and  $\sqrt{I_p}$  as functions of  $V$ . Both types of diagrams exhibit a linear dependence on  $V$  indicating a quadratic variation with  $V$  for large enough applied voltages  $V$ . However, as found in other liquids, the curves  $I_p/V$  are better suited to determine the threshold voltage  $V_{th}(d)$  and give values of  $V_{th}$  slightly higher than the ones given by the  $\sqrt{I_p}$  representation. The threshold voltage  $V_{th}$  does not significantly vary with  $d$  and is  $V_{th} = 7 \pm 1$  kV. This is compatible with a constant critical field on the tip [6]  $E_{th} = 2V_{th}/[r_o \log(4d/r_o)]$  for the onset of strong injection (presumably involving electronic processes in the liquid). With  $r_o \approx 3$  mm and  $d = 12$  mm, we get  $E_{th} = 4.8 \pm 0.7$  MV/cm, a value close to the observed ones [16, 17] in various liquids. The similarity of the  $I_p(V)$  curves with those for gases suggests a similar influence of the injected space charge and a similar behavior of the field on the needle tip which should remain nearly constant ( $\approx E_{th}$ ) independent of the current. Let us also mention that for  $V \gg V_{th}$  there are current impulses with a non-regular frequency and also a weak blue light emission near the tip.

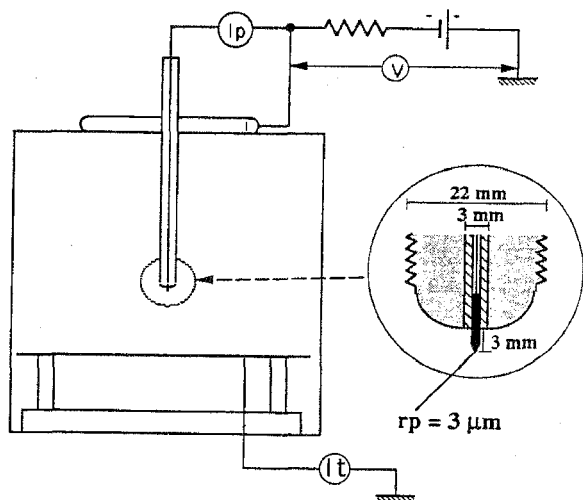


Figure 3. Experimental setup and measurement of electrical currents. Inset zoom view of the tip. The dashed annulus around the dark tip is an insulating material which allows us to measure only the current injected by the tip.

The law  $\sqrt{I_p} = A(V - V_{th})$  seems to be the most appropriate one to look at variations of  $I$  well above the threshold and to deduce from this linear variation an equivalent mobility  $K_{eq}$ . When subtracting the low ohmic current from  $I_p$  we get the variation  $I_p = A^2(d)(V - V_{th})^2$  shown in Figure 5. By plotting  $A^2$  as a function of  $d$  (Figure 6) it shows a  $d^{-1}$  dependence for small distances ( $d < 20$  mm) which would correspond to a constant 'equivalent mobility'  $K_{eq} = 1.25 \times 10^{-7} \text{ m}^2/(\text{Vs})$  as deduced from (1). This equivalent

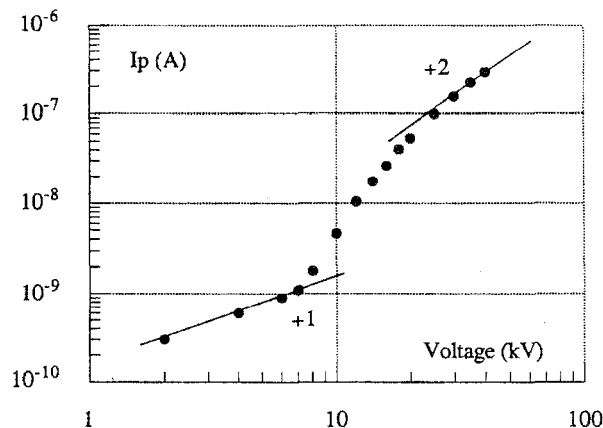


Figure 4. Typical current-voltage curve showing the ohmic part, a steep increase, and a  $V^2$  variation, at  $d = 42$  mm.

mobility value is close to  $K_H = \sqrt{\epsilon/\rho} = 1.5 \times 10^{-7} \text{ m}^2/(\text{Vs})$ , i.e.  $K_{eq} \approx 0.8K_H$ . In the case  $d = 20$  mm and  $V = 20$  kV ( $\bar{E} = V/d = 10$  kV/cm) this equivalent mobility corresponds to a typical velocity  $w \approx 0.12$  m/s. This would be the drift velocity of charge carriers if the charge distribution would be identical to that in gases. But here in liquid, the width of the plume is much narrower and the current results from the convection of charge carriers which are located in a thin core. We thus expect a higher charge density in the core and a clearly higher velocity (a velocity of 1 m/s seems plausible). Figure 6 indicates a tendency towards a law  $A^2 \propto d^{-\alpha}$  with a different value for  $\alpha$ :  $\alpha \approx 0.7$  for  $d > 20$  mm. This change in this law for  $A$  has never been reported and must be confirmed experimentally by refined investigations including the use of other liquids (such a change might be due to a change in the charge distribution or to a laminar-turbulent transition of the plume).

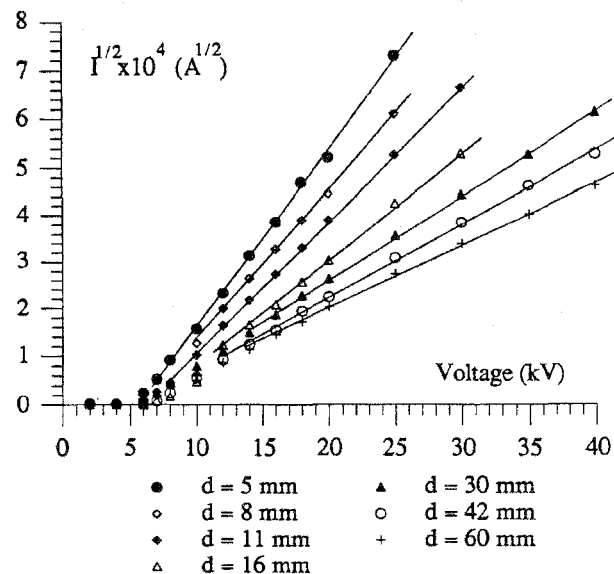


Figure 5. Variation with applied voltage  $V$  of the square root of the injected current  $I_p^{1/2}$ .

By using the Schlieren technique [18] for  $V > V_{th}$ , we observed relatively thin plumes with typical diameters of the order of  $\leq 1$  mm.

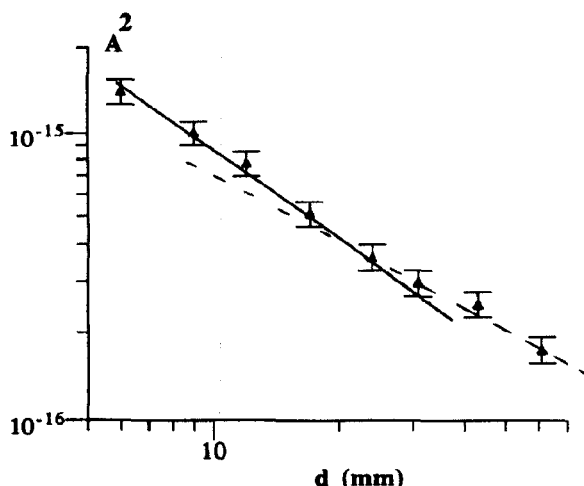


Figure 6. Variation with distance of the ratio  $A^2 = I_p / (V - V_{th})^2$ . The continuous line indicates a  $d^{-1}$  variation law, the dashed one a variation like  $d^{-0.7}$ .

Moreover, the plume is somewhat unsteady. When arriving on the plate, the plume spreads over it and induces a recirculating large toroidal eddy. The determination of the velocity field is very hard to perform because of the unsteadiness of the plume. In order to test whether or not the plume unsteadiness is due to this recirculating eddy, we replaced the solid plate by a metallic grid offering a negligible hydrodynamic resistance (below the grid a metallic mesh was placed in order to collect ions and also to damp out the flow). Indeed the plume vacillation remained as well as the recirculation but with a lower amplitude. We measured  $I_p$  in this configuration: there was no significant difference with the current characterizing the plate.

We think however that the mesh was too dense and the eddy not enough damped to prevent interaction with the plume. The other possibility is that the vacillation could be due to an instability mechanism of the plume itself (this instability has been studied theoretically in the case of the 2D blade-plane plume [13]). But in this case one would see a stable first part of the plume near the needle followed by the vacillation (the disturbances need time to develop), which is not the case here where the vacillation takes place along the plume.

### 3. PREVIOUS THEORETICAL APPROACHES ON CHARGE PLUMES

Theoretical treatments of charge plumes have been attempted by different authors based on the analogous case of laminar thermal plumes. All these approaches are of the boundary layer approximation type. They all started with the assumptions of the space charge being confined in a narrow region around the axis of symmetry (and the plane of symmetry for a 2D plume) and of a velocity profile extending in a larger charge-free region.

The first attempt is due to Zhakin [11] who considered 2D (blade-plane) as well as 3D (point-plane) problems and assumed the electric field  $E$  to vary as  $z^{-\beta}$ ,  $\beta \geq 0$  along the axis. Neglecting the charge repulsion, he retained the diffusion term and sought self-similar solutions [19]. Let us remark that the hypotheses made by Zhakin are not

physically sound because diffusion is usually fully negligible [14] whereas Coulomb repulsion between charges plays the major role in the spreading of the charge core. Moreover, several order of magnitude estimations are erroneous. McCluskey and Perez have obtained independently an approximate analytical solution for the 2D case [12], by prescribing a particular profile for the velocity field (exponential decay) in the plume. Unfortunately such an approximation cannot be extended to the axisymmetric case (3D). Here too the Coulomb repulsion is not explicitly assumed.

Using self-similar methods, Castellanos *et al.* [13] obtained the same result for the 2D case. Note that the distribution of charge is singular taking the form of a Dirac distribution. In a more recent paper [14] they discussed the extension to the axisymmetric case where the singular distribution of charge implies a singular distribution for the velocity profile. Recognizing that the singularities occur because diffusion and Coulomb repulsion were neglected, they obtained an approximate self-similar solution by imposing a finite radius for the charge core [14]. They showed that the singularity obtained when the radius tends to zero is a logarithmic one and in this case the EHD plume has to be compared with the thermal one in the case of an infinite Prandtl number.

Let us mention also an order of magnitude analysis [15] based on the assumption of a broadening of the hydrodynamic plume due to viscosity (laminar case). Solutions similar to those of [11–14] for 2D are proposed for the 3D case. This analysis also has been extended to an electric field varying as  $z^{-\beta}$ . The case of turbulent plumes was also treated.

Finally, we have to mention the numerical attempt by Takashima *et al.* [20] which is interesting because these authors clearly took into account the effect of liquid motion on charge transport. Their results, however, are of limited relevance because they imposed a cylindrical shape of constant (and high) radius for the charge tube and they assumed space charge limited injection which is not a realistic guess. The problem with such numerical attempts is that it is difficult to extract approximate expressions and to gain clear insight into the phenomena.

### 4. GENERAL CONSIDERATIONS

Before examining the axisymmetric charge plume in a detailed way, it is worthwhile to look at some orders of magnitude. From the constancy of threshold voltage, we deduce a constant field on the point  $E_{th} \simeq 4$  MV/cm. This field and the results of [8, 16, 17] appear consistent with the assumption of electronic phenomena occurring in the liquid in the very vicinity of the point. These phenomena will be referred to as 'corona effect' in the liquid for sake of simplicity. In liquids, as in gases, the field on the needle tip remains nearly constant as the voltage is increased because the injection law  $q = f(E)$  is such that  $df/dE$  is very high for  $E \geq E_{th}$ .

Because of the abrupt character of the injection law as a function of the field, the angular extension of the injecting zone on the needle will be limited and, therefore, in the following the radial distribution of charge around the axis will be approximated by a rectangular function. To estimate an order of magnitude of the charge density at the injector, we assume that the injection process is restricted to the spherical part of the

tip. The mean injected charge density  $q_o$  can be deduced from the electrical current  $I$  flowing from tip to plane

$$I \approx 2\pi r_o^2 K q_o E_{th} \quad (2)$$

In the experiments (Section 2)  $I$  varies from  $10^{-8}$  to  $10^{-7}$  A, depending on the distance and the applied potential. From  $E_{tip} \approx 4$  MV/cm we deduce that  $q_o$  ranges from 250 to 2500 C/m<sup>3</sup> which are quite huge charge density values.

It is clear that the Coulomb repulsion will play the major role in the decrease of the charge density  $q$  along the axis. The evolution of  $q$  is easily obtained as a function of time. The equation of conservation of charge is

$$\frac{\partial q}{\partial t} + \nabla \cdot \vec{j} = 0 \quad \vec{j} = q(K\vec{E} + \vec{V}) \quad (3)$$

Taking into account Poisson's equation, and writing the equation for  $q$  when following the charge carriers during their motion, leads to the total derivative

$$\frac{Dq}{Dt} = -\frac{K}{\varepsilon} q^2 = \frac{\partial q}{\partial t} + w \frac{\partial q}{\partial z} \quad (4)$$

This gives the variation law

$$q(t) = \frac{q_o}{1 + t/\tau_o} \quad \tau_o = \frac{\varepsilon}{Kq_o} \quad (5)$$

From the estimates of  $q_o$  we obtain  $\tau_o$  ranging from 8 to 80  $\mu$ s. This implies a steep decrease of  $q$  along the axis. For  $t \gg \tau_o$  (in practice,  $\tau \geq 10\tau_o$ ), and taking  $K \approx 10^{-9}$  m<sup>2</sup>/(Vs) we have the asymptotic law

$$q(t) \approx \frac{\varepsilon}{Kt} \approx \frac{2 \times 10^{-2}}{t} \quad (6)$$

The question now is to determine the abscissa  $z$  where the charge lies, at time  $t$  after injection. There are two components in the velocity of ions with respect to the electrodes, the drift velocity  $K\vec{E}$  and the convective velocity  $\vec{V}$  of the liquid:  $\vec{v}_{ion} = K\vec{E} + \vec{V}$ . We expect that the convective component will be predominant in the major part of the plume. It is only near the injecting electrode that the drift velocity can dominate over  $w$ . It is interesting to estimate the distance  $z_1$  where the drift velocity  $KE_o$  and the axial liquid velocity  $w_o$  will take the same value:  $w_o(z) = KE_o(z)$  with the notations  $w_o = w_o(z) = w(0, z)$  and  $E_o = E_o(z) = E(0, z)$ .

In the immediate vicinity of the needle tip, the electric field  $E$  takes values of the order of the threshold field. A rough estimate of the hydrodynamic velocity is  $w \sim \sqrt{\varepsilon/\rho} E$  for turbulent motion induced by injected space charge [1]. Another expression has been proposed from a rough analysis of the laminar charge plume [15]

$$w_o \sim \left( \frac{IE_o}{\eta} \right)^{1/2} \quad (7)$$

With a typical value of ion mobility  $K \approx 10^{-9}$  m<sup>2</sup>/(Vs) for the transformer oil used in the experiment, we obtain the following values for  $KE_o$  and  $w_o$  calculated for different electrical field values in the vicinity of the tip (Table 2).

From Table 2 it appears that the velocity  $w_o$  is 2 orders of magnitude larger than  $KE_o$ . The boundary layer should be of very small thickness  $z_1$ . We can obtain an order of magnitude of  $z_1$  by making very

Table 2. Estimates of ionic and hydrodynamic velocities near the needle tip. \*  $I = 10^{-8}$  A, +  $I = 10^{-7}$  A

$E$ MV/cm	$KE$ m/s	$\sqrt{\varepsilon/\rho}E$ m/s	$(IE/\eta)^{1/2}$ m/s
5	0.5	50	16* 50+
2	0.2	20	10* 32*
1	0.1	10	7* 22*

crude assumptions. What accelerates the liquid on the axis, is the difference between the Coulomb force  $F_1$  on the axis and  $F_2$  nearby. Taking  $F_1 - F_2 \approx \beta F_1$  with  $\beta < 1$  (in the following we shall take  $\beta \sim 10^{-1}$ ) and assuming that the volume of accelerated liquid is  $10 \times$  larger than the one which experiences the force, we obtain the acceleration  $\gamma \sim \beta F_1 / 10\rho = qE / 100\rho \sim q_o E_{th} / 100\rho$ . From (2) we deduce  $\gamma \sim I / 1000r_o^2 \rho K$ ; in our case  $r_o \approx 3 \mu$ m and we obtain  $\gamma \sim 10^6$  to  $10^7$  m/s<sup>2</sup>. Hence the time  $t_1$  to obtain a velocity of 1 m/s would be  $10^{-7}$  to  $10^{-6}$  s, which would correspond to  $z_1 = 10^{-1}$  to 1  $\mu$ m. But at this scale the viscous effects can be the dominant ones and an estimate has also to be derived in this case. The electro-viscous time  $\tau = \eta/\varepsilon E^2$  is between  $4 \times 10^{-9}$  and  $10^{-7}$  s. Assuming that for reaching a velocity of the order of  $KE_o$  we need a time  $10\tau$  to  $100\tau$ , gives  $z_1 \sim 0.05$  to 10  $\mu$ m (we find the same order of magnitude for  $z_1$ ). When we also take into account the physical processes occurring in the liquid such as electronic avalanches, which develop along a path of the order of 1  $\mu$ m, it appears that this zone  $z < z_1$  can be neglected. The transport of electric current is then due to convection alone all along the axis except in a very thin zone on the collecting plate.

## 5. ANALYSIS OF LAMINAR PLUMES

In Section 4 we have used a very rough estimate of the velocity  $w_o$  on the axis (Equation (7)). A better approximation can be obtained when taking into account the structure of the laminar plume with an axial core, of radius  $a$ , carrying the charge and a larger velocity plume of characteristic radius  $\delta$ , where there is no charge (Figure 2). As was done for two-dimensional thermal [9, 10] as well as charged plumes [11, 14], this boundary layer-like problem can be split into an 'inner' and an 'outer' parts. In polar coordinates  $(r, \theta, z)$ , the Navier-Stokes equation for the axial velocity component  $w$  ( $z$ -direction) can be written, in steady conditions  $\frac{\partial}{\partial t} = 0$  with the assumption of axial symmetry

$$\rho \left[ u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right] = \eta \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2} \right] + qE \quad (8)$$

where  $u$  is the radial velocity component. In (8) the pressure perturbation has been neglected, as it is the case in all boundary layer approximations.

The mass conservation law, under steady conditions, is

$$\frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0 \quad (9)$$

Equations (8) and (9) are the two governing hydrodynamic equations. In order to close the studied problem, we need the electrical equations

coupled with the previous ones, namely the charge conservation (3) and the Poisson equations.

We can first try to derive exact relations before doing any approximations. We integrate, between 0 and infinity, Equation (8) after multiplication by  $r$

$$\rho \int_0^\infty \left[ ru \frac{\partial w}{\partial r} + rw \frac{\partial w}{\partial z} \right] dr = \eta \int_0^\infty \left[ \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + r \frac{\partial^2 w}{\partial z^2} \right] dr + \int_0^\infty rqE dr \quad (10)$$

By adding to (10) the continuity Equation (9) multiplied by  $rw$  and integrated between 0 and infinity we obtain

$$\rho \int_0^\infty \left[ ru \frac{\partial w}{\partial r} + w \frac{\partial(ru)}{\partial r} + 2rw \frac{\partial w}{\partial z} \right] dr = \eta \int_0^\infty \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) dr + \eta \int_0^\infty r \frac{\partial^2 w}{\partial z^2} dr + \frac{1}{2\pi} E_o Q \quad (11)$$

where  $Q(z)$  is defined by

$$Q(z) = 2\pi \int_0^a rq(r, z) dr \quad (12)$$

and where we have supposed that the variation of  $E$  with  $r$  between  $r = 0$  and  $r = a$  is negligible

$$E(r, z) \simeq C^{st} = E(0, z) = E_o(z) \quad r \leq a \quad (13)$$

Then

$$\rho \int_0^\infty \frac{\partial(ruw)}{\partial r} dr + \rho \int_0^\infty r \frac{\partial w^2}{\partial z} dr = \eta r \frac{\partial w}{\partial r} \Big|_0^\infty + \eta \int_0^\infty r \frac{\partial^2 w}{\partial r^2} dr + \frac{1}{2\pi} E_o Q \quad (14)$$

or, with the condition  $r \frac{\partial w}{\partial r} \rightarrow 0$  when  $r \rightarrow \infty$

$$\rho(ruw)|_0^\infty + \rho \int_0^\infty r \frac{\partial(w^2)}{\partial z} dr = \eta \int_0^\infty r \frac{\partial^2 w}{\partial z^2} dr + \frac{1}{2\pi} E_o Q \quad (15)$$

For  $r = 0$ , we have  $u = 0$  and for  $r \rightarrow \infty$ ,  $w \rightarrow 0$ , the first term cancels. We finally obtain

$$\rho \frac{\partial}{\partial z} \left[ \int_0^\infty rw^2 dr \right] = \eta \frac{\partial^2}{\partial z^2} \left[ \int_0^\infty rw dr \right] + \frac{1}{2\pi} E_o Q \quad (16)$$

We can estimate the ratio of the first two terms of Equation (16)

$$\frac{\rho \int_0^\infty r \frac{\partial(w^2)}{\partial z} dr}{\eta \int_0^\infty r \frac{\partial^2 w}{\partial z^2} dr} \sim \frac{1}{v} \frac{(w_o^2/L)}{(w_o/L^2)} \sim \frac{w_o L}{v} \quad (17)$$

where  $L$  is a scale corresponding to the variation of  $w_o$  with  $z$ , that is typically the distance  $d$  between the two electrodes. This ratio is of the order of the Reynolds number using  $d$  and  $w_o$ . With the typical values found in the experiments, i.e.  $w_o \sim 30$  cm/s,  $d \sim 3$  cm and  $v = 25 \times 10^{-6}$  m<sup>2</sup>/s we obtain a Reynolds number of  $w_o d/v \sim 400$ . The second term in (16) is then negligible compared with the first one and we finally have

$$\rho \frac{\partial}{\partial z} \left[ \int_0^\infty rw^2 dr \right] = \frac{1}{2\pi} E_o Q \quad (18)$$

Let us consider now the inner core ( $r \leq a(z)$ ). Integration of (8) (multiplied by  $r$ ) between 0 and  $a$ , and taking into account (9), leads to

$$\rho \int_0^a \frac{\partial(ruw)}{\partial r} dr + \rho \int_0^a r \frac{\partial w^2}{\partial z} dr = \eta a \frac{\partial w}{\partial r} \Big|_{r=a} + \eta \int_0^a r \frac{\partial^2 w}{\partial z^2} dr + \frac{1}{2\pi} E_o Q \quad (19)$$

or, by taking into account the conditions  $u = 0$  and  $\frac{\partial w}{\partial r} = 0$  on the axis

$$\rho a(uw)|_{r=a} + \rho \int_0^a r \frac{\partial w^2}{\partial z} dr = \eta a \frac{\partial w}{\partial r} \Big|_{r=a} + \eta \int_0^a r \frac{\partial^2 w}{\partial z^2} dr + \frac{1}{2\pi} E_o Q \quad (20)$$

If the core radius  $a$  is small enough, then we can neglect to a first approximation the variations of  $w$  and  $E$  as a function of the radial coordinate  $r$  ( $r \leq a$ )

$$w(r, z) \simeq C^{st} = w(0, z) = w_o(z) \\ E(r, z) \simeq C^{st} = E(0, z) = E_o(z) \quad r \leq a \quad (21)$$

The first, second and fourth terms in (20) are proportional to  $a^2$  and therefore are negligible compared with the two remaining ones. Equation (20) reduces now to the simpler form

$$va \frac{\partial w}{\partial r} \Big|_{r=a} + \frac{QE_o}{2\pi\rho} = 0 \quad (22)$$

At this stage, in the 2D problem (blade-plane), the first type of classical treatment consists in prescribing the radial profile of the velocity component  $w$  outside the core in the form  $\exp(-x/\delta(z))$  [9, 10, 12],  $\delta$  being the plume half-width. In the second type of approach the profile is determined assuming that  $w(x, z)$  has a self-similar behavior [11, 13, 14]. These approaches are not so successful in the axisymmetric problem: a radial exponential decrease for  $w(r, z)$  is too drastic an approximation and the self-similar approach leads to singular solutions when neglecting Coulomb repulsion [14].

We now examine the 3D problem with the approximation of a self-similar velocity profile when retaining the Coulomb repulsion which plays a major role in this problem of charge plumes. With

$$w(r, z) = w_o(z) f \left( \frac{r}{\delta(z)} \right) \quad (23)$$

we have

$$\int_0^\infty rw^2(r, z) dr = w_o^2 \delta^2 \int_0^\infty \frac{r}{\delta} f^2 \left( \frac{r}{\delta} \right) d \left( \frac{r}{\delta} \right) = A w_o^2 \delta^2 \quad (24)$$

$$A = \int_0^\infty \xi f^2(\xi) d\xi$$

Equation (18) can now be written as

$$A \frac{\partial(w_o^2 \delta^2)}{\partial z} = \frac{1}{2\pi\rho} E_o Q \quad (25)$$

taking an exponential decay for  $f$  gives  $A = 1/4$ . From (23) we also deduce

$$\frac{\partial w}{\partial r} \Big|_{r=a} = \frac{w_o(z) \frac{df}{d\xi}(\xi_a)}{\delta(z)} \quad \xi_a = \frac{a}{\delta}(z) \quad (26)$$

The derivative of  $f(\xi)$  with respect to  $\xi$  is of order of  $-1$  and in the following we retain the approximate relation

$$v \left( \frac{a}{\delta} \right) w_o \simeq \frac{QE_o}{2\pi\rho} \quad (27)$$

As the velocity  $w_o$  is much higher than the drift velocity  $KE_o$ , the total current  $I = Q(K E_o(z) + w_o(z))$  can be written  $I \simeq Qw_o$  and (25) and (27) become

$$w_o \frac{\partial(w_o^2 \delta^2)}{\partial z} = \frac{1}{A} \frac{IE_o}{2\pi\rho} \quad (28)$$

$$v \left( \frac{a}{\delta} \right) w_o^2 \simeq \frac{E_o I}{2\pi\rho} \quad (29)$$

We have two Equations (28) and (29) relating the 3 unknown variables  $a$ ,  $\delta$  and  $w_o$  if we assume the current  $I$  and the field distribution on the axis  $E_o(z)$  to be known. A third relation can be obtained from the conservation of charge in the case of laminar plume; in the core we have

$$\begin{aligned} I &= \int_0^a j(r, z) 2\pi r dr \\ &\simeq \pi a^2(z) j(0, z) \\ &\simeq \pi a^2(z) q(0, z) w_o(z) \end{aligned} \quad (30)$$

The charge density  $q(0, z)$  is given by relation (6) which expresses the decrease in time under the action of Coulomb repulsion, when accompanying the charge carriers in their movement. We then have

$$a^2(z) \simeq \frac{IKt}{\pi\varepsilon w_o(z)} \quad (31)$$

We eliminate the variable  $t$  by differentiating the relation (31) and using the relation

$$\frac{\partial}{\partial t} = w_o \frac{\partial}{\partial z} \quad (32)$$

when neglecting the drift velocity  $KE_o$  compared with the velocity  $w_o$ . We finally obtain

$$w_o \frac{\partial}{\partial z} [a^2(z) w_o(z)] = \frac{IK}{\pi\varepsilon} = Cst \quad (33)$$

With the 3 relations (28), (29) and (32) the problem is closed: assuming the current  $I$  and the field distribution on the axis  $E_o(z)$  to be known, the 3 unknown variables  $a$ ,  $\delta$  and  $w_o$  can be obtained by solving the system of these 3 ordinary differential equations. In practice, without any further approximations, the problem has no simple solutions.

## 6. SIMPLIFIED TREATMENT

We can greatly simplify the problem and get semi-quantitative estimates by assuming a plausible variation law for  $\delta$ . The simplest one is to prescribe  $\delta$  to be proportional with  $a$ . In this case we recover, from (29), relation (7) obtained by rough arguments

$$w_o \sim \left[ \frac{IE_o}{2\pi\eta} \right]^{1/2} \quad (34)$$

Another approximation, which should be asymptotically valid, is to state that, similarly to other boundary value problems, the increase in plume radius  $\delta$  comes from the diffusion of vorticity, *i.e.*

$$\delta = c(vt)^{1/2} \quad (35)$$

$c$  being a constant of order 1. Eliminating the time  $t$  between (35) and (31), we get

$$\frac{\delta^2}{a^2} \sim \frac{c^2 \pi v \varepsilon w_o}{KI} \quad (36)$$

and from (29) we finally obtain

$$w_o \sim c^{2/3} \left[ \frac{\varepsilon I E_o^2}{4\pi\rho K \eta} \right]^{1/3} \quad (37)$$

This expression of  $w_o$  which is different from (34), very likely is more realistic asymptotically because the assumption  $a/\delta = Cst$  is not supported by any physical argument. If we suppose that  $E_o(z)$  is constant,  $w_o(z)$  is also constant and this implies that  $a/\delta$  is constant and then, the other expression (34) for  $w_o$  is also verified. In this case we obtain a linear relation between  $I$  and  $E$  which is not experimentally verified. In practice  $E(z)$  has the same variation law as the harmonic field in the vicinity of the point, but it can increase near the plate due to the action of space charge and therefore cannot be considered as constant.

It is interesting to estimate an order of magnitude of the radius  $a$  of the core containing the charge. From (31), with

$$t = \int_0^z \frac{dx}{w_o(x)} \quad (38)$$

and taking the lower bound  $V/d = \bar{E}$  for  $E(z)$ , which is the case for  $z < z_2$ , ( $z_2/d \sim 0.1$ ), and thus taking  $w_o(E) \geq w_o(\bar{E})$ , we obtain an upper bound for  $a$

$$a^2 \leq \frac{KIz}{\pi\varepsilon w_o^2(\bar{E})} \quad (39)$$

With expression (37) for  $w_o$  where we take  $E_o = \bar{E}$  we finally obtain

$$\frac{a}{d} \leq c^{-1/3} \left[ \frac{5KId}{\varepsilon v^2} \right]^{1/6} T^{-2/3} \left( \frac{z}{d} \right)^{1/2} \quad (40)$$

where  $T = \varepsilon V / K \eta$  is the instability parameter [2]. This upper bound leads to rather low  $a$  values; for instance in the case  $V = 30$  kV,  $d = 3$  cm in a typical viscous liquid like the oil used in the experiments ( $v \approx 25 \times 10^{-6}$  m<sup>2</sup>/s) with a typical value of mobility  $K \approx 10^{-9}$  m<sup>2</sup>/Vs) and with  $I = 10^{-7}$  A, taking  $z = d/2$ , (40) with  $c = 1$  gives  $a/d \leq 2.5 \times 10^{-3} \rightarrow a \approx 75$   $\mu$ m. Using (36) and (37), we obtain  $w_o \approx 2$  m/s and  $\delta \approx 400$   $\mu$ m. All these values justify the above analysis assuming the charge to be confined in a very narrow region. Note that  $a/\delta \approx 0.2$  is not very small.

## 7. DISCUSSION AND CONCLUSION

If they are confirmed, the foregoing theoretical estimates make it understandable why it is difficult to perform an accurate experimental investigation of these plumes: their small diameter, the high velocity gradients (and also their unsteady character) do not allow the determination of velocity profiles unless very powerful and sophisticated techniques are used. We can only compare the order of magnitudes of  $w_o$  and  $\delta$ . The observations by Schlieren techniques [18] indicate that the plume diameter  $\approx 2\delta$  is of the order of 1 mm, which appears consistent with the estimates given by (40).

For the velocity, there appears a discrepancy between  $(w_o)_{th} \approx 2$  m/s and  $(w_o)_{exp} \approx 0.3$  m/s in the typical conditions considered. A possible explanation is that the plume is not laminar but turbulent.

In the latter case, estimations of turbulent velocities using either hydrodynamic mobility or the expression given by the analysis in [15] lead to the same value of 15 cm/s, the low value is explained by the fact that a turbulent plume spreads over a larger zone, typically  $\delta/d \approx 0.1$ . The experimental observations of a larger velocity and a thin plume strongly suggest that, in the experiments, we deal with a laminar plume, also the Reynolds number  $w_o \delta / \nu$  takes values of  $\sim 10$  ( $w_o \approx 0.3$  m/s) or 70 ( $w_o \approx 2$  m/s).

Taking for granted that the observed plumes are laminar, why is there such a difference between the estimate ( $w_o \approx 2$  m/s) and the visual observations ( $w_o \approx 0.3$  m/s)? First, relation (37) gives an order of magnitude of  $w_o$  and a refined analysis might result in lower predicted values when taking into account numerical factors which have been omitted here. On the other hand, the small value of  $\delta$  implies that only a very small volume of liquid moves with a velocity  $\geq 1$  m/s for instance, when  $w_o = 2$  m/s. This fact combined with the rather low sensitivity of the thermally induced refractive index gradients and the global and inaccurate character of direct visual observation may explain a severe underestimate of the maximum velocity of a sharp profile.

In conclusion, we feel that the arguments developed in this work and the preliminary observations in a viscous liquid strongly support the basic feature of a very thin charged plume convecting the charge injected by a needle at a rather high velocity. Of course this has to be confirmed by a refined analysis and a more detailed experimental investigation.

Relations (34) and (37) give  $w_o$  as a function of  $I$  and  $E_o$ . This is only a part of the studied problem; in order to solve it completely, *i.e.* to deduce the field modified by space charge and then the characteristic relation between current and voltage, it is necessary to add Poisson's equation. This leads to a problem which has no analytic solution even after making some rough approximations and simplifications. In an attempt to derive an expression for the current  $I$ , we assume that the electric field  $E_o(0)$  on the needle tip remains very close to  $E_{th}$  whatever the current  $I$  may be. Expressing the charge  $Q = I/w_o$  with (7) for instance one obtains

$$E_{th} = \frac{2V}{r_o \ln(4d/r_o)} - \left[ \frac{\eta I}{8\pi\epsilon^2} \right]^{1/2} \int_0^d \frac{g(x)}{[E(x)]^{1/2}} dx \quad (41)$$

where  $g(x)$  expresses the influence of the charge  $Q(x)$  in the given geometry. The main difficulty here is to determine a good approximation for  $g(x)$ . Without such an expression it is not possible to make predictions for  $I$ , in particular as a function of  $d$ . The problem of determining  $g(x)$  and then to calculate  $E(z)$  and  $I(V)$  requires further investigations.

Comparison between experiments and theory to be made is, up to now, somewhat impossible to do, because what is easily obtained experimentally (the  $I - (V, d)$  characteristic curves) corresponds to a very difficult analytical problem to solve, and, conversely, what can be obtained by theoretical relations (velocity profile and width of the plume) is hard to determine experimentally because of the unsteadiness of the plume.

In order to refine the analysis, we need new experimental results allowing for the determination of  $w_o$  and of the velocity profile. This might be done using particle tracking analysis [21] but remains not easy in our case of very narrow width. Another experimental measurement which

could be of some help is the determination of the field near the plate by the Kerr effect. This requires complicated deconvolution techniques, for instance the use of an Abel transform as proposed by Zahn *et al.* [22–25]. This technique has just begun to be applied to experiments [24, 25].

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