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Indifference valuation in incomplete binomial models

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Abstract

The indifference valuation problem in incomplete binomial models is analyzed. The model is more general than the ones studied so far, because the stochastic factor, which generates the market incompleteness, may affect the transition propabilities and/or the values of the traded asset as well as the claim's payoff. Two pricing algorithms are constructed which use, respectively, the minimal martingale and the minimal entropy measures. We study in detail the interplay among the different kinds of market incompleteness, the pricing measures and the price functionals. The dependence of the prices on the choice of the trading horizon is discussed. The family of "almost complete" (reduced) binomial models is also studied. It is shown that the two measures and the associated price functionals coincide, and that the effects of the horizon choice dissipate.

1. Introduction

This paper is a contribution to indifference valuation in incomplete binomial models under exponential preferences. Market incompleteness stems from the presence of a stochastic factor which may affect the transition probabilities of the traded asset or/and its values. It may also affect the payoff of the claim in consideration. The model is, thus, more general than all binomial models considered so far in exponential indifference valuation (see, among others, [1], [18] and [29]).

The aim is to construct valuation algorithms for the indifference prices and provide a detailed study of their properties and structure. We construct two such algorithms. They are both iterative and resemble the ones introduced in [29] and [18]. However, all existing pricing schemes are applicable only when the stochastic factor affects exclusively the claim's payoff. When the factor affects the dynamics and/or the values of the traded asset, the situation is much more complex, for internal market incompleteness emerges which needs to be priced together with the one coming from the claim's payoff. The algorithms herein exhibit how the pricing of both kinds of incompleteness is carried out and the interplay among the incompleteness, the pricing measures and the price functionals.

In both algorithms, the indifference price is calculated via iterative valuation schemes which are applied backwards in time, starting at the claim's maturity. The schemes have local and dynamic properties. Dynamically, the associated pricing functionals are similar in that, at each time interval, the price is computed via the single-step pricing operators, applied to the end of the period payoff. The latter turns out to be the indifference price at the next time step,

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yielding prices consistent across times. Locally, valuation is executed in two steps, in analogy to the single-period counterpart (see (3.4)). In the first sub-step, the end of the period payoff is altered via a non-linear functional and the conditioning on the information generated by an appropriately chosen filtration. The new intermediate payoff is in turn, priced by expectation.

There are important differences between the algorithms, both with respect to the pricing measures and the form of the non-linear price functionals. The first algorithm uses the minimal martingale measure. This measure has the intuitively pleasing property of preserving the conditional distribution of the stochastic factor, given the stock price, in terms of its historical counterpart. However, the form of the associated pricing functional has no apparent natural structure. The situation is reversed in the second algorithm which uses the minimal entropy measure. We show that the density of this measure has no intuitively pleasing form, in contrast to the relevant functional which does.

The forms of the non-linear pricing functionals motivate us to investigate two important questions. Firstly, we study whether these functionals provide a natural extension to the classical static certainty equivalent pricing rule. We show that both price functionals fail to provide such connection. Secondly, we study how the indifference prices are affected by the choice of the trading horizon, the point at which the underlying exponential utility is pre-specified. We show that prices are significantly affected by the horizon choice and provide this "horizon disparity" in closed form.

Lastly, we investigate how the above results simplify when the model reduces to the one that has been studied so far, i.e. when the stochastic factor affects solely the claim's payoff. We call such a model *reduced.* We show that, as expected, there is a unique pricing measure, for the nested model is now complete. We also show that the price functionals become identical. A direct and important consequence of these simplifications is that the indifference prices are now independent on the choice of the trading horizon.

Besides our findings on indifference prices, we also provide results for the minimal martingale and the minimal entropy measures. Both these measures have been extensively analyzed by a number of authors and for more general market settings. However, our model-specific results are, to the best of our knowledge, new and provide interesting perspectives on the structure and relation of these two martingale measures. We compute the densities in closed form. We also construct, through an iterative scheme, the so-called aggregate minimal entropy process which plays a central role in the representation of the value function process and the indifference price, as well as the "quantification" of market incompleteness.

We were motivated to consider this binomial framework and study the problems at hand for various reasons. Firstly, indifference valuation has by now become one of the central theoretical pricing methodologies in incomplete markets[1]. It is based on fundamental economic principles which are universally applicable, independently of both the individual utility function and the market model. However, the underlying maximal expected utility problems are so complex that it is very difficult to extract any information about the form of the prices, let alone to even prove existence and uniqueness of solutions to these problems (see, among others, [13], [14], [24] and [32]).

More transparent results for indifference prices have been obtained when risk preferences are exponential. Indeed, for this class of utilities, certain additive scaling properties with respect to the wealth argument facilitate the solution of the underlying optimization problems and, in turn, the construction of exponential indifference prices. There is a plethora of results for continuous-time models, derived either using duality theory or PDE techniques for Markovian models (see, among others, [4], [13], [9], [17] and [23]). In some simple cases - specifically, when the nested model is complete - indifference prices can be constructed explicitly ([18]).

[1]For a concise exposition of the theory of indifference prices, we refer the reader to the recent book [2].

Obtaining explicit representations for indifference prices is desirable, for it helps us to carry out sensitivity analysis, compare prices and hedging policies for different market opportunity sets, different trading horizons, etc. In continuous-time models such studies are lacking, mainly because explicit solutions are either not available or can be found only for very simple incomplete models in which, however, the essential effects of market incompleteness are not present. The work herein, albeit in a simplified framework, contributes considerably in this direction. The binomial model we consider is tractable while allowing for internal market incompleteness. To our knowledge, this is the only setup in which indifference prices can be calculated so transparently, despite the stochasticity of the market opportunity set. In addition, the model allows us to investigate, among others, the structural properties of the price in terms of the two measures, decompose the price in hedgeable and non-hedgeable parts, explicitly quantify the effects of varying trading horizons, etc.

Despite their theoretical foundation and tractability for specific utilities and market models, the applicability of indifference prices has been so far very limited, if any. There are several reasons for this. Firstly, it is difficult to determine the "utility function" of a certain activity, a desk or, in general, a firm. While there have been some results in this direction in the areas of Decision Analysis, Real Options and Insurance, where the concept of perfect replication is not central, utility specification has not been addressed satisfactorily, if at all, in the area of derivatives. Secondly, determining the indifference price requires solving the underlying expected utility models for general utility functions. There are many challenges for this. From one hand, solving these problems requires knowledge of the mean rate of return of the traded securities. The estimation issues for this input are well known. On the other hand, as it was mentioned earlier, these stochastic optimization problems are typically fully non-linear and degenerate and, for this, no general theory can be applied, even in order to establish the mere existence and uniqueness of their solution. This poses, in turn, many difficulties for the numerical computation of the latter (see, for example, [32]).

An interesting direction of research would be to compare the indifference prices herein to other prices which are obtained by alternative criteria based, for example, on linear pricing rules that use one of the (many) martingale measures. We initiate this line by comparing our results to the ones in [9] and [11].

Lastly, we mention that the next task is the specification of the associated (indifference) hedging strategies. These are naturally defined as the difference between the optimal investment policies with and without the claim at hand. However, constructing this pair of policies is rather difficult, given the complexity of the underlying expected utility problems. In addition, two challenging questions arise. Firstly, is there an analogue of a "payoff decomposition" in terms of its indifference price, indifference hedge and a residual term? Secondly, what is the role of the latter and, in particular, what is its indifference price as seen as a claim? Both question could be potentially important in practical applications where indifference prices and indifference decomposition could be used for higher order approximations. These questions have been addressed in simple continuous-time models (see, for example, [17]) but not in more complex incomplete models.

The paper is organized as follows. In section 2, we introduce the incomplete (non-reduced) model and provide results on the two pricing measures and the exponential value function process. In section 3, we construct the valuation algorithms and discuss their properties. We also investigate the analogies of the price functionals with the static certainty equivalent and their dependence on the point at which risk preferences are (pre)set. In section 4 we analyze the reduced binomial models. We conclude with section 5 where we provide numerical results.

2. The model and auxiliary results on the pricing measures

In a trading horizon, $[0, T]$, two securities are available for trading, a riskless bond and a risky stock. The time T is arbitrary but fixed. The bond offers zero interest rate. The values of the stock, denoted by S_t, $t = 0, 1, ..., T$, satisfy $S_t > 0$ and are given by

$$\xi_{t+1} = \frac{S_{t+1}}{S_t}, \quad \xi_{t+1} = \xi_{t+1}^d, \ \xi_{t+1}^u \quad \text{with} \quad 0 < \xi_{t+1}^d < 1 < \xi_{t+1}^u. \tag{2.1}$$

Incompleteness is generated by a non-traded factor, denoted by Y_t, $t = 0, 1, ..., T$, whose levels satisfy $Y_t \neq 0$ and are given by

$$\eta_{t+1} = \frac{Y_{t+1}}{Y_t}, \ \eta_{t+1} = \eta_{t+1}^d, \ \eta_{t+1}^u \quad \text{with} \ \ 0 < \eta_{t+1}^d < \eta_{t+1}^u. \tag{2.2}$$

We, then, view $\{(S_t, Y_t) : t = 0, 1, ...\}$ as a two-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The filtration $\mathcal{F}_t$ is generated by the random variables S_i and Y_i, or, equivalently, by ξ_i and η_i, for $i = 0, 1, ..., T$. We, also, consider the filtration $\mathcal{F}_t^S$ generated only by S_i, for $i = 0, 1, ..., T$. The real (historical) probability measure on Ω and $\mathcal{F}$ is denoted by $\mathbb{P}$.

We assume that the values ξ_{t+1}^d, ξ_{t+1}^u of the $\mathcal{F}_{t+1}-$measurable random variable ξ_{t+1} satisfy

$$\xi_{t+1}^d \in \mathcal{F}_t \qquad \text{and} \quad \xi_{t+1}^u \in \mathcal{F}_t \,. \tag{2.3}$$

An investor starts at $t = 0, 1, ..., T$ with initial endowment $X_t = x \in \mathbb{R}$ and trades between the stock and the bond, following self-financing strategies. The number of shares held in his portfolio over the time period $[i-1, i)$, $i = t+1, t+2, ..., T$, is denoted by α_i. It is throughout assumed that $\alpha_i \in \mathcal{F}_{i-1}$. The individual's aggregate wealth is, then, given by

$$X_s = x + \sum_{i=t+1}^{s} \alpha_i \bigtriangleup S_i, \tag{2.4}$$

where $\bigtriangleup S_i = S_i - S_{i-1}$ and $s = t+1, ..., T$.

The performance of the implemented investment strategies is measured via an expected exponential utility criterion applied to the terminal wealth that these portfolios generate. The maximal expected utility (value function) is, then, given by the solution of the stochastic optimization problem

$$V_t(x) = \sup_{\alpha_{t+1}, \cdots, \alpha_T} E_{\mathbb{P}} \left(-e^{-\gamma X_T} \left| \mathcal{F}_t \right. \right), \tag{2.5}$$

$t = 0, 1, ..., T$ with $\gamma > 0$ and X_T as in (2.4), $X_t = x$. This process has been extensively analyzed for general market settings (see, for example, [4], [7], [13] and [23]).

The goal is to carry out a detailed study of the indifference prices under the preference criterion (2.5). We stress that the model we consider is quite more general than the binomial models that have been, so far, analyzed in the context of indifference valuation[2]; see, among others, [5], [18], [29] and [30]. Indeed, in these works, the nested model is complete, with the non-traded factor affecting only the claim's payoff but not the transition probability or the values of the traded asset. In such "almost complete" models, considerable simplifications take place. We revisit these cases in section 5.

In the extended framework herein, additional pricing features emerge due to the *internal market incompleteness*. For their study, we will employ the minimal martingale and the minimal entropy measures. As the analysis will show, these measures turn out to be natural pricing ingredients, for they clearly expose how this incompleteness is processed by indifference valuation.

[2]While preparing the final version of this manuscript, the recent paper [16] was brought to the attention of the authors. Therein, the utility is of power type and the model more general than the one considered herein.

We start with some key properties for their densities of the minimal martingale and the minimal entropy measures and a parity result between them. To our knowledge, for the binomial model at hand, these results are new.

To this end, we let $\mathcal{Q}_T$ be the set of martingale measures restricted on $\mathcal{F}_T$. With a slight abuse of notation, we denote by $\mathbb{Q}$ its generic element.

We introduce, for $t = 0, 1, ..., T$, the sets

$$A_t = \{\omega : \xi_t(\omega) = \xi_t^u\} \quad \text{and} \quad B_t = \{\omega : \eta_t(\omega) = \eta_t^u\}. \tag{2.6}$$

Note that for all $\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}_T$,

$$\mathbb{Q}(A_t | \mathcal{F}_{t-1}) = \mathbb{Q}'(A_t | \mathcal{F}_{t-1}). \tag{2.7}$$

Definition 2.1. Let ξ_t, $t = 0, 1, .., T$, be as in (2.1) and consider the risk neutral probabilities

$$q_t = \frac{1 - \xi_t^d}{\xi_t^u - \xi_t^d}.$$

The local entropy process h_t, $t = 1, ..., T$, is defined by

$$h_t = q_t \ln \frac{q_t}{\mathbb{P}(A_t | \mathcal{F}_{t-1})} + (1 - q_t) \ln \frac{1 - q_t}{1 - \mathbb{P}(A_t | \mathcal{F}_{t-1})}, \tag{2.8}$$

where A_t as in (2.6), $\mathbb{P}$ is the historical probability measure and $\mathcal{F}_t$ is the filtration generated by the random variables S_i and Y_i, for $i = 0, 1, ..., T$.

Lemma 2.2. *The local entropy process h_t is $\mathcal{F}_t$-predictable, i.e., for $t = 1, ..., T$, $h_t \in \mathcal{F}_{t-1}$. Moreover, for all $\mathbb{Q} \in \mathcal{Q}_T$,*

$$h_t = \mathbb{Q}(A_t | \mathcal{F}_{t-1}) \ln \frac{\mathbb{Q}(A_t | \mathcal{F}_{t-1})}{\mathbb{P}(A_t | \mathcal{F}_{t-1})} + (1 - \mathbb{Q}(A_t | \mathcal{F}_{t-1})) \ln \frac{1 - \mathbb{Q}(A_t | \mathcal{F}_{t-1})}{1 - \mathbb{P}(A_t | \mathcal{F}_{t-1})}.$$

We recall the elementary fact

$$\mathbb{Q}(\xi_{t+1}, ..., \xi_T, \eta_{t+1}, ..., \eta_T | \mathcal{F}_t) = \prod_{s=t}^{T-1} \mathbb{Q}(\xi_{s+1}, \eta_{s+1} | \mathcal{F}_s),$$

which gives the useful simplification

$$\ln \frac{\mathbb{Q}(\xi_{t+1}, ..., \xi_T, \eta_{t+1}, ..., \eta_T | \mathcal{F}_t)}{\mathbb{P}(\xi_{t+1}, ..., \xi_T, \eta_{t+1}, ..., \eta_T | \mathcal{F}_t)} = \sum_{s=t}^{T-1} \ln \frac{\mathbb{Q}(\xi_{s+1}, \eta_{s+1} | \mathcal{F}_s)}{\mathbb{P}(\xi_{s+1}, \eta_{s+1} | \mathcal{F}_s)}. \tag{2.9}$$

2.1. **The minimal martingale measure**

The minimal martingale measure, $\mathbb{Q}^{mm}(\cdot | \mathcal{F}_t)$, $t = 1, ..., T$, is defined on $\mathcal{F}_T$ as the minimizer of $H_{t,T}^{mm}$, where

$$H_{t,T}^{mm}(\mathbb{Q}(\cdot | \mathcal{F}_t) \mid \mathbb{P}(\cdot | \mathcal{F}_t)) = E_{\mathbb{P}}\left(-\ln \frac{\mathbb{Q}(\cdot | \mathcal{F}_t)}{\mathbb{P}(\cdot | \mathcal{F}_t)} | \mathcal{F}_t\right),$$

for $t = 1, ..., T$ and $\mathbb{Q} \in \mathcal{Q}_T$, i.e.

$$H_{t,T}^{mm}(\mathbb{Q}^{mm}(\cdot | \mathcal{F}_t) \mid \mathbb{P}(\cdot | \mathcal{F}_t)) = \min_{\mathbb{Q} \in \mathcal{Q}_T} H_{t,T}^{mm}(\mathbb{Q}(\cdot | \mathcal{F}_t) \mid \mathbb{P}(\cdot | \mathcal{F}_t)). \tag{2.10}$$

It was introduced in [6] (see, also among others, [25], [26], [28] and [15]).

The next result highlights an important property of the minimal martingale measure. Specifically, it shows that under this measure, the conditional distribution of the non-traded factor process, given the stock price, is preserved in relation to its historical counterpart[3].

[3]For the single period case, see [18].

Proposition 2.3. *The minimal martingale measure has the property*

$$\mathbb{Q}^{mm}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)=\mathbb{P}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right), \tag{2.11}$$

for $t=1,...,T,$ *or, equivalently,*

$$\frac{\mathbb{Q}^{mm}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{mm}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{mm}\left(A_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)} \tag{2.12}$$

and

$$\frac{\mathbb{Q}^{mm}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{mm}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{mm}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\,\mathcal{F}_{t-1}\right)},$$

with the sets A_t,B_t *as in (2.6).*

Proof. Since the rest of the proof follows along similar arguments, we only show that

$$\frac{\mathbb{Q}^{mm}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{mm}\left(A_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}. \tag{2.13}$$

We use induction. At $t=T$,

$$\begin{aligned}E_{\mathbb{P}}\left(-\ln\frac{\mathbb{Q}\left(\xi_T,\eta_T\,|\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(\xi_T,\eta_T\,|\mathcal{F}_{T-1}\right)}\,|\mathcal{F}_{T-1}\right)&=-\mathbb{P}\left(A_TB_T|\,\mathcal{F}_{T-1}\right)\ln\frac{\mathbb{Q}\left(A_TB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_TB_T|\,\mathcal{F}_{T-1}\right)}\\&-\mathbb{P}\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)\ln\frac{\mathbb{Q}\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)}\\&-\mathbb{P}\left(A_TB_T^c|\,\mathcal{F}_{T-1}\right)\ln\frac{\mathbb{Q}\left(A_T|\,\mathcal{F}_{T-1}\right)-\mathbb{Q}\left(A_TB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_TB_T^c|\,\mathcal{F}_{T-1}\right)}\\&-\mathbb{P}\left(A_T^cB_T^c|\,\mathcal{F}_{T-1}\right)\ln\frac{\mathbb{Q}\left(A_T^c|\,\mathcal{F}_{T-1}\right)-\mathbb{Q}\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_T^cB_T^c|\,\mathcal{F}_{T-1}\right)},\end{aligned}$$

and direct differentiation yields the claimed equality. Next, we assume that (2.13) holds for $t+1,..,T$ and show its validity for t. We have

$$E_{\mathbb{P}}\left(-\ln\frac{\mathbb{Q}\left(\cdot|\,\mathcal{F}_t\right)}{\mathbb{P}\left(\cdot|\,\mathcal{F}_t\right)}\middle|\,\mathcal{F}_t\right)$$

$$=-E_{\mathbb{P}}\left(\ln\left(\prod_{i=t+1}^{T-1}\frac{\mathbb{Q}\left(\xi_{i+1},\eta_{i+1}|\,\mathcal{F}_i\right)}{\mathbb{P}\left(\xi_{i+1},\eta_{i+1}|\,\mathcal{F}_i\right)}\right)\middle|\,\mathcal{F}_t\right)-E_{\mathbb{P}}\left(\ln\frac{\mathbb{Q}\left(\xi_{t+1},\eta_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(\xi_{t+1},\eta_{t+1}|\,\mathcal{F}_t\right)}\middle|\,\mathcal{F}_t\right).$$

Combining the single-period arguments used to establish (2.13) for $t=T$ and the fact that the second term above depends only on $\mathbb{Q}\left(\xi_{t+1}\eta_{t+1}|\,\mathcal{F}_t\right)$, we easily conclude. □

The above property can be, also, deduced from existing results on the minimal martingale measure. As an example, we consider its explicit characterization as derived in [26]. Therein, it is shown that

$$\left.\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}}\right|_{\mathcal{F}_T}=\prod_{t=1}^{T}\frac{1-\lambda_t\left(S_t-S_{t-1}\right)}{1-\lambda_t\left(m_t-m_{t-1}\right)}, \tag{2.14}$$

where

$$\lambda_t=\frac{m_t-m_{t-1}}{E_{\mathbb{P}}\left(\left(S_t-S_{t-1}\right)^2\middle|\,\mathcal{F}_{t-1}\right)}$$

and $m_t-m_{t-1}=E_{\mathbb{P}}\left(S_t-S_{t-1}|\,\mathcal{F}_{t-1}\right),\ m_0=0$ (see, also, [15]).

We, then, easily obtain the following result.

Corollary 2.4. *The representations (2.11) and (2.14) are equivalent.*

Proof. We only show that (2.11) implies (2.14). Using (2.12) and (2.9) we have

$$\left.\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \prod_{i=1}^{T} \frac{\mathbb{Q}^{mm}\left(\xi_i\right|\mathcal{F}_{i-1})}{\mathbb{P}\left(\xi_i\right|\mathcal{F}_{i-1})}.$$

Let $\Delta S_t = S_t - S_{t-1}$ and $\Delta A_t = m_t - m_{t-1}$. Because

$$\frac{\mathbb{Q}^{mm}\left(\xi_i\right|\mathcal{F}_{i-1})}{\mathbb{P}\left(\xi_i\right|\mathcal{F}_{i-1})} = \begin{cases} \frac{1-\xi_i^d}{\mathbb{P}\left(\xi_i=\xi_i^u\right|\mathcal{F}_{i-1})(\xi_i^u-\xi_i^d)}, & \xi_i = \xi_i^u, \\ \\ \frac{\xi_i^u-1}{(1-\mathbb{P}\left(\xi_i=\xi_i^u\right|\mathcal{F}_{i-1}))(\xi_i^u-\xi_i^d)}, & \xi_i = \xi_i^d, \end{cases}$$

it remains to show that the ratio $\frac{1-\lambda_t\Delta S_t}{1-\lambda_t\Delta m_t}$ equals the right-hand side of the above equality. Direct calculations yield that

$$\frac{1-\lambda_t\Delta S_t}{1-\lambda_t\Delta A_t} = \frac{E_{\mathbb{P}}\left(\left(\Delta S_t\right)^2\middle|\,\mathcal{F}_{t-1}\right) - \left(\Delta S_t\right)\left(\Delta m_t\right)}{E_{\mathbb{P}}\left(\left(\Delta S_t\right)^2\middle|\,\mathcal{F}_{t-1}\right) - \left(\Delta m_t\right)^2}.$$

On the other hand,

$$\begin{gathered} E_{\mathbb{P}}\left(\left(\Delta S_t\right)^2\middle|\,\mathcal{F}_{t-1}\right) - \left(\Delta m_t\right)^2 \\ = \mathbb{P}(\xi_t = \xi_t^u|\,\mathcal{F}_{t-1})(1-\mathbb{P}(\xi_t = \xi_t^u|\,\mathcal{F}_{t-1}))(\xi_t^u - \xi_t^d)^2. \end{gathered}$$

Moreover, on the sets A_t and A_t^c we have

$$\begin{gathered} E_{\mathbb{P}}\left(\left(\Delta S_t\right)^2\middle|\,\mathcal{F}_{t-1}\right) - \left(\Delta S_t\right)\left(\Delta m_t\right) \\ = \begin{cases} (1-\mathbb{P}(\xi_t = \xi_t^u|\,\mathcal{F}_{t-1}))(\xi_t^d - \xi_t^u)(\xi_t^d - 1), & \xi_t = \xi_t^u, \\ \\ \mathbb{P}(\xi_t = \xi_t^u|\,\mathcal{F}_{t-1})(\xi_t^u - \xi_t^d)(\xi_t^u - 1), & \xi_t = \xi_t^d. \end{cases} \end{gathered}$$

Combining the above we easily conclude. □

2.2. **The minimal entropy measure**

The minimal entropy measure, $\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right)$, is defined on $\mathcal{F}_T$ as the minimizer of $H^{me}_{t,T}$, where

$$H^{me}_{t,T}\left(\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)|\;\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = E_{\mathbb{Q}}\left(\ln\frac{\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)}{\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right),$$

for $t = 1, ..., T$, and $\mathbb{Q} \in \mathcal{Q}_T$, i.e.,

$$H^{me}_{t,T}\left(\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right)|\;\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = \min_{\mathbb{Q}\in\mathcal{Q}_T} H^{me}_{t,T}\left(\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)|\;\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right). \tag{2.15}$$

We refer the reader to [7] (see, also, [4], [8], [13] and [23]) for its properties and the role of this measure in stochastic optimization problems of exponential utility.

To facilitate the presentation, we will be using the condensed notation

$$\mathcal{H}^{me}_{t,T} = H^{me}_{t,T}\left(\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right)|\;\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) \tag{2.16}$$

and referring to $\mathcal{H}^{me}_{t,T}$ as the *minimal aggregate entropy.*

Next, we provide an explicit representation for the minimal entropy measure which, to the best of our knowledge, is new. The construction is based on an iterative procedure which yields the conditional distribution $\mathbb{Q}^{me}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}^S_t\right)$ in terms of its historical counterpart $\mathbb{P}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}^S_t\right)$ and the conditional on $\mathcal{F}_{t-1}$ minimal aggregate entropy $\mathcal{H}^{me}_{t-1,T}$. The latter term is constructed through an independent iterative procedure which involves the minimal martingale measure. To ease the presentation, we present the construction of $\mathcal{H}^{me}_{t,T}$ separately (see Proposition 9).

We stress that the arguments below are recursive but not tautological because the construction of $\mathbb{Q}^{me}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)$ (and similarly for the sets $A_tB_t^c$, $A_t^cB_t$ and $A_t^cB_t^c$) involves the values of $\mathcal{H}^{me}_{t,T}$ and *not* $\mathcal{H}^{me}_{t-1,T}$.

Proposition 2.5. *The minimal entropy measure satisfies, for $t=1,...,T$,*

$$\frac{\mathbb{Q}^{me}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(\left.A_t\right|\mathcal{F}_{t-1}\right)}=\frac{\frac{\mathbb{P}\left(A_tB_t|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,uu}_{t,T}}}{\frac{\mathbb{P}\left(A_tB_t|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,uu}_{t,T}}+\frac{\mathbb{P}\left(A_tB_t^c|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,ud}_{t,T}}}, \tag{2.17}$$

$$\frac{\mathbb{Q}^{me}\left(\left.A_tB_t^c\right|\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(\left.A_t\right|\mathcal{F}_{t-1}\right)}=\frac{\frac{\mathbb{P}\left(A_tB_t^c|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,ud}_{t,T}}}{\frac{\mathbb{P}\left(A_tB_t|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,uu}_{t,T}}+\frac{\mathbb{P}\left(A_tB_t^c|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,ud}_{t,T}}},$$

$$\frac{\mathbb{Q}^{me}\left(\left.A_t^cB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(\left.A_t^c\right|\mathcal{F}_{t-1}\right)}=\frac{\frac{\mathbb{P}\left(A_t^cB_t|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,du}_{t,T}}}{\frac{\mathbb{P}\left(A_t^cB_t|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,du}_{t,T}}+\frac{\mathbb{P}\left(A_t^cB_t^c|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,dd}_{t,T}}}$$

and

$$\frac{\mathbb{Q}^{me}\left(\left.A_t^cB_t^c\right|\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(\left.A_t^c\right|\mathcal{F}_{t-1}\right)}=\frac{\frac{\mathbb{P}\left(A_t^cB_t^c|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,dd}_{t,T}}}{\frac{\mathbb{P}\left(A_t^cB_t|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,du}_{t,T}}+\frac{\mathbb{P}\left(A_t^cB_t^c|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}^{me,dd}_{t,T}}},$$

where A_t, B_t are as in (2.6) and $\mathcal{H}^{me,uu}_{t,T}, \mathcal{H}^{me,ud}_{t,T}, \mathcal{H}^{me,du}_{t,T}, \mathcal{H}^{me,dd}_{t,T}$ are the values of the $\mathcal{F}_t-$measurable random variable $\mathcal{H}^{me}_{t,T}$, (cf. (2.16)), conditional on $\mathcal{F}_{t-1}$. The explicit form of $\mathcal{H}^{me}_{t,T}$ is given in Proposition 9.

Proof. We only show (2.17) since the rest can be proved along similar arguments. We first observe that

$$E_{\mathbb{Q}}\left(\ln\frac{\mathbb{Q}\left(\cdot\,|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\cdot\,|\mathcal{F}_{t-1}\right)}\,|\mathcal{F}_{t-1}\right)=E_{\mathbb{Q}}\left(\ln\frac{\mathbb{Q}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)}\,|\mathcal{F}_{t-1}\right)$$
$$+E_{\mathbb{Q}}\left(E_{\mathbb{Q}}\left(\ln\left(\prod_{i=t+1}^{T}\frac{\mathbb{Q}\left(\xi_i,\eta_i\,|\mathcal{F}_{i-1}\right)}{\mathbb{P}\left(\xi_i,\eta_i\,|\mathcal{F}_{i-1}\right)}\right)\middle|\,\mathcal{F}_t\right)\middle|\,\mathcal{F}_{t-1}\right).$$

Recalling the definition of the minimal entropy measure (cf. (2.15)), the first term needs to be minimized over $\mathbb{Q}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)$; in the second term, we first need to minimize the nested conditional expectation over $\mathbb{Q}\left(\xi_i,\eta_i\,|\mathcal{F}_{i-1}\right)$, $i=t+1,...,T$, and, in turn, the outer expectation over $\mathbb{Q}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)$. Using (2.15), we deduce that it suffices to calculate

$$\min_{\mathbb{Q}\left(\xi_t\eta_t|\mathcal{F}_{t-1}\right)}\left(E_{\mathbb{Q}}\left(\ln\frac{\mathbb{Q}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)}\,|\mathcal{F}_{t-1}\right)+E_{\mathbb{Q}}\left(\left.\mathcal{H}^{me}_{t,T}\right|\mathcal{F}_{t-1}\right)\right).$$

Expanding yields

$$E_{\mathbb{Q}}\left(\ln\frac{\mathbb{Q}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\xi_t,\eta_t\,|\mathcal{F}_{t-1}\right)}\,|\mathcal{F}_{t-1}\right)+E_{\mathbb{Q}}\left(\left.\mathcal{H}^{me}_{t,T}\right|\mathcal{F}_{t-1}\right)$$
$$=\mathbb{Q}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\ln\frac{\mathbb{Q}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}+\mathbb{Q}\left(\left.A_t^cB_t\right|\mathcal{F}_{T-1}\right)\ln\frac{\mathbb{Q}\left(\left.A_t^cB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left.A_t^cB_t^c\right|\mathcal{F}_{t-1}\right)}$$
$$+\left(\mathbb{Q}\left(\left.A_t\right|\mathcal{F}_{t-1}\right)-\mathbb{Q}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\right)\ln\frac{\mathbb{Q}\left(\left.A_t\right|\mathcal{F}_{t-1}\right)-\mathbb{Q}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left.A_tB_t^c\right|\mathcal{F}_{t-1}\right)}$$
$$+\left(\mathbb{Q}\left(\left.A_t^c\right|\mathcal{F}_{t-1}\right)-\mathbb{Q}\left(\left.A_t^cB_t\right|\mathcal{F}_{t-1}\right)\right)\ln\frac{\mathbb{Q}\left(\left.A_t^c\right|\mathcal{F}_{t-1}\right)-\mathbb{Q}\left(\left.A_t^cB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left.A_t^cB_t^c\right|\mathcal{F}_{t-1}\right)}$$
$$+\mathbb{Q}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\mathcal{H}^{me,uu}_{t,T}+\left(\mathbb{Q}\left(\left.A_t\right|\mathcal{F}_{t-1}\right)-\mathbb{Q}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\right)\mathcal{H}^{me,ud}_{t,T}$$

$$+\mathbb{Q}\left(A_t^d B_t\middle|\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{me,du}+\left(\mathbb{Q}\left(A_t^c\middle|\mathcal{F}_{t-1}\right)-\mathbb{Q}\left(A_t^c B_t\middle|\mathcal{F}_{t-1}\right)\right)\mathcal{H}_{t,T}^{me,dd}.$$

Differentiating with respect to $\mathbb{Q}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)$ and rearranging terms yields that at the optimum

$$\ln\frac{\mathbb{Q}^{me}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)}-\ln\frac{\mathbb{Q}^{me}\left(A_t\middle|\mathcal{F}_{t-1}\right)-\mathbb{Q}^{me}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t^c\middle|\mathcal{F}_{t-1}\right)}$$
$$+\mathcal{H}_{t,T}^{me,uu}-\mathcal{H}_{t,T}^{me,ud}=0,$$

and we conclude. $\square$

Remark 2.6. To preserve the analogy with formulae (2.13) and (2.12), we provide an alternative to (2.17) representation. For $t=1,...,T$, we have

$$\frac{\mathbb{Q}^{me}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{me}\left(A_t\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,uu}}}{\mathbb{P}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,uu}}+\mathbb{P}\left(A_tB_t^c\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,ud}}} \tag{2.18}$$

$$\frac{\mathbb{Q}^{me}\left(A_tB_t^c\middle|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t^c\middle|\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{me}\left(A_t\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,ud}}}{\mathbb{P}\left(A_tB_t\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,uu}}+\mathbb{P}\left(A_tB_t^c\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,ud}}}$$

$$\frac{\mathbb{Q}^{me}\left(A_t^cB_t\middle|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t\middle|\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{me}\left(A_t^c\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,du}}}{\mathbb{P}\left(A_t^cB_t\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,du}}+\mathbb{P}\left(A_t^cB_t^c\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,dd}}}$$

and

$$\frac{\mathbb{Q}^{me}\left(A_t^cB_t^c\middle|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t^c\middle|\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{me}\left(A_t^c\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,dd}}}{\mathbb{P}\left(A_t^cB_t\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,du}}+\mathbb{P}\left(A_t^cB_t^c\middle|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,dd}}}.$$

Remark 2.7. Notice that if $\mathcal{H}_{t,T}^{me,uu}=\mathcal{H}_{t,T}^{me,ud}$ equality (2.18) reduces to (2.12). This observation will play a key role in the analysis of the reduced binomial model (see Proposition 34 herein).

We continue with an explicit construction of the minimal aggregate entropy $\mathcal{H}_{t,T}^{me}$. We, first, introduce the following nonlinear entropic functionals.

Definition 2.8. Let Z be a random variable on $(\Omega,\mathcal{F},\mathbb{P})$. For $s=0,1,...,T-1$, $t=s+1,...,T$ and $\mathbb{Q}\in\mathcal{Q}_T$, define the single- and multi-step entropic nonlinear functionals

$$\mathcal{J}_{\mathbb{Q}}^{(s,s+1)}(Z)=E_{\mathbb{Q}}\left(\ln E_{\mathbb{Q}}\left(e^Z\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)|\mathcal{F}_s\right) \tag{2.19}$$

and

$$\mathcal{J}_{\mathbb{Q}}^{(s,t)}(Z)=\mathcal{J}_{\mathbb{Q}}^{(s,s+1)}\left(\mathcal{J}_{\mathbb{Q}}^{(s+1,s+2)}\left(...\mathcal{J}_{\mathbb{Q}}^{(t-1,t)}(Z)\right)\right). \tag{2.20}$$

Herein, $\mathcal{F}_s$ and $\mathcal{F}_s^S$ are the filtrations generated, respectively, by (S_i,Y_i) and S_i for $i=1,...,s$.

We are now ready to provide the iterative algorithm for the minimal aggregate entropy. Notice that the involved measure is the minimal martingale one, given in Proposition 3.

Proposition 2.9. *The minimal aggregate entropy is given by*

$$\mathcal{H}_{T,T}^{me}=0\qquad\text{and}\quad\mathcal{H}_{T-1,T}^{me}=h_T \tag{2.21}$$

and, for $t=0,1,...,T-2$, *by the iterative schemes*

$$\mathcal{H}_{t,T}^{me}=h_{t+1}-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(-\mathcal{H}_{t+1,T}^{me}\right) \tag{2.22}$$

and

$$\mathcal{H}_{t,T}^{me}=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}\left(-\sum_{i=t+1}^{T}h_i\right). \tag{2.23}$$

Herein h_{t+1} *is defined in (2.8), and* $\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}$ *and* $\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}$ *are given, respectively, in (2.19) and (2.20) with the measure* $\mathbb{Q}^{mm}$ *being used.*

Proof. The first equality in (2.21) is immediate while the second one follows from the definition of $\mathcal{H}^{me}_{T-1,T}$ and h_T. We now establish (2.22) for $t = 0, 1, ..., T-2$, i.e., that

$$\mathcal{H}^{me}_{t,T} = h_{t+1} - E_{\mathbb{Q}^{mm}}\left(\ln E_{\mathbb{Q}^{mm}}\left(e^{-\mathcal{H}^{me}_{t+1,T}}\,\middle|\,\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right)|\mathcal{F}_t\right). \tag{2.24}$$

We have

$$\begin{aligned}\mathcal{H}^{me}_{t,T} &= E_{\mathbb{Q}^{me}}\left(\ln\frac{\mathbb{Q}^{me}\left(\xi_{t+1},\eta_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right)\\ &+E_{\mathbb{Q}^{me}}\left(E_{\mathbb{Q}^{me}}\left(\ln\prod_{i=t+2}^{T}\frac{\mathbb{Q}^{me}\left(\xi_i,\eta_i|\,\mathcal{F}_{i-1}\right)}{\mathbb{P}\left(\xi_i,\eta_i\,|\mathcal{F}_{i-1}\right)}\,\middle|\,\mathcal{F}_{t+1}\right)|\mathcal{F}_t\right)\\ &= E_{\mathbb{Q}^{me}}\left(\ln\frac{\mathbb{Q}^{me}\left(\xi_{t+1},\eta_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t\right)}\,\middle|\,\mathcal{F}_t\right) + E_{\mathbb{Q}^{me}}\left(\mathcal{H}^{me}_{t+1,T}\,|\mathcal{F}_t\right),\end{aligned}$$

where we used the definition of the aggregate minimal entropy (cf. (2.16)). We introduce the random variables

$$Z^u_t = \frac{\mathbb{P}\left(A_{t+1}B_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1}\,|\mathcal{F}_t\right)}e^{-\mathcal{H}^{me,uu}_{t+1,T}} + \frac{\mathbb{P}\left(A_{t+1}B^c_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1}\,|\mathcal{F}_t\right)}e^{-\mathcal{H}^{me,ud}_{t+1,T}}$$

and

$$Z^d_t = \frac{\mathbb{P}\left(A^c_{t+1}B_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A^c_{t+1}\,|\mathcal{F}_t\right)}e^{-\mathcal{H}^{me,du}_{t+1,T}} + \frac{\mathbb{P}\left(A^c_{t+1}B^c_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A^c_{t+1}\,|\mathcal{F}_t\right)}e^{-\mathcal{H}^{me,dd}_{t+1,T}},$$

where $\mathcal{H}^{me,uu}_{t+1,T}, \mathcal{H}^{me,ud}_{t+1,T}, \mathcal{H}^{me,du}_{t+1,T}$, and $\mathcal{H}^{me,dd}_{t+1,T}$ are the values of the $\mathcal{F}_{t+1}$-measurable random variable $\mathcal{H}^{me}_{t+1,T}$ conditional on $\mathcal{F}_t$.

>From Proposition 5, we have

$$\begin{aligned}\mathcal{H}^{me}_{t,T} &= \mathbb{Q}^{me}\left(A_{t+1}B_{t+1}|\,\mathcal{F}_t\right)\ln\left(\frac{\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)e^{-\mathcal{H}^{me,uu}_{t+1,T}}}{\mathbb{P}\left(A_{t+1}|\,\mathcal{F}_t\right)Z^u_t}\right)\\ &+\mathbb{Q}^{me}\left(A_{t+1}B^c_{t+1}|\,\mathcal{F}_t\right)\ln\left(\frac{\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)e^{-\mathcal{H}^{me,ud}_{t+1,T}}}{\mathbb{P}\left(A_{t+1}|\,\mathcal{F}_t\right)Z^u_t}\right)\\ &+\mathbb{Q}^{me}\left(A^c_{t+1}B_{t+1}|\,\mathcal{F}_t\right)\ln\left(\frac{\mathbb{Q}^{me}\left(A^c_{t+1}|\,\mathcal{F}_t\right)e^{-\mathcal{H}^{me,du}_{t+1,T}}}{\mathbb{P}\left(A^c_{t+1}|\,\mathcal{F}_t\right)Z^d_t}\right)\\ &+\mathbb{Q}^{me}\left(A^c_{t+1}B^c_{t+1}|\,\mathcal{F}_t\right)\ln\left(\frac{\mathbb{Q}^{me}\left(A^c_{t+1}|\,\mathcal{F}_t\right)e^{-\mathcal{H}^{me,dd}_{t+1,T}}}{\mathbb{P}\left(A^c_{t+1}|\,\mathcal{F}_t\right)Z^d_t}\right) + E_{\mathbb{Q}^{me}}\left(\mathcal{H}^{me}_{t+1,T}\,\middle|\,\mathcal{F}_{T-2}\right).\end{aligned}$$

Further simplification and rearrangement of terms yield

$$\begin{aligned}\mathcal{H}^{me}_{t,T} &= -\mathcal{H}^{me,uu}_{t-1,T}\mathbb{Q}^{me}\left(A_{t+1}B_{t+1}|\,\mathcal{F}_t\right) - \mathcal{H}^{me,ud}_{t-1,T}\mathbb{Q}^{me}\left(A_{t+1}B^c_{t+1}|\,\mathcal{F}_t\right)\\ &-\mathcal{H}^{me,du}_{t-1,T}\mathbb{Q}^{me}\left(A^c_{t+1}B_{t+1}|\,\mathcal{F}_t\right) - \mathcal{H}^{me,dd}_{t-1,T}\mathbb{Q}^{me}\left(A^c_{t+1}B^c_{t+1}|\,\mathcal{F}_t\right)\\ &+\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)\left(\ln\frac{\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1}|\,\mathcal{F}_t\right)} - \ln Z^u_t\right)\\ &+\mathbb{Q}^{me}\left(A^c_{t+1}|\,\mathcal{F}_t\right)\left(\ln\frac{\mathbb{Q}^{me}\left(A^c_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A^c_{t+1}|\,\mathcal{F}_t\right)} - \ln Z^d_t\right) + E_{\mathbb{Q}^{me}}\left(\mathcal{H}^{me}_{t-1,T}\,\middle|\,\mathcal{F}_t\right)\\ &= \mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)\ln\frac{\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1}|\,\mathcal{F}_t\right)}\\ &+\left(1-\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)\right)\ln\frac{1-\mathbb{Q}^{me}\left(A_{t+1}|\,\mathcal{F}_t\right)}{1-\mathbb{P}\left(A_{t+1}|\,\mathcal{F}_t\right)}\end{aligned}$$

$$-\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_t\right)\ln Z_t^u-\mathbb{Q}^{me}\left(\left.A_{t+1}^c\right|\mathcal{F}_t\right)\ln Z_t^d.$$

Using (2.8) we obtain

$$\mathcal{H}_{t,T}^{me}=h_{t+1}-\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_t\right)\ln Z_t^u-\mathbb{Q}^{me}\left(\left.A_{t+1}^c\right|\mathcal{F}_t\right)\ln Z_t^d. \tag{2.25}$$

Observe, however, that because of (2.11),

$$Z_t^u=E_{\mathbb{Q}^{mm}}\left(\left.e^{-\mathcal{H}_{t+1,T}^{me}}\right|\mathcal{F}_t\vee A_{t+1}\right)$$

and

$$Z_t^d=E_{\mathbb{Q}^{mm}}\left(\left.e^{-\mathcal{H}_{t+1,T}^{me}}\right|\mathcal{F}_t\vee A_{t+1}^c\right).$$

The above and (2.7) yield

$$\begin{aligned}&\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_t\right)\ln Z_t^u+\mathbb{Q}^{me}\left(\left.A_{t+1}^c\right|\mathcal{F}_t\right)\ln Z_t^d\\&=E_{\mathbb{Q}^{mm}}\left(\left.\ln E_{\mathbb{Q}^{mm}}\left(\left.e^{-\mathcal{H}_{t+1,T}^{me}}\right|\mathcal{F}_t\vee\mathcal{F}_{t+1}^S\right)\right|\mathcal{F}_t\right),\end{aligned}$$

and (2.22) follows.

Assertion (2.23) follows from (2.22) and Lemma 2. □

The following process, yielding the sum of the aggregate entropies, turns out to be useful in establishing parity relations between the two measures.

Corollary 2.10. *For $s=1,..,T,$ define*

$$M_{s,T}=\Sigma_{i=1}^s h_i+\mathcal{H}_{s,T}^{me}. \tag{2.26}$$

Then, $M_{s,T}\in\mathcal{F}_s$ and for $t=s+1,...,T,$

$$M_{s,T}=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,t)}\left(-M_{t,T}\right) \tag{2.27}$$

with $\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,t)}$ given in (2.20).

Proof. We first show (2.27) for $s=t-1$. Using the measurability of h (see Lemma 2), we rewrite (2.22) as

$$\begin{aligned}\Sigma_{i=1}^{t-1}h_i+\mathcal{H}_{t-1,T}^{me}&=\Sigma_{i=1}^{t-1}h_i+h_t-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\mathcal{H}_{t,T}^{me}\right)\\&=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\Sigma_{i=1}^{t}h_i-\mathcal{H}_{t,T}^{me}\right),\end{aligned} \tag{2.28}$$

and (2.27) follows. Using similar arguments, we deduce

$$M_{t-2,T}=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-M_{t-1,T}\right). \tag{2.29}$$

Combining (2.20), (2.28) and (2.29), we obtain (2.27) for $s=t-2$. For $s<t-2$, we proceed similarly. □

2.3. A parity result between the minimal martingale and the minimal entropy measures

In the previous two sections we obtained representations for the densities of the minimal martingale and minimal entropy measures. It is easy to see, by comparing (2.11) to (2.17) and using (2.21), that these measures coincide *only* at expiration,

$$\mathbb{Q}^{mm}\left(\left.Y_T\right|\mathcal{F}_{T-1}\vee\mathcal{F}_T^S\right)=\mathbb{Q}^{me}\left(\left.Y_T\right|\mathcal{F}_{T-1}\vee\mathcal{F}_T^S\right). \tag{2.30}$$

The fact that they differ at previous times, however, has important consequences on the upcoming representations under these measures of both the exponential value functions and the related indifference prices. Understanding how these two measures are related to each other is helpful in exploring the specific features of the pricing algorithms.

>From the computational point of view, it turns out that the connection between the two measures is most transparent through the minimal aggregate entropy, $\mathcal{H}^{me}_{t,T}$, when computed under $\mathbb{Q}^{mm}\left(\cdot|\,\mathcal{F}_t\right)$ and $\mathbb{Q}^{me}\left(\cdot|\,\mathcal{F}_t\right)$, respectively. We explore this next. We start by investigating the connection between the nonlinear entropic functionals $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}$ and $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}}$, and their multi-step analogues $\mathcal{J}^{(t,s)}_{\mathbb{Q}^{mm}}$ and $\mathcal{J}^{(t,s)}_{\mathbb{Q}^{me}}$ (see Definition 8).

Proposition 2.11. *Let $s=0,...,T-1$, and $Z\in\mathcal{F}_{s+1}$. Let also $\mathcal{H}^{me}_{s+1,T}$, $\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ and $M_{s,T}$ be as in (2.16), (2.19) and (2.26), respectively. Then,*

$$\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{me}}\left(Z\right)=\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(Z-\mathcal{H}^{me}_{s+1,T}\right)-\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(-\mathcal{H}^{me}_{s+1,T}\right) \tag{2.31}$$

$$=\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(Z-M_{s+1,T}\right)+M_{s,T}. \tag{2.32}$$

Moreover, for $Z\in\mathcal{F}_t$, and $t=s+1,..,T$,

$$\mathcal{J}^{(s,t)}_{\mathbb{Q}^{me}}\left(Z\right)=\mathcal{J}^{(s,t)}_{\mathbb{Q}^{mm}}\left(Z-M_{t,T}\right)+M_{s,T}. \tag{2.33}$$

Proof. Note that (2.32) follows easily from (2.31) as we can see using the measurability of the local entropy (see Lemma 2) and Corollary 2.10. For this, we only show (2.31) and (2.33). We start with the former. From the definition of $\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ (cf. (2.19)) and using (2.7) and (2.13), we have

$$\begin{aligned}
&\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(Z\right)\\
&=\mathbb{Q}^{mm}\left(A_{s+1}|\,\mathcal{F}_s\right)\ln\left(\frac{\mathbb{Q}^{mm}\left(A_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{mm}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{uu}}+\frac{\mathbb{Q}^{mm}\left(A_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{mm}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{ud}}\right)\\
&+\mathbb{Q}^{mm}\left(A^c_{s+1}|\,\mathcal{F}_s\right)\ln\left(\frac{\mathbb{Q}^{mm}\left(A^c_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{mm}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{du}}+\frac{\mathbb{Q}^{mm}\left(A^c_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{mm}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{dd}}\right)\\
&=\mathbb{Q}^{me}\left(A_{s+1}|\,\mathcal{F}_s\right)\ln\left(\frac{\mathbb{P}\left(A_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{uu}}+\frac{\mathbb{P}\left(A_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{ud}}\right)\\
&+\mathbb{Q}^{me}\left(A^c_{s+1}|\,\mathcal{F}_s\right)\ln\left(\frac{\mathbb{P}\left(A^c_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{du}}+\frac{\mathbb{P}\left(A^c_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{dd}}\right)
\end{aligned}$$

where $Z^{uu},Z^{ud},Z^{du},Z^{dd}$ are the values of the random variable Z conditional on $\mathcal{F}_t$. Propositions 3 and 5, then, yield

$$\begin{aligned}
\mathcal{J}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(Z\right)&=\mathbb{Q}^{me}\left(A_{s+1}|\,\mathcal{F}_s\right)\ln\Bigg(\frac{\mathbb{Q}^{me}\left(A_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{me}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{uu}+\mathcal{H}^{me,uu}_{s+1,T}}\times\\
&\times\left(\frac{\mathbb{P}\left(A_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{-\mathcal{H}^{me,uu}_{s+1,T}}+\frac{\mathbb{P}\left(A_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{-\mathcal{H}^{me,ud}_{s+1,T}}\right)\\
&+\frac{\mathbb{Q}^{me}\left(A_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{me}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{ud}+\mathcal{H}^{me,ud}_{s+1,T}}\times\\
&\times\left(\frac{\mathbb{P}\left(A_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{-\mathcal{H}^{me,uu}_{s+1,T}}+\frac{\mathbb{P}\left(A_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A_{s+1}|\,\mathcal{F}_s\right)}e^{-\mathcal{H}^{me,ud}_{s+1,T}}\right)\Bigg)\\
&+\mathbb{Q}^{me}\left(A^c_{s+1}|\,\mathcal{F}_s\right)\ln\Bigg(\frac{\mathbb{Q}^{me}\left(A^c_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{me}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{du}+\mathcal{H}^{me,du}_{s+1,T}}\times\\
&\times\left(\frac{\mathbb{P}\left(A^c_{s+1}B_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{-\mathcal{H}^{me,du}_{s+1,T}}+\frac{\mathbb{P}\left(A^c_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{P}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{-\mathcal{H}^{me,dd}_{s+1,T}}\right)\\
&+\frac{\mathbb{Q}^{me}\left(A^c_{s+1}B^c_{s+1}|\,\mathcal{F}_s\right)}{\mathbb{Q}^{me}\left(A^c_{s+1}|\,\mathcal{F}_s\right)}e^{Z^{dd}+\mathcal{H}^{me,dd}_{s+1,T}}\times
\end{aligned}$$

$$\times\left(\frac{\mathbb{P}\left(\left.A_{s+1}^{c}B_{s+1}\right|\mathcal{F}_{s}\right)}{\mathbb{P}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)}e^{-\mathcal{H}_{s+1,T}^{me,du}}+\frac{\mathbb{P}\left(\left.A_{s+1}^{c}B_{s+1}^{c}\right|\mathcal{F}_{s}\right)}{\mathbb{P}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)}e^{-\mathcal{H}_{s+1,T}^{me,dd}}\right)\Bigg).$$

Therefore,

$$\begin{aligned}\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)=\mathbb{Q}^{me}\left(\left.A_{s+1}\right|\mathcal{F}_{s}\right)\ln&\left(\frac{\mathbb{Q}^{me}\left(\left.A_{s+1}B_{s+1}\right|\mathcal{F}_{s}\right)}{\mathbb{Q}^{me}\left(\left.A_{s+1}\right|\mathcal{F}_{s}\right)}e^{Z^{uu}+\mathcal{H}_{s+1,T}^{me,uu}}\right.\\&\left.+\frac{\mathbb{Q}^{me}\left(\left.A_{s+1}B_{s+1}^{c}\right|\mathcal{F}_{s}\right)}{\mathbb{Q}^{me}\left(\left.A_{s+1}\right|\mathcal{F}_{s}\right)}e^{Z^{ud}+\mathcal{H}_{s+1,T}^{me,ud}}\right)\\+\mathbb{Q}^{me}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)\ln&\left(\frac{\mathbb{Q}^{me}\left(\left.A_{s+1}^{c}B_{s+1}\right|\mathcal{F}_{s}\right)}{\mathbb{Q}^{me}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)}e^{Z^{du}+\mathcal{H}_{s+1,T}^{me,du}}\right.\\&\left.+\frac{\mathbb{Q}^{me}\left(\left.A_{s+1}^{c}B_{s+1}^{c}\right|\mathcal{F}_{s}\right)}{\mathbb{Q}^{me}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)}e^{Z^{dd}+\mathcal{H}_{s+1,T}^{me,dd}}\right)\\+\mathbb{Q}^{me}\left(\left.A_{s+1}\right|\mathcal{F}_{s}\right)\ln&\left(\frac{\mathbb{P}\left(\left.A_{s+1}B_{s+1}\right|\mathcal{F}_{s}\right)}{\mathbb{P}\left(\left.A_{s+1}\right|\mathcal{F}_{s}\right)}e^{-\mathcal{H}_{s+1,T}^{me,uu}}+\frac{\mathbb{P}\left(\left.A_{s+1}B_{s+1}^{c}\right|\mathcal{F}_{s}\right)}{\mathbb{P}\left(\left.A_{s+1}\right|\mathcal{F}_{s}\right)}e^{-\mathcal{H}_{s+1,T}^{me,ud}}\right)\\+\mathbb{Q}^{me}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)\ln&\left(\frac{\mathbb{P}\left(\left.A_{s+1}^{c}B_{s+1}\right|\mathcal{F}_{s}\right)}{\mathbb{P}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)}e^{-\mathcal{H}_{s+1,T}^{me,du}}+\frac{\mathbb{P}\left(\left.A_{s+1}^{c}B_{s+1}^{c}\right|\mathcal{F}_{s}\right)}{\mathbb{P}\left(\left.A_{s+1}^{c}\right|\mathcal{F}_{s}\right)}e^{-\mathcal{H}_{s+1,T}^{me,dd}}\right),\end{aligned}$$

and (2.31) follows.

To prove (2.33), we first note that it is equivalent to

$$\mathcal{J}_{\mathbb{Q}^{me}}^{(s,t)}(Z)$$

$$=\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,t)}\left(Z-\mathcal{H}_{t,T}^{me}-\sum_{i=s+2}^{t}h_i\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,t)}\left(-\mathcal{H}_{t,T}^{me}-\sum_{i=s+2}^{t}h_i\right). \tag{2.34}$$

This follows from Lemma 2 and Corollary 2.10. Next, we show the above equality (2.34). For $s=t-1$, it was shown in (2.31). For $s=t-2$, we observe

$$\begin{aligned}\mathcal{J}_{\mathbb{Q}^{me}}^{(t-2,t)}(Z)&=\mathcal{J}_{\mathbb{Q}^{me}}^{(t-2,t-1)}\left(\mathcal{J}_{\mathbb{Q}^{me}}^{(t-1,t)}(Z)\right)\\&=\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{J}_{\mathbb{Q}^{me}}^{(t-1,t)}(Z)-\mathcal{H}_{t-1,T}^{me}\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\mathcal{H}_{t-1,T}^{me}\right)\\&=\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{J}_{\mathbb{Q}^{me}}^{(t-1,t)}(Z)-h_t+\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\mathcal{H}_{t,T}^{me}\right)\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\mathcal{H}_{t-1,T}^{me}\right)\\&=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z-\mathcal{H}_{t,T}^{me}\right)-h_t\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\mathcal{H}_{t-1,T}^{me}\right)\\&=\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(Z-\mathcal{H}_{t,T}^{me}-h_t\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\mathcal{H}_{t-1,T}^{me}\right)\\&=\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(Z-\mathcal{H}_{t,T}^{me}-h_t\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-h_t+\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\mathcal{H}_{t,T}^{me}\right)\right)\\&=\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(Z-\mathcal{H}_{t,T}^{me}-h_t\right)-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-h_t-\mathcal{H}_{t,T}^{me}\right),\end{aligned}$$

where we used (2.31) for times t and $t-1$, and the recursive formula (2.22) for the minimal aggregate entropy. To establish (2.34) for $s<t-2$, we work by induction. The arguments are lengthy but routine and are, thus, omitted. □

A consequence of the above findings is the following result.

Corollary 2.12. *For $t=0,1,..,T$, let $\mathcal{H}_{t,T}^{me}$ be the minimal aggregate entropy and $\mathcal{J}_{\mathbb{Q}^{me}}^{(t,t+1)},\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}$ be as in (2.19) with $\mathbb{Q}=\mathbb{Q}^{me},\mathbb{Q}^{mm}$. Then,*

$$\mathcal{J}_{\mathbb{Q}^{me}}^{(t,t+1)}\left(\mathcal{H}_{t+1,T}^{me}\right)=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(-\mathcal{H}_{t+1,T}^{me}\right). \tag{2.35}$$

The next result yields an explicit one-step representation of the minimal aggregate entropy in terms of the non-linear functional $\mathcal{J}_{\mathbb{Q}}^{(t,t+1)}$, evaluated at $\mathbb{Q}=\mathbb{Q}^{me}$. The assertion follows from (2.22) and (2.35).

Proposition 2.13. *The minimal aggregate entropy $\mathcal{H}_{t,T}^{me}$ (cf. (2.16)) is given by the iterative scheme*

$$\mathcal{H}_{T,T}^{me}=0 \quad \text{and} \quad \mathcal{H}_{T-1,T}^{me}=h_T$$

and

$$\mathcal{H}_{t,T}^{me}=h_{t+1}+\mathcal{J}_{\mathbb{Q}^{me}}^{(t,t+1)}\left(\mathcal{H}_{t+1,T}^{me}\right), \quad t=0,1,...,T-2, \tag{2.36}$$

with h_{t+1} and $\mathcal{J}_{\mathbb{Q}^{me}}^{(t,t+1)}$ defined, respectively, in (2.8) and (2.19).

Proceeding iteratively in (2.36) and using Lemma 2 we obtain the (multi-step) representation of the minimal aggregate entropy in terms of the minimal entropy measure. Combining it with (2.23), we obtain the following parity result.

Theorem 2.14. *For $t=0,1,...,T$, let $\mathbb{Q}^{mm}(\cdot|\,\mathcal{F}_t)$ and $\mathbb{Q}^{me}(\cdot|\,\mathcal{F}_t)$ be, respectively, the minimal martingale and the minimal entropy measure, $\mathcal{H}_{t,T}^{me}$ the minimal aggregate entropy and h_i, $i=t+1,...,T$, the local entropy. Then,*

$$\mathcal{H}_{t,T}^{me}=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}\left(-\sum_{i=t+1}^{T}h_i\right)=\mathcal{J}_{\mathbb{Q}^{me}}^{(t,T)}\left(\sum_{i=t+1}^{T}h_i\right), \tag{2.37}$$

where $\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}$ and $\mathcal{J}_{\mathbb{Q}^{me}}^{(t,T)}$ are given in (2.20) for $\mathbb{Q}=\mathbb{Q}^{mm}$,$\mathbb{Q}^{me}$.

The reader is invited to compare the above parity representations with the ones proved in [31] for the case of a diffusion model with stochastic volatility.

Remark 2.15. It is worth commenting on some distinct features of the minimal martingale and the minimal entropy measures. Firstly, we recall that the density of the former (see (2.11)) has the intuitively pleasing property of preserving the conditional distribution of the non-traded factor, given the stock price, in terms of its historical counterpart. In essence, this property states that the unhedgeable risks, given the hedgeable ones, are viewed in the same manner under $\mathbb{P}$ and $\mathbb{Q}^{mm}$.

The minimal entropy measure, however, albeit its predominant role in exponential utility maximization, appears to be lacking an intuitively pleasing structure, as the formulae in Proposition 5 show. Secondly, we observe the dependence of this measure on the horizon choice, T, as reflected by the $T-$dependent values $\mathcal{H}_{t,T}^{me}$ in (2.17). In Section 4, we will see how the indifference prices inherit, in turn, this dependence. Note, however, that the minimal martingale measure does *not* depend on the specific horizon choice as (2.11) shows.

We finish with representation results for the value function process $V_t(x)$, defined in (2.5). The first formula is well known (see, for example, [4], [13] and [23]) while formulae (2.39) and (2.40) are, to the best of our knowledge, new. They follow from Theorem 14.

Proposition 2.16. *The value function process $V_t(x)$ is given, for $x\in\mathbb{R}$ and $t=0,1,...,T$, by*

$$V_t(x)=-e^{-\gamma x-\mathcal{H}_{t,T}^{me}} \tag{2.38}$$

$$=-\exp\left(-\gamma x-\mathcal{J}_{\mathbb{Q}^{me}}^{(t,T)}\left(\sum_{i=t+1}^{T}h_i\right)\right) \tag{2.39}$$

$$=-\exp\left(-\gamma x+\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}\left(-\sum_{i=t+1}^{T}h_i\right)\right), \tag{2.40}$$

with h_t as in (2.8), and $\mathcal{J}_{\mathbb{Q}}^{(t,T)}$ defined in (2.20), for $\mathbb{Q}=\mathbb{Q}^{mm},\mathbb{Q}^{me}$.

3. Indifference valuation algorithms

In this section, we review the notion of indifference price and provide two iterative algorithms for its construction. The claim to be priced is written at time t_0 on both the traded stock and the non-traded factor. For simplicity, we assume that $t_0 = 0$. The claim matures at $t = 1, ..., T$, yielding payoff C_t, represented as an $\mathcal{F}_t$-measurable random variable. We are interested in computing its indifference price in reference to the exponential criterion (2.5). For the moment, we price a single claim and present the results on the multi-claim case afterwards. For convenience, we eliminate the "exponential" terminology. We recall the familiar definition of indifference price (see, for example, [4] and [23]).

Definition 3.1. Consider a claim, written at time $t_0 = 0$ and yielding at t payoff $C_t \in \mathcal{F}_t$, $t = 0, 1, ..., T$. Let $V_t(x)$ be the value function process (2.5). The claim's indifference price is defined as the amount $\nu_s(C_t)$, $s = 0, 1, ..., t$, for which

$$V_s(x - \nu_s(C_t)) = \sup_{\alpha_{s+1},\dots,\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right)|\mathcal{F}_s\right), \tag{3.1}$$

for all initial wealth levels $X_s = x \in \mathbb{R}$.

We remark that the alignment of the expiry of the claim with the time at which the value function process is calculated in the right hand side of the pricing condition (3.1) is chosen for mere convenience. Indeed, the above definition can be directly extended to times beyond the claim's maturity in that (3.1) can be replaced by

$$V_s\left(X_s - \nu_s\left(C_t\right)\right) = \sup_{\alpha_{s+1},\dots,\alpha_{t'}} E_{\mathbb{P}}\left(V_{t'}\left(X_{t'} - C_t\right)|\,\mathcal{F}_s\right), \tag{3.2}$$

for $t' = t+1, ..., T-1, T$. This follows easily from (3.1), the dynamic programming principle and the fact that $C_t \in \mathcal{F}_{t'}$. Observe, however, that this cannot be done for times t' exceeding T.

Next, we review the price representation obtained for the single-period case in [18] (see, also, [19]). Therein, the claim's indifference price is represented as a non-linear expectation of its payoff, providing the incomplete market analogue of the linear arbitrage-free pricing rule. We refer the reader to these papers for a detailed discussion on the nature and properties of the pricing formula. For indifference prices in single-period models for utilities different than the exponential, see [5].

Proposition 3.2. *(Single-period model) Let $\mathbb{Q}$ be the martingale measure under which the conditional distribution of the non-traded factor, given the traded asset, is preserved with respect to the historical measure $\mathbb{P}$, i.e.,*

$$\mathbb{Q}(Y_T\,|S_T) = \mathbb{P}(Y_T\,|S_T). \tag{3.3}$$

Let $C_T = C(S_T, Y_T)$ be the claim to be priced under exponential preferences with risk aversion coefficient γ. Then, its indifference price, $\nu_0\left(C_T\right)$, is given by

$$\nu_0(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}}\left(e^{\gamma C_T}\,|S_T\right)\right). \tag{3.4}$$

As the above result shows, the underlying indifference pricing blocks are the non-linear expectation $\mathcal{E}_{\mathbb{Q}}$ and the pricing measure $\mathbb{Q}$. For the multi-period case, we need to build their appropriate multi-period analogues. We stress that due to the inherent nonlinearities of the problem, together with the fact that the model at hand is non-reduced (i.e., the nested model is not complete), it is *not at all* clear how these analogues should be constructed. Notice, for example, that property (3.3) is satisfied by both the minimal martingale and minimal entropy measures, $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$, but only at expiration (see (2.30)). For times before $T-1$, the two measures differ and property (3.3) is held by $\mathbb{Q}^{mm}$, and not $\mathbb{Q}^{me}$, which is the natural martingale measure in exponential utility maximization. This important difference motivates us to look for algorithmic price representations under each of these two measures.

Next, we introduce two non-linear functionals. In the sequel, they will be evaluated on $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$ for the construction of the pricing algorithms.

Definition 3.3. Let $T > 0$ and Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. For $s = 0, 1, ..., T-1$, $t = s+1, ..., T$ and $\mathbb{Q} \in \mathcal{Q}_T$, define the single- and multi-step functionals

$$\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z) = \frac{1}{\gamma} E_{\mathbb{Q}}\left(\ln E_{\mathbb{Q}}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)|\mathcal{F}_s\right) \tag{3.5}$$

and

$$\mathcal{E}_{\mathbb{Q}}^{(s,t)}(Z) = \mathcal{E}_{\mathbb{Q}}^{(s,s+1)}\left(...\mathcal{E}_{\mathbb{Q}}^{(t-1,t)}(Z)\right). \tag{3.6}$$

We caution the reader that, for $t > s+1$,

$$\mathcal{E}_{\mathbb{Q}}^{(s,t)}(Z) \neq \frac{1}{\gamma} E_{\mathbb{Q}}\left(\ln E_{\mathbb{Q}}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_t^S\right.\right)|\mathcal{F}_s\right).$$

Definition 3.4. Let Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. For $s = 0, 1, ..., T-1$ and $t = s+1, ..., T$, define the nonlinear single- and multi-step price functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,s)}$ by

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) \tag{3.7}$$

and

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}(Z) = \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(...\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-1,t)}(Z)\right), \tag{3.8}$$

with $\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ given in (3.5) for $\mathbb{Q} = \mathbb{Q}^{mm}$.

The following lemma provides the explicit form of the multi-step functional $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}$.

Lemma 3.5. *Let Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for $s < t-1$,*

$$\begin{aligned}&\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}(Z)\\ &= \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}\sum_{i=s+2}^{t} h_i\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}\sum_{i=s+2}^{t} h_i\right).\end{aligned} \tag{3.9}$$

Proof. We establish (3.9) only for $s = t-2$ since the rest of the proof follows along similar arguments. We need to show that

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-2,t)}(Z) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}h_t\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}h_t\right). \tag{3.10}$$

Using (3.7) and (3.8), we write

$$\begin{aligned}&\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-2,t)}(Z) = \mathcal{P}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-1,t)}(Z)\right)\\ &= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)\\ &\quad -\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right).\end{aligned} \tag{3.11}$$

On the other hand, (2.36) yields

$$-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me} = -\frac{1}{\gamma}h_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)$$

and the second term in (3.11) becomes

$$-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right) = -\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}h_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right)$$

$$= -\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}h_t - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right),$$

where we used the measurability properties of h_t. Similarly, the first term in (3.11) becomes

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)$$

$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right.$$

$$\left.-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \frac{1}{\gamma}h_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right)$$

$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \frac{1}{\gamma}h_t\right)$$

$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}h_t\right)\right).$$

Combining the above, (3.10) follows. □

We are now ready to provide the pricing algorithms for the indifference price. The first algorithm uses the minimal martingale measure and the pricing functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}$ (cf. (3.7) and (3.8)) while the second one uses the minimal entropy measure and the pricing functionals $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,t)}$ (cf. (3.5) and (3.6)).

To ease the presentation, we first state the main theorems and, then, provide their proofs and discussion.

Theorem 3.6. *Consider a claim written at $t_0 = 0$ and expiring at t yielding payoff $C_t \in \mathcal{F}_t$. For $t = 1, ..., T$ and $s = 0, 1, ..., t-1$, the following statements are true:*

i) The indifference price $\nu_s(C_t)$, defined in (3.1), is given by the algorithm

$$\nu_t(C_t) = C_t, \tag{3.12}$$

$$\nu_s(C_t) = \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(\nu_{s+1}(C_t)), \tag{3.13}$$

where $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ is the single-step pricing functional defined in (3.7).

ii) The indifference price $\nu_s(C_t) \in \mathcal{F}_s$ is given by

$$\nu_s(C_t) = \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}(C_t) \tag{3.14}$$

$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(C_t - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}\sum_{i=s+2}^{t} h_i\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}\sum_{i=s+2}^{t} h_i\right), \tag{3.15}$$

with the multi-step price functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}$ and $\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}$ defined, respectively, in (3.8) and (3.6) for $\mathbb{Q} = \mathbb{Q}^{mm}$.

iii) The pricing algorithm is consistent across time in that, for $0 \le s \le s' \le t$, the semigroup property

$$\nu_s(C_t) = \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s')}(\mathcal{P}_{\mathbb{Q}^{mm}}^{(s',t)}(C_t)) = \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s')}(\nu_{s'}(C_t)) = \nu_s(\mathcal{P}_{\mathbb{Q}^{mm}}^{(s',t)}(C_t)) \tag{3.16}$$

holds.

Theorem 3.7. *Consider a claim written at $t_0 = 0$ and expiring at t yielding payoff $C_t \in \mathcal{F}_t$. For $t = 1, ..., T$ and $s = 0, 1, ..., t-1$, the following statements are true:*

i) The indifference price $\nu_s(C_t)$, defined in (3.1), is given by the algorithm

$$\nu_t(C_t) = C_t,$$

$$\nu_s(C_t) = \mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(\nu_{s+1}(C_t)), \tag{3.17}$$

where $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ is the single-step price functional defined in (3.5) for $\mathbb{Q} = \mathbb{Q}^{me}$.

ii) The indifference price process is given by

$$\nu_s(C_t) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}(C_t), \tag{3.18}$$

with the multi-step price functional $\mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}$ *defined in (3.6) for* $\mathbb{Q} = \mathbb{Q}^{me}$.

iii) The pricing algorithm is consistent across time in that, for $0 \le s \le s' \le t$, *the semigroup property*

$$\nu_s(C_t) = \mathcal{E}^{(s,s')}_{\mathbb{Q}^{me}}(\mathcal{E}^{(s',t)}_{\mathbb{Q}^{me}}(C_t)) = \mathcal{E}^{(s,s')}_{\mathbb{Q}^{me}}(\nu_{s'}(C_t)) = \nu_s(\mathcal{E}^{(s',t)}_{\mathbb{Q}^{me}}(C_t)) \tag{3.19}$$

holds.

We proceed with the following auxiliary result.

Lemma 3.8. *Let* $s = 0, 1, ..., T-1$, $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ *be defined in (3.5) for* $\mathbb{Q} = \mathbb{Q}^{mm}$ *and* $Z \in \mathcal{F}_{s+1}$. *Then,*

$$\sup_{\alpha_{s+1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{s+1}-Z)}|\mathcal{F}_s\right) = -e^{-\gamma\left(X_s - \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}(Z)\right) - h_{s+1}}, \tag{3.20}$$

with h_s *as in (2.8).*

Proof. With A_{s+1} as in (2.6) we have

$$\begin{aligned}
&\sup_{\alpha_{s+1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{s+1}-Z)}|\mathcal{F}_s\right)\\
&= -e^{-\gamma X_s}\left(\mathbb{P}(A_{s+1}|\mathcal{F}_s)e^{-\gamma\alpha_{s+1}S_s(\xi^u_{s+1}-1)}E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s \vee A_{s+1}\right)\right.\\
&\quad \left.+(1-\mathbb{P}(A_{s+1}|\mathcal{F}_s))e^{-\gamma\alpha_{s+1}S_s(\xi^d_{s+1}-1)}E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s \vee A^c_{s+1}\right)\right).
\end{aligned}$$

Differentiating with respect to α_{s+1} yields that the optimum occurs at

$$\alpha_{s+1} = \frac{1}{\gamma S_s\left(\xi^u_{s+1} - \xi^d_{s+1}\right)} \ln\left(\frac{E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s \vee A_{s+1}\right)\mathbb{P}(A_{s+1}|\mathcal{F}_s)\left(\xi^u_{s+1}-1\right)}{E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s \vee A^c_{s+1}\right)\left(1-\mathbb{P}(A_{s+1}|\mathcal{F}_s)\right)\left(1-\xi^d_{s+1}\right)}\right).$$

Using the form of the density of the minimal martingale measure (see (2.11)) we obtain

$$\begin{aligned}
&\sup_{\alpha_{s+1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{s+1}-Z)}|\mathcal{F}_s\right)\\
&= -\exp\left(-\gamma X_s + \mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)\ln E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s \vee A_{s+1}\right)\right.\\
&\quad \left.+\left(1-\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)\right)\ln E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s \vee A^c_{s+1}\right)\right)\times\\
&\times\left(\frac{\mathbb{P}(A_{s+1}|\mathcal{F}_s)}{\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}\right)^{\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}\left(\frac{1-\mathbb{P}(A_{s+1}|\mathcal{F}_s)}{1-\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}\right)^{1-\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}.
\end{aligned}$$

Using once again the form of the density of the minimal martingale measure (2.11) and the definition of $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ (cf. (3.5)), (3.20) follows. □

We are now ready to prove Theorem 3.6.

Proof. i) Equality (3.12) is immediate. We prove (3.13) for $s = t-1$. From (2.38) we have

$$\begin{aligned}
&\sup_{\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right)|\,\mathcal{F}_{t-1}\right)\\
&= \sup_{\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma\left(X_t - \left(C_t - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right)}\middle|\,\mathcal{F}_{t-1}\right).
\end{aligned}$$

Using Lemma 3.8 for $s = t-1$ and $Z = C_t - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}$, we get

$$\sup_{\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right)|\,\mathcal{F}_{t-1}\right) = -e^{-\gamma\left(X_{t-1} - \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(C_t - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right) - h_t}.$$

Combining the above with (3.1) and formula (2.38) for V_{t-1}, we deduce

$$\nu_{t-1}(C_t) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) + \frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me} - \frac{1}{\gamma}h_t$$
$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right), \tag{3.21}$$

where we used (2.22) for $\mathcal{H}_{t-1,T}^{me}$.

For $s = t-2$, we have

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right)\middle|\, \mathcal{F}_{t-2}\right)$$
$$= \sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})+\alpha_t(S_t-S_{t-1})-\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right)}\middle|\, \mathcal{F}_{t-2}\right)$$
$$= \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2}))}\right.$$
$$\left.\times \sup_{\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_t(S_t-S_{t-1})-\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right)}\middle|\, \mathcal{F}_{t-1}\right)\middle|\, \mathcal{F}_{t-2}\right).$$

Using Lemma 3.8 for $s = t-1$ and $Z = C_t - \frac{1}{\gamma}\mathcal{H}_{t,T}^{me}$, and (2.22) and (3.21) we deduce

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right)\middle|\, \mathcal{F}_{t-2}\right)$$
$$= \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right)-h_t}\middle|\, \mathcal{F}_{t-2}\right)$$
$$= \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})-\left(\nu_{t-1}(C_t)+\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}h_t\right)\right)}\middle|\, \mathcal{F}_{t-2}\right)$$
$$= \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})-\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)\right)}\middle|\, \mathcal{F}_{t-2}\right).$$

Using Lemma 3.8 once again, this time for $s = t-2$ and $Z = \nu_{t-1}(C_t) - \frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}$, we obtain

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right)\middle|\, \mathcal{F}_{t-2}\right)$$
$$= -e^{-\gamma\left(X_{t-2}-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)\right)-h_{t-1}}. \tag{3.22}$$

On the other hand, (2.38) yields,

$$V_{t-2}\left(X_{t-2} - \nu_{t-2}(C_t)\right) = -e^{-\gamma(X_{t-2}-\nu_{t-2}(C_t))-\mathcal{H}_{t-2,T}^{me}}.$$

Comparing the above to (3.22), using the definition of the indifference price (3.1) and formula (2.22), we deduce

$$\nu_{t-2}(C_t) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\nu_{t-1}(C_t) - \frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right) - \frac{1}{\gamma}h_{t-1} + \frac{1}{\gamma}\mathcal{H}_{t-2,T}^{me}$$
$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\nu_{t-1}(C_t) - \frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right) - \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right), \tag{3.23}$$

and we conclude. For $s = 0, ..., t-3$, (3.13) follows along similar arguments.

ii) In view of property (3.9), assertions (3.14) and (3.15) are equivalent. We only show (3.15). For $s = t-1$, (3.15) follows trivially. To show (3.15) for $s = t-2$, we work as follows. We first observe that (2.22) together with the measurability properties of the local entropy process h_t yield

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right) - \frac{1}{\gamma}h_t\right)$$
$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me} - \frac{1}{\gamma}h_t\right).$$

On the other hand, using (3.13) for $\nu_{t-1}(C_t)$ and (2.22), we deduce

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right.$$

$$\left.+\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}h_t\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}h_t\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right)\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right).$$

Combining the above with (3.23) yields

$$\nu_{t-2}(C_t)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right),$$

and we deduce (3.15). For $s=0,...,t-3$, we work similarly.

The semigroup property (3.16) follows easily. □

We continue with the proof of Theorem 3.7.

Proof. We only need to establish that

$$\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)-\frac{\mathcal{H}_{s+1,T}^{me}}{\gamma}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{\mathcal{H}_{s+1,T}^{me}}{\gamma}\right),$$

since all assertions of the theorem would follow by straightforward arguments.

To this end, let $Z=\gamma\nu_{s+1}\left(C_t\right)-\mathcal{H}_{s+1,T}^{me}$. Then (2.31) yields

$$\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)-\mathcal{H}_{s+1,T}^{me}\right)=\mathcal{J}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)\right)+\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\mathcal{H}_{s+1,T}^{me}\right)$$

and, in turn,

$$\frac{1}{\gamma}\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)-\mathcal{H}_{s+1,T}^{me}\right)$$

$$=\frac{1}{\gamma}\mathcal{J}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)\right)+\frac{1}{\gamma}\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\mathcal{H}_{s+1,T}^{me}\right).$$

We easily conclude. □

Discussion on the pricing algorithms: The indifference price is calculated via the iterative pricing schemes (3.13) and (3.17), applied backwards in time, starting at the claim's maturity. The schemes have local and dynamic properties.

Dynamically, the pricing functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,t)}$ are similar. Specifically, at each time interval, say $(s,s+1)$, the price $\nu_s(C_t)$ is computed via the single-step pricing operators, $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$, applied to the end of the period payoff. The latter turns out to be the indifference price, $\nu_{s+1}(C_t)$, yielding prices consistent across time.

Locally, however, the pricing roles of $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ and $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}$ are very different both in structure and the associated measures. We start with the latter price functional since it has the simpler of the two forms. Valuation is executed in two steps, in analogy to the single-period counterpart (3.4). In the first sub-step, the end of the period payoff, $\nu_{s+1}(C_t)$, is altered via a non-linear functional and the conditioning on the information generated by $\mathcal{F}_s \vee \mathcal{F}^S_{s+1}$. The new payoff,

$$\tilde{\nu}_{s+1}(C_t) = \frac{1}{\gamma} \ln E_{\mathbb{Q}^{me}} \left(e^{\gamma \nu_{s+1}(C_t)} \middle| \mathcal{F}_s \vee \mathcal{F}^S_{s+1} \right) \tag{3.24}$$

emerges which is, in turn, priced by expectation. The indifference price is, then, given by

$$\nu_s(C_t) = E_{\mathbb{Q}^{me}}(\tilde{\nu}_{s+1}(C_t) \,| \mathcal{F}_s). \tag{3.25}$$

While structure-wise the price functional $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}$ has a simple and intuitive form, the employed measure $\mathbb{Q}^{me}$ does *not*, as it can be seen from (2.17).

The situation is reversed in the first algorithm. Specifically, the pricing functional $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ has no transparent form while the used measure, $\mathbb{Q}^{mm}$, has the intuitively pleasing property (2.11). Indeed, $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ incorporates the minimal aggregate entropy $\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T}$ in a "palindromic" manner. Namely, at each time step, the end of the period payoff $\nu_{s+1}(C_t)$ is reduced by $\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T}$ and priced, yielding the indifference price

$$\nu^1_{s+1} = \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}} \left(\nu_{s+1}(C_t) - \frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T} \right).$$

In turn, the payoff

$$\nu^2_{s+1} = -\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}} \left(-\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T} \right)$$

is added. Both quantities ν^1_{s+1} and ν^2_{s+1} are calculated via the two-step procedure similar to the one described in (3.24) and (3.25). Notice that due to the non-linear character of the indifference price, the entropic liability $-\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T}$ could *not* be factored out. This is a direct effect of the internal market incompleteness in the model herein.

Therefore,

$$\nu_s(C_t) = \mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}} (\nu_{s+1}(C_t))$$

$$= \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}} \left(\nu_{s+1}(C_t) - \frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T} \right) - \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}} \left(-\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T} \right)$$

$$\neq \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}} (\nu_{s+1}(C_t)).$$

The following results follow easily from the above pricing algorithms.

Corollary 3.9. *Let the payoff C_t be of the form*

$$C_t = Y_t + Z_t$$

with $Y_t \in \mathcal{F}_t$ and Z_t being such that there exist $Z_s \in \mathcal{F}^S_s$ and $\alpha_i \in \mathcal{F}^S_{i-1}$, $i = s+1, ..., t$, satisfying $Z_t = Z_s + \Sigma^t_{i=s+1} \alpha_i \Delta S_i$, a.e. Then,

$$\nu_s(C_t) = \nu_s(Y_t + Z_t) = \nu_s(Y_t) + Z_s$$

$$= \mathcal{P}^{(s,t)}_{\mathbb{Q}^{mm}}(Y_t) + E_{\mathbb{Q}^{mm}} \left(Z_t | \, \mathcal{F}^S_s \right) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}(Y_t) + E_{\mathbb{Q}^{me}} \left(Z_t | \, \mathcal{F}^S_s \right).$$

3.1. Multiple claims

We provide the pricing algorithms for the multi-claim case. For convenience, we assume that in the interval $[0, n+1]$ with $n+1 \leq T$, we price a collection of $n+2$ claims, $C_0, C_1, ..., C_j, ...C_{n+1}$, with each generic claim maturing at time j, $j = 0, 1, ..., n+1$ and yielding payoff $C_j \in \mathcal{F}_j$. The result follows easily from the earlier and, for this, we do not provide its proof.

Theorem 3.10. *Consider a collection of $n+2$ claims, written at $t_0 = 0$, yielding payoffs $C_j \in \mathcal{F}_j$, with $j = 0, 1, ..., n+1$. The following statements hold:*

i) The indifference price $\nu_s(\Sigma_{j=s}^{n+1} C_j)$, is given, for $s = 0, 1, ..., n+1$, by the iterative algorithm

$$\nu_{n+1}(C_{n+1}) = C_{n+1},$$

$$\begin{aligned}\nu_s(C_s + \Sigma_{j=s+1}^{n+1} C_j) &= C_s + \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(C_{s+1} + \nu_{s+1}(\Sigma_{j=s+2}^{n+1} C_j)) \\ &= C_s + \mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(C_{s+1} + \nu_{s+1}(\Sigma_{j=s+2}^{n+1} C_j)),\end{aligned}$$

with $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ as in (3.7) and (3.5).

ii) The indifference price process $\nu_s(C_s + \Sigma_{j=s+1}^{n+1} C_j) \in \mathcal{F}_s$ and satisfies, for $s = 0, 1, ..., n+1$,

$$\nu_s(C_s + \Sigma_{j=s+1}^{n+1} C_j)$$

$$= C_s + \mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(C_{s+1} + \mathcal{P}_{\mathbb{Q}^{mm}}^{(s+1,s+2)}\left(C_{s+2} + ...\mathcal{P}_{\mathbb{Q}^{mm}}^{(n-1,n)}\left(C_n + \mathcal{P}_{\mathbb{Q}^{mm}}^{(n,n+1)}(C_{n+1})\right)\right)\right)$$

$$= C_s + \mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(C_{s+1} + \mathcal{E}_{\mathbb{Q}^{me}}^{(s+1,s+2)}\left(C_{s+2} + ...\mathcal{E}_{\mathbb{Q}^{me}}^{(n-1,n)}\left(C_n + \mathcal{E}_{\mathbb{Q}^{me}}^{(n,n+1)}(C_{n+1})\right)\right)\right).$$

3.2. The static certainty equivalent and the indifference price

The definition of indifference price motivates us to ask whether there is a natural connection between the dynamic price $\nu_s(C_t)$ and the static certainty equivalent pricing rule. The latter is given, say for a random variable Z, by

$$\mathcal{C}(Z) = -u^{-1} E_{\mathbb{P}}(u(-Z)), \tag{3.26}$$

with u being an increasing and concave utility function. Notice that this pricing rule is defined in the *absence* of any trading activity.

Is the indifference price the dynamic analogue of the above static pricing rule? We will see that, surprisingly, it is not!

To this end, we first introduce the auxiliary process $V_s^{-1}(x)$, $s = 0, 1, ..., T$, denoting the spatial inverse of the value function process (2.5), given by

$$V_s^{-1}(x) = -\frac{\ln(-x)}{\gamma} - \frac{\mathcal{H}_{s,T}^{me}}{\gamma}, \qquad x \in \mathbb{R}^-. \tag{3.27}$$

Inspecting (3.26), we are motivated to define the following process, which we will be referring to as the *conditional certainty equivalent.*

Definition 3.11. Let Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ and $V_s(x)$ and $V_s^{-1}(x)$, $s = 0, 1, ..., T$, be, respectively, the value function process and its inverse (cf. (2.5) and (3.27)). For $\mathbb{Q} \in \mathcal{Q}_T$, define the conditional certainty equivalent $\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z)$ by

$$\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z) = -V_{s+1}^{-1}\left(E_{\mathbb{Q}}\left(V_{s+1}(-Z)|\, \mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right)\right). \tag{3.28}$$

The following lemma follows from direct arguments.

Lemma 3.12. *Let the conditional certainty equivalent* $\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}$, $s=0,1,...,T$, *be defined in (3.28), and* $\mathbb{Q}^{mm}$ *and* $\mathbb{Q}^{me}$ *be the minimal martingale and minimal entropy measures. Then,*

$$\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(0)\neq 0 \qquad \text{and} \quad \mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)\neq \mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z).$$

Moreover,

$$\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(Z+\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\neq \mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z)+\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)$$

with $Z\in(\Omega,\mathcal{F},\mathbb{P})$.

We note that there are particular cases when the above inequalities become equalities. These cases are discussed in Proposition 4.4 and Theorem 4.5.

We are now ready to explore the analogies of the indifference price and the static certainty equivalent.

Proposition 3.13. *Let* $\nu_{s+1}(C_t)$ *be the indifference price of the claim at time* $s=0,1,...,t$. *Let also* $\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ *and* $\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}$ *be as in (3.28) for* $\mathbb{Q}=\mathbb{Q}^{mm},\mathbb{Q}^{me}$ *and* $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ *and* $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ *be as in (3.7) and (3.5). Then, for* $s=0,1,...,T$,

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(\nu_{s+1}(C_t))=E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(\nu_{s+1}(C_t))\middle|\mathcal{F}_s\right)-E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(0)\middle|\mathcal{F}_s\right). \tag{3.29}$$

Similarly,

$$\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(\nu_{s+1}(C_t))=E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\nu_{s+1}(C_t)+\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\mathcal{F}_s\right) \tag{3.30}$$
$$-E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\mathcal{F}_s\right).$$

Proof. To prove (3.29), we use Definition 3.11 to obtain

$$\begin{aligned}
&E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(\nu_{s+1}(C_t))\middle|\mathcal{F}_s\right)\\
&=E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^{mm}}\left(e^{\gamma\nu_{s+1}(C_t)-\mathcal{H}_{s+1,T}^{me}}\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)\middle|\mathcal{F}_s\right)\\
&\quad+E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\mathcal{F}_s\right)\\
&=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\nu_{s+1}(C_t)-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)+E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\mathcal{F}_s\right).
\end{aligned} \tag{3.31}$$

For $Z=0$, we have

$$E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(0)\middle|\mathcal{F}_s\right)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\tfrac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)+E_{\mathbb{Q}^{mm}}\left(\tfrac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\mathcal{F}_s\right). \tag{3.32}$$

Subtracting (3.32) from (3.31) and using (3.7) yields (3.29).

To prove (3.30), we work similarly. To this end, we have

$$\begin{aligned}
&E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\nu_{s+1}(C_t)+\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\mathcal{F}_s\right)\\
&=E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^{mm}}\left(e^{\gamma\left(Z+\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)-\mathcal{H}_{s+1,T}^{me}}\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)\middle|\mathcal{F}_s\right)\\
&\quad+E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\mathcal{F}_s\right)\\
&=E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^{mm}}\left(e^{\gamma\nu_{s+1}(C_t)}\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)\middle|\mathcal{F}_s\right)+E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\mathcal{F}_s\right)\\
&=\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z)+E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\mathcal{F}_s\right).
\end{aligned}$$

We also have

$$E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\Big|\,\mathcal{F}_s\right)=E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\Big|\,\mathcal{F}_s\right).$$

Subtracting the above we conclude. □

As the analysis above shows, there is *no* direct connection between the indifference price and the conditional certainty equivalent. Indeed, using (3.14) and (3.29), we easily see

$$\nu_s\left(C_t\right)\neq E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)\right)\Big|\,\mathcal{F}_s\right). \tag{3.33}$$

Respectively, (3.18) and (3.30) yield

$$\nu_s\left(C_t\right)\neq E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)\right)\Big|\,\mathcal{F}_s\right). \tag{3.34}$$

These findings are direct consequences of the form of the investment performance process $V_t\left(x\right)$ (see (2.40) and (2.39)), as well as the measurability properties of the minimal aggregate entropy $\mathcal{H}_{t,T}^{me}$ (see, for example, (2.23) or (2.36)).

Remark 3.14. A direct analogy between the indifference price and the classical certainty equivalent is present in two cases. Specifically, it holds when the binomial model is of reduced form. This case is analyzed in detail in Section 4. It is, also, present in an alternative kind of indifference prices built in reference to a new framework for portfolio choice in which the value function process, $V_s\left(x\right)$, is replaced by its "forward" analogue (we refer the reader to [21] for further details).

3.3. **Risk preference normalization points and the related indifference prices**

So far, we have derived indifference prices associated with an exponential utility function (pre)set at time T. An important implicit assumption in the entire construction is that the claims we consider mature *before* this exogenously chosen horizon. We will refer to the instant T as the *risk preference normalization point.*

Two questions then arise: i) how the indifference prices depend on the choice of the risk preference normalization point? and ii) can this dependence be relaxed? Herein, we only address the first question and refer the reader to [21] for the second one.

In order to emphasize the dependence on the horizon choice, we introduce the notations $V_{t,T}\left(x\right)$ and $\nu_s\left(C_t;T\right)$ for the value function and the indifference price, respectively. We will be referring to $\nu_s\left(C_t;T\right)$ as the indifference price normalized at T.

Theorem 3.15. *Let $\hat{T}$ and T be two normalization points with $\hat{T}>T$, and let $\mathcal{H}_{s,T}^{me}$ and $\mathcal{H}_{s,\hat{T}}^{me}$ be the associated minimal aggregate entropy processes. Consider a claim written at $t_0=0$ and maturing at $t=0,1,...,T$, yielding payoff $C_t\in\mathcal{F}_t$. Let $\nu_s\left(C_t;\hat{T}\right)$ and $\nu_s\left(C_t;T\right)$ be the indifference prices normalized at $\hat{T}$ and T, respectively. Then, for $0\leq s\leq t\leq T$,*

$$\nu_s\left(C_t;\hat{T}\right)=\nu_s\left(C_t-Z_t;T\right)+Z_s \tag{3.35}$$

where, for $u=s,...,t$,

$$Z_u=\frac{1}{\gamma}\left(\mathcal{H}_{u,\hat{T}}^{me}-\mathcal{H}_{u,T}^{me}\right). \tag{3.36}$$

Proof. Consider the normalization point $\hat{T}$. Then, (2.38) yields

$$\sup_{\alpha_{s+1},\ldots,\alpha_t}E_{\mathbb{P}}\left(V_{t,\hat{T}}\left(X_t-C_t\right)\Big|\,\mathcal{F}_s\right)=$$

$$=\sup_{\alpha_{s+1},\ldots,\alpha_t}E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t-C_t\right)-\mathcal{H}_{t,\hat{T}}^{me}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - C_t\right) - \left(\mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T}\right) - \mathcal{H}^{me}_{t,T}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - \left(C_t - \frac{1}{\gamma}\left(\mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T}\right)\right)\right) - \mathcal{H}^{me}_{t,T}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - \left(C_t - Z_t\right)\right) - \mathcal{H}^{me}_{t,T}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,T}\left(X_t - \left(C_t - Z_t\right)\right)\right|\mathcal{F}_s\right),$$

where we used (3.36). From (3.1), we have

$$\sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,\hat{T}}\left(X_t - C_t\right)\Big|\,\mathcal{F}_s\right) = -\exp\left(-\gamma\left(x - \nu_s\left(C_t;\hat{T}\right)\right) - \mathcal{H}^{me}_{s,\hat{T}}\right)$$

and, similarly,

$$\sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,T}\left(X_t - \left(C_t - Z_t\right)\right)\right|\mathcal{F}_s\right) = -\exp\left(-\gamma\left(x - \nu_s\left(C_t - Z_t;T\right)\right) - \mathcal{H}^{me}_{s,T}\right).$$

Combining the above we easily conclude. □

We conclude with an interesting parity result for the indifference price of the differential of the minimal aggregate entropies.

Corollary 3.16. *Consider a claim written at $t_0 = 0$ and yielding at t, $t = 1, \ldots, T$, payoff (cf. (3.36))*

$$Z_t = \frac{1}{\gamma}\left(\mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T}\right),$$

with $\hat{T} > T$. Then, for $s = 0, 1, \ldots, t$,

$$Z_s = \nu_s\left(-Z_t;T\right) = -\mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}_{T}}\left(-Z_t\right). \tag{3.37}$$

On the other hand,

$$Z_s = \nu_s\left(Z_t;\hat{T}\right) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}_{\hat{T}}}\left(Z_t\right). \tag{3.38}$$

4. Reduced incomplete binomial models

We focus on an important special case of the incomplete binomial model introduced in Section 2. Specifically, we assume that neither the values nor the transition probabilities of the stock price process are affected by the non-traded factor process, i.e. for $t = 0, 1, \ldots, T-1$,

$$\xi^u_{t+1} \in \mathcal{F}^S_t \quad \text{and} \quad \xi^d_{t+1} \in \mathcal{F}^S_t, \tag{4.1}$$

and

$$\mathbb{P}\left(\xi_{t+1} = \xi^u_{t+1}\left|\mathcal{F}_t\right.\right) = \mathbb{P}\left(\xi_{t+1} = \xi^u_{t+1}\left|\mathcal{F}^S_t\right.\right). \tag{4.2}$$

We will call such an incomplete binomial model *reduced.*

Notice that under (4.1) and (4.2) the *nested* model becomes *complete* and market incompleteness is generated only through the presence of the non-traded risk factor in the claim's payoff. To our knowledge, this is the only case analyzed so far in exponential indifference pricing in binomial models (see, among others, [1], [18], [30] and [29])).

As it is expected, the minimal martingale and minimal entropy measures must coincide since there is now a unique (nested) martingale measure. We denote this measure by $\mathbb{Q}\left(\cdot\right|\mathcal{F}_t)$, $t = 0, 1, \ldots, T$. The interesting fact is that the minimal aggregate entropy looses its non-linear character and reduces to a mere conditional expectation of the aggregate local entropy.

Lemma 4.1. *Under assumptions (4.1) and (4.2), the local entropy process is $\mathcal{F}^S_t$-predictable, i.e., $h_t \in \mathcal{F}^S_{t-1}$, $t = 1, \ldots, T$.*

Proposition 4.2. *In the reduced binomial model, the minimal martingale and minimal entropy measures coincide, i.e. for* $t = 0, 1, ..., T$,

$$\mathbb{Q}\left(\cdot|\,\mathcal{F}_t\right) = \mathbb{Q}^{mm}\left(\cdot|\,\mathcal{F}_t\right) = \mathbb{Q}^{me}\left(\cdot|\,\mathcal{F}_t\right). \tag{4.3}$$

Moreover, the minimal aggregate entropy $\mathcal{H}^{me}_{t,T}$ *becomes*

$$\mathcal{H}^{me}_{t,T} = E_{\mathbb{Q}}\left(\sum_{i=t+1}^{T} h_i \middle| \mathcal{F}^S_t\right), \tag{4.4}$$

with h_i, $i = t+1, ..., T$, *as in (2.8).*

Proof. The proof is based on iterative arguments. Equality (4.3) holds trivially for $t = T$ while for $t = T-1$ it was shown in (2.30). Equality (4.4) also holds for $t = T, T-1$, as it can be shown using (2.21) and (2.30).

Next, we observe that Lemma 33 yields $\mathcal{H}^{me}_{T-1,T} \in \mathcal{F}^S_{T-1}$ and, thus, the (conditional on $\mathcal{F}_{T-2}$) values $\mathcal{H}^{me,uu}_{T-1,T}$ and $\mathcal{H}^{me,ud}_{T-1,T}$ coincide. Therefore, (2.17) and (2.11) imply

$$\mathbb{Q}^{me}(A_{T-1}B_{T-1}|\mathcal{F}_{T-2}) = \mathbb{Q}^{mm}(A_{T-1}B_{T-1}|\mathcal{F}_{T-2}).$$

Similar arguments yield analogous equalities for the values of $\mathbb{Q}^{me}(\cdot|\mathcal{F}_{T-2})$ and $\mathbb{Q}^{mm}(\cdot|\mathcal{F}_{T-2})$ on the sets $A^c_{T-1}B_{T-1}$, $A_{T-1}B^c_{T-1}$ and $A^c_{T-1}B^c_{T-1}$. Thus, (4.3) is shown for $t = T-2$.

Using once more that $H^{me}_{T-1,T}$ is $\mathcal{F}^S_{T-1}$-measurable, (2.22) implies

$$\mathcal{H}^{me}_{T-2,T} = h_{T-1} + \mathbb{Q}^{mm}(A_{T-1}|\mathcal{F}_{T-2})\mathcal{H}^{me,uu}_{T-1,T} + \mathbb{Q}^{mm}(A^c_{T-1}|\mathcal{F}_{T-2})\mathcal{H}^{me,du}_{T-1,T}.$$

In turn,

$$\mathcal{H}^{me}_{T-2,T} = h_{T-1} + E_{\mathbb{Q}^{mm}}(\mathcal{H}^{me}_{T-1,T}|\mathcal{F}_{T-2}) = E_{\mathbb{Q}^{mm}}(h_{T-1} + h_T|\mathcal{F}_{T-2}),$$

and (4.4) follows for $t = T-2$. The rest of the proof follows by similar arguments which are omitted. □

Combining (4.4) with Proposition 2.16, we deduce the following result.

Proposition 4.3. *Under assumptions (4.1) and (4.2), the value function process* $V_t(x)$ *is* $\mathcal{F}^S_t-$*adapted and given by*

$$V_t(x) = -\exp\left(-\gamma x - E_{\mathbb{Q}}\left(\sum_{i=t+1}^{T} h_i \middle| \mathcal{F}^S_t\right)\right),$$

with $\mathbb{Q}$ *as in (4.3) and* h *as in (2.8).*

The next result shows that in the reduced binomial model, the pricing functionals $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}}$ and $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}}$ coincide. Moreover, they are equal to the conditional expectation of the conditional certainty equivalent $\mathcal{C}^{(s,s+1)}_{\mathbb{Q}}$.

Proposition 4.4. *Let* $\mathbb{Q}$ *be as in (4.3) and* Z *be a random variable in* $(\Omega, \mathcal{F}, \mathbb{P})$. *For* $s = 0, 1, ..., T-1$, *the following statements are true.*

i) The single-step pricing functionals $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ *and* $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}(Z)$ *(cf. (3.7) and (3.5)) coincide*

$$\mathcal{P}^{(s,s+1)}_{\mathbb{Q}}(Z) = \mathcal{E}^{(s,s+1)}_{\mathbb{Q}}(Z).$$

ii) Moreover, the conditional certainty equivalence defined in (3.28) satisfies

$$E_{\mathbb{Q}}\left(\mathcal{C}^{(s,s+1)}_{\mathbb{Q}}(Z)\middle|\mathcal{F}_s\right) = \mathcal{P}^{(s,s+1)}_{\mathbb{Q}}(Z) = \mathcal{E}^{(s,s+1)}_{\mathbb{Q}}(Z).$$

Proof. i) From (3.7) we have

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right).$$

Property (4.4) implies $\mathcal{H}_{s+1,T}^{me}\in\mathcal{F}_{s+1}^{S}$, for $s=0,1,...,T-1$, and, thus,

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)-E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right).$$

Similarly,

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)=-E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right).$$

Combining the above with (4.3) we easily conclude.

ii) We only show that

$$E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z)\middle|\mathcal{F}_s\right)=\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z),$$

since the rest of the statements follow easily. To this end, using (4.3), (2.38) and (3.27), we deduce

$$\begin{aligned}E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z)\middle|\mathcal{F}_s\right)&=E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}}\left(e^{\gamma Z-\mathcal{H}_{s+1,T}^{me}}\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^{S}\right)\middle|\mathcal{F}_s\right)\\&\quad+E_{\mathbb{Q}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right).\end{aligned}$$

Using that in the reduced model $\mathcal{H}_{s+1,T}^{me}\in\mathcal{F}_{s+1}^{S}$, we obtain

$$\begin{aligned}&E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}}\left(e^{\gamma Z-\mathcal{H}_{s+1,T}^{me}}\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^{S}\right)\middle|\mathcal{F}_s\right)\\&=\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z)-E_{\mathbb{Q}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right),\end{aligned}$$

and the assertion follows. □

We are now ready to state the main theorems of this section. The first theorem, a direct consequence of the above result, states that in the reduced model the two pricing algorithms (presented on Theorems 22 and 23) coincide. It, also, states that the single-step indifference price functional yields a natural stochastic extension of the classical certainty equivalent rule.

The second theorem shows that in the reduced model the indifference price is *not* affected by the risk preference normalization point. The intuition behind this property is the following. In the general model, there are two sources of market incompleteness, one coming from the payoff and the other from the model itself. The latter affects the form of the value function which is also affected by the choice of the normalization point. Once the internal incompleteness is removed, the measurability of the minimal aggregate entropy reduces and scaling simplifications take place.

Theorem 4.5. *In the reduced binomial model, the indifference price $\nu_s(C_t)$ satisfies*

$$\nu_t(C_t)=C_t,$$

and, for $s=0,1,...,t$,

$$\begin{aligned}\nu_s(C_t)&=\mathcal{P}_{\mathbb{Q}}^{(s,s+1)}(\nu_{s+1}(C_t))=\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(\nu_{s+1}(C_t))\\&=E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(\nu_{s+1}(C_t))\middle|\mathcal{F}_s\right).\end{aligned}$$

Theorem 4.6. *In the reduced binomial model, the indifference prices are invariant with respect to the choice of the risk preference normalization point. Specifically, consider a claim written at* $t_0 = 0$ *and maturing at* $t = 0, 1, ..., T$, *yielding payoff* $C_t \in \mathcal{F}_t$. *Let* T, $\hat{T}$ *be two normalization points with* $\hat{T} > T$ *and* $\nu_s(C_t; T)$, $\nu_s\left(C_t; \hat{T}\right)$, $s = 0, 1, ..., t,$, *be the associated indifference prices. Then,*

$$\nu_s(C_t; T) = \nu_s\left(C_t; \hat{T}\right). \tag{4.5}$$

Proof. Using (4.4) we have $\mathcal{H}^{me}_{s,T'} \in \mathcal{F}^S_s$, for $T' = T$, $\hat{T}$ and $s \leq t \leq T$. Therefore, the claim $Z_t = \mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T} \in \mathcal{F}^S_t$ and, in turn, (3.37) implies, for $s = 0, 1, ..., t$, $Z_s = E_{\mathbb{Q}}(Z_t | \mathcal{F}_s)$. Equation (3.35), then, yields

$$\begin{aligned} \nu_{t-1}\left(C_t; \hat{T}\right) &= \nu_{t-1}(C_t - Z_t; T) + Z_{t-1} \\ &= \nu_{t-1}(C_t; T) - E_{\mathbb{Q}}(Z_t | \mathcal{F}_{t-1}) + Z_{t-1} = \nu_{t-1}(C_t; T), \end{aligned}$$

with $\mathbb{Q}$ as in (4.3). Similarly, for $s = t - 2$, we deduce, using (3.35),

$$\begin{aligned} \nu_{t-2}\left(C_t; \hat{T}\right) &= \nu_{t-2}\left(\nu_{t-1}(C_t; T); \hat{T}\right) \\ &= \nu_{t-2}(\nu_{t-1}(C_t; T) - Z_{t-1}; T) + Z_{t-2} \\ &= \nu_{t-2}(\nu_{t-1}(C_t; T); T) - E_{\mathbb{Q}}(Z_{t-1} | \mathcal{F}_{t-2}) + Z_{t-2} \\ &= \nu_{t-2}(\nu_{t-1}(C_t; T); T) = \nu_{t-2}(C_t; T). \end{aligned}$$

Proceeding iteratively and using similar to the above arguments, we obtain (4.5) for $s = 0, 1, ..., t - 2$. ☐

5. Numerical results

We study numerically the dependence of the indifference price on the risk preference normalization point T and on the risk aversion parameter γ. We consider a non-reduced incomplete model in which the stochastic factor affects both the claim's payoff and the transition probabilities of the stock price process.

Specifically, we assume that the values ξ^u_t, ξ^d_t, η^u_t and η^d_t, $t = 0, 1, ..., T$, (cf. (2.1) and (2.2)) are given by

$$\xi^u_t = 1 + \mu dt + \sigma\sqrt{dt} \quad \text{and} \quad \xi^d_t = 1 + \mu dt - \sigma\sqrt{dt},$$

and

$$\eta^u_t = 1 + bdt + a\sqrt{dt} \quad \text{and} \quad \eta^u_t = 1 + bdt - a\sqrt{dt},$$

with the constants σ, μ, a and b satisfying $-\sigma < \mu\sqrt{dt} < \sigma$ and $-a < b\sqrt{dt} < a$.

The time increment dt is given by $dt = \frac{1}{N}T$ where T and N represent, respectively, the risk preference normalization point and the number of periods in $[0, T]$. For $t = 0, 1, ..., T$, we choose

$$\mathbb{P}(Y_t = Y^u_t | \mathcal{F}_{t-1}) = 0.5,$$

$$\mathbb{P}(S_t = S^u_t | \mathcal{F}_{t-1}) = \begin{cases} 0.75, & Y_{t-1} \geq Y_0, \\ 0.5, & \text{otherwise} \end{cases}$$

and

$$Cor(\Delta S_t, \Delta Y_t | \mathcal{F}_{t-1}) = 0.5.$$

We consider a call option written only on the stochastic factor and maturing at T, i.e. $C_T = (Y_T - K)^+$. The model parameters are chosen as $\sigma = 0.2$, $a = 0.5$, $b = \mu = 0$ and $S_0 = Y_0 = K = 10$.

Figures 1 and 2 show, respectively, the dependence of the option's price on the risk preference normalization time, T, and the risk aversion coefficient, γ.

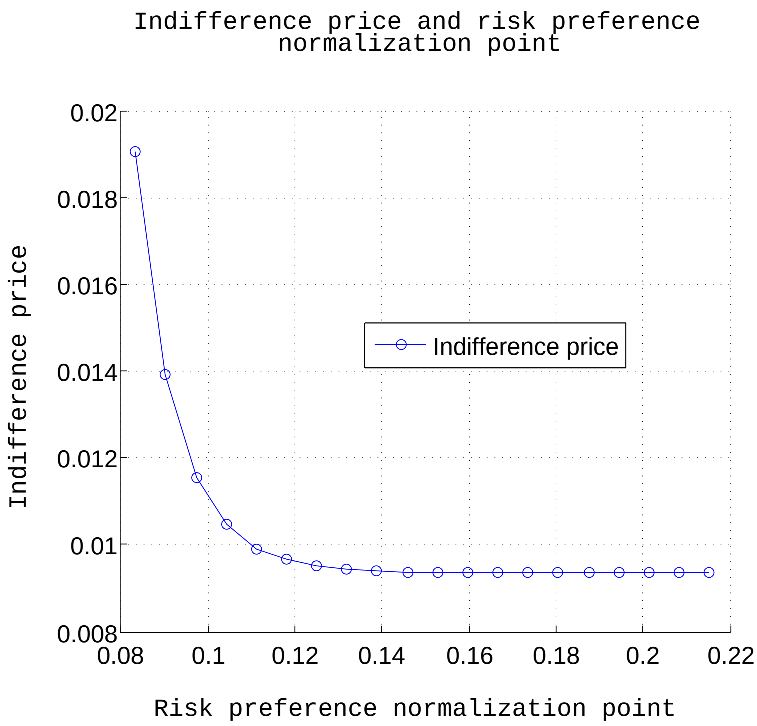

FIGURE 1. Dependence of the indifference price on the risk preference normalization point.

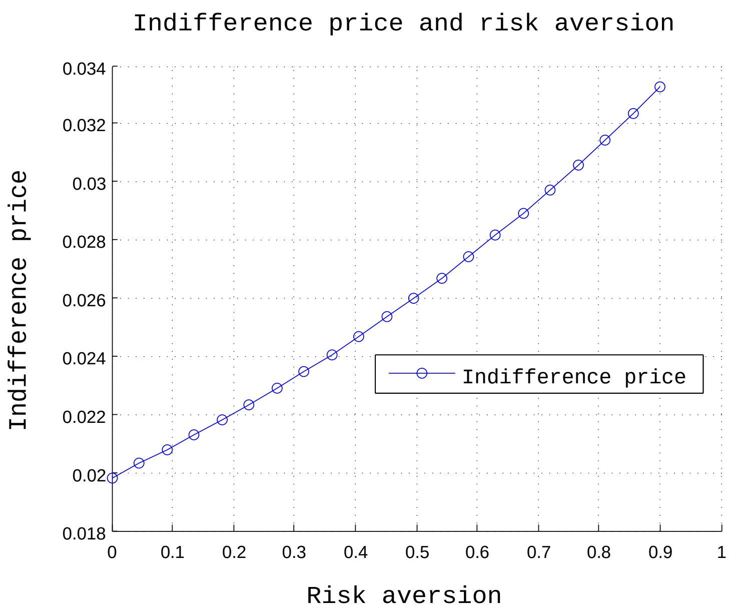

FIGURE 2. Dependence of the indifference price on the risk aversion.

In Figure 1, γ is fixed at 0.2. The number of time increments, N, varies from 60 to 155 in 5 unit-increments, and T varies from 0.083 to 0.215. The claim's expiration time is fixed at 0.083 years.

In Figure 2, $N = 115$, T=0.4792, the claim's expiration time is set at 0.25 years and γ varies from 0.001 to 0.901, with 0.045 increments.

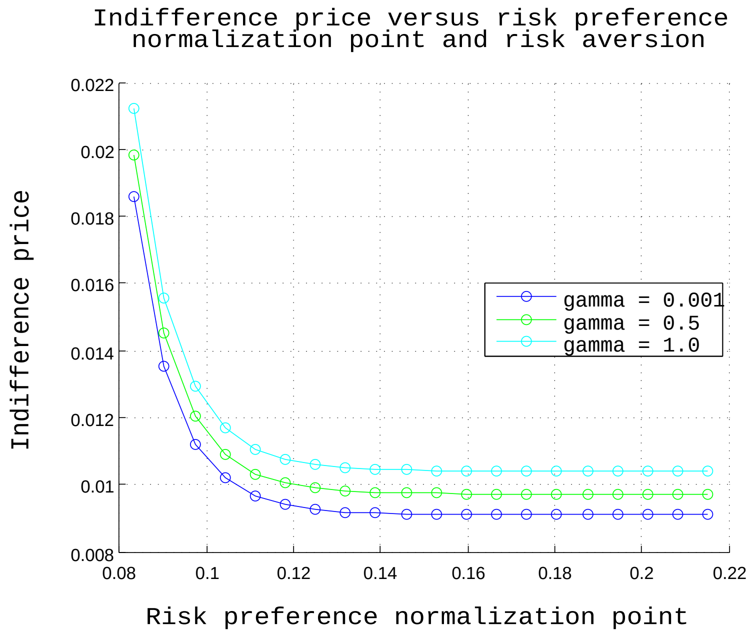

Figure 3. Dependence of the indifference price on the risk preference normalization point and the risk aversion.

Finally, Figure 3 incorporates changes in both T and γ. Therein, N and T are varying as in Figure 1.

As discussed earlier, the indifference price changes with the risk preference normalization point. For the chosen example, the price decreases as the normalization point moves further away from the expiration time. This dependence dissipates considerably when the normalization point is set at more than twice the contract's expiration. It is worth noticing that Figure 1 suggests that the price has a finite limit as $T \to +\infty$. Two interesting questions arise: i) does the price limit as $T \to +\infty$ coincide with a price obtained from any known pricing methodology and ii) do such limits exist for other, more general, contingent claims? We plan to address these questions in another paper.

Figure 2 shows the dependence of the indifference price on the risk aversion for a fixed horizon choice T. The latter is taken to be different that the claim's expiration time. One easily sees the well known result that the price is monotone with respect to γ.

Figure 3 displays results for various levels of risk aversion, namely, when $\gamma = 0.001$, 0.5, and 1.0. The graph highlights the interplay between the risk preference normalization point and the risk aversion coefficient. As it is well known, when the risk aversion approaches zero, the indifference price becomes linear.

Based on the latter observation, one may wrongly expect that the dependence on T vanishes for small values of γ. This is not, however, what the graph shows. For example, when $\gamma = 0.001$, significant dependence on T is still present on the price. In our opinion, this dependence may be attributed to the fact that while the pricing functional (3.5) is independent of the horizon choice, the associated pricing measure, the minimal entropy one, is (cf. (2.17)). As discussed earlier, this dependence is reversed if one uses the pricing algorithm in Theorem 23, in that now the pricing functional (3.7) depends on the normalization point while the pricing measure, the minimal martingale one, does not (cf. (2.11)). In both cases, the corresponding dependences prevail even if the risk aversion coefficient becomes very small.

5.1. **Comparison to linear pricing rules**

By far, both in theory and practice, the most popular pricing approach in incomplete markets is based on the construction of the price as the conditional expectation of the (discounted) payoff under a specific martingale measure. This approach produces the price in an ad hoc way and not as the outcome of replicating strategies, for perfect replication is not feasible. On the other hand, the deduced price inherits all desirable features of the one in complete markets, especially its semi-group property and linearity with respect to the claim's payoff[4].

How to choose this measure is the topic of very active research. In continuous time models with stochastic volatility that is correlated with the stock diffusion process - which is the closest analogue of our binomial model - a convenient representation of all equivalent martingale measures is via the family of the so-called $q-$optimal measures (see, among others, [9] and [12]). Three values are the most popular, namely, $q=0$, $q=1$ and $q=2$. The choice $q=0$ yields the minimal martingale measure while $q=1$ generates the minimal entropy measure. For $q=2$, the associated measure is the variance-optimal measure (see below).

The parameter q, then, becomes a useful index and a natural question arises, whether the associated prices are monotone with respect to it. The answer is affirmative and we refer the reader to [9] and [11]. To our knowledge, a similar analysis for non-reduced binomial models has not been carried out. This requires extensive work and will be addressed in the future.

Next, we take a preliminary step and only study numerically two families of "prices". The first one consists of expectations of the payoff under the minimal martingale and minimal entropy measures, $\mathbb{Q}^{mm}$ $(q=0)$ and $\mathbb{Q}^{me}$ $(q=1)$, and the minimal variance measure, denoted by analogy by $\mathbb{Q}^{mv}$ $(q=2)$. In an self-evident notation, we denote these quantities by

$$\pi_s^0\left(C_t\right)=E_{\mathbb{Q}^{mm}}^{(s,t)}\left(C_t\right),\ \pi_s^1\left(C_t\right)=E_{\mathbb{Q}^{me}}^{(s,t)}\left(C_t\right)\quad\text{and}\quad\pi_s^2\left(C_t\right)=E_{\mathbb{Q}^{mv}}^{(s,t)}\left(C_t\right). \tag{5.1}$$

The second family consists of their non-linear counterparts, namely,

$$\hat{\pi}_s^0\left(C_t\right)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(C_t\right),\ \hat{\pi}_s^1\left(C_t\right)=\mathcal{E}_{\mathbb{Q}^{me}}^{(s,t)}\left(C_t\right)\quad\text{and}\quad\hat{\pi}_s^2\left(C_t\right)=\mathcal{E}_{\mathbb{Q}^{mv}}^{(s,t)}\left(C_t\right), \tag{5.2}$$

where the nonlinear functionals are given in Definition 19 for $\mathbb{Q}=\mathbb{Q}^{mm},\mathbb{Q}^{me}$ and $\mathbb{Q}^{mv}$. Clearly the indifference price computed herein satisfies

$$\nu_s\left(C_t\right)=\hat{\pi}_s^1\left(C_t\right). \tag{5.3}$$

In analogy to the results in Propositions 3 and 5 where we provide, respectively the densities of the minimal martingale and minimal entropy measures ($q=0$ and $q=1$), we construct the density of the measure ($q=2$). We first recall the definition of the latter (see, for example, [3] and [27]).

The variance-optimal measure, $\mathbb{Q}^{mv}\left(\cdot\,|\mathcal{F}_t\right)$, $t=1,...,T$, is defined on $\mathcal{F}_T$ as the minimizer of $H_{t,T}^{mv}$ where

$$H_{t,T}^{mv}\left(\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)\mid\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right)=E_{\mathbb{P}}\left(\frac{\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)^2}{\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right),$$

for $t=1,...,T$ and $\mathbb{Q}\in\mathcal{Q}_T$, i.e.

$$\mathcal{H}_{t,T}^{mv}\left(\mathbb{Q}^{mv}\left(\cdot\,|\mathcal{F}_t\right)\mid\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right)=\min_{\mathbb{Q}\in\mathcal{Q}_T}H_{t,T}^{mv}\left(\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)\mid\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right). \tag{5.4}$$

The following results are, to the best, of our knowledge, new. They resemble the ones derived for the minimal entropy measure (see Propositions 5 and 9). With a slight abuse of notation, we refer to $\mathcal{H}_{t,T}^{mv}$ as the minimal entropy. The calculations to derive the formulae below are rather tedious and are omitted for the sake of presentation.

[4] Implicitly, the price operator is also independent of any individual investment horizon.

Proposition 5.1. *The variance-optimal measure, $\mathbb{Q}^{mv}$, satisfies, for $t = 1, ..., T$,*

$$\frac{\mathbb{Q}^{mv}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)} = \frac{\mathbb{Q}^{mv}\left(A_t|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,ud}}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,ud} + \mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,uu}}, \tag{5.5}$$

$$\frac{\mathbb{Q}^{mv}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)} = \frac{\mathbb{Q}^{mv}\left(A_t|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,uu}}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,ud} + \mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,uu}},$$

$$\frac{\mathbb{Q}^{mv}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)} = \frac{\mathbb{Q}^{mv}\left(A_t^c|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,dd}}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,dd} + \mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,du}}$$

and

$$\frac{\mathbb{Q}^{mv}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)} = \frac{\mathbb{Q}^{mv}\left(A_t^c|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,du}}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,dd} + \mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)\mathcal{H}_{t,T}^{mv,du}}$$

where A_t, B_t are as in (2.6) and $\mathcal{H}_{t,T}^{mv,uu}, \mathcal{H}_{t,T}^{mv,ud}, \mathcal{H}_{t,T}^{mv,du}, \mathcal{H}_{t,T}^{(mv,dd}$ are the values of the $\mathcal{F}_t-$measurable random variable $\mathcal{H}_{t,T}^{mv}$ (cf. (5.4)) conditional on $\mathcal{F}_{t-1}$.

Next, we provide an iterative scheme for the computation of the minimal entropy $\mathcal{H}_{t,T}^{mv}$.

Proposition 5.2. *The minimal entropy $\mathcal{H}_{t,T}^{mv}$ is given by the iterative scheme*

$$\mathcal{H}_{T,T}^{mv} = 0 \qquad \text{and} \qquad \mathcal{H}_{T-1,T}^{mv} = \frac{\mathbb{Q}^{mv}(A_T|\,\mathcal{F}_{T-1})}{\mathbb{P}(A_T|\,\mathcal{F}_{T-1})} + \frac{(1-\mathbb{Q}^{mv}(A_T|\,\mathcal{F}_{T-1}))^2}{1-\mathbb{P}(A_T|\,\mathcal{F}_{T-1})}$$

and

$$\begin{aligned}\mathcal{H}_{t-1,T}^{mv} = {} & \frac{\mathbb{Q}^{mv}(A_t|\,\mathcal{F}_{t-1})}{\mathbb{P}(A_t|\,\mathcal{F}_{t-1})}\frac{\mathcal{H}_{t,T}^{mv,ud}\mathcal{H}_{t,T}^{mv,uu}}{\frac{\mathbb{P}(A_tB_t|\mathcal{F}_{t-1})}{\mathbb{P}(A_t|\mathcal{F}_{t-1})}\mathcal{H}_{t,T}^{mv,ud} + \frac{\mathbb{P}(A_tB_t^c|\mathcal{F}_{t-1})}{\mathbb{P}(A_t|\mathcal{F}_{t-1})}\mathcal{H}_{t,T}^{mv,uu}} \\ & + \frac{(1-\mathbb{Q}^{mv}(A_t|\,\mathcal{F}_{t-1}))^2}{1-\mathbb{P}(A_t|\,\mathcal{F}_{t-1})}\frac{\mathcal{H}_{t,T}^{mv,dd}\mathcal{H}_{t,T}^{mv,du}}{\frac{\mathbb{P}(A_t^cB_t|\mathcal{F}_{t-1})}{1-\mathbb{P}(A_t|\mathcal{F}_{t-1})}\mathcal{H}_{t,T}^{mv,dd} + \frac{\mathbb{P}(A_t^cB_t^c|\mathcal{F}_{t-1})}{1-\mathbb{P}(A_t|\mathcal{F}_{t-1})}\mathcal{H}_{t,T}^{mv,du}}.\end{aligned}$$

We are now ready to compute the quantities in (5.1) and (5.2).

Figure 4 shows the dependence of the linear prices in (5.1) on the risk preference normalization point T. Naturally, the minimal martingale measure yields prices independent of T. To the contrary, the minimal entropy and the variance-optimal measure exhibit monotonically decreasing dependence on T. Moreover, the graph shows that prices are monotone with respect to parameter q, with smaller values of q giving higher prices. Thus the q-monotonicity result of [9] appears to be valid for the binomial model at hand as well.

Figure 5 describes the nonlinear prices in (5.2). All three prices depend on the risk aversion parameter, as expected from the definition of the nonlinear price functional. Moreover, for all three values $q = 0, 1, 2$, the dependence on the risk preference normalization point, T, bears the same character as for the linear pricing operators. Finally, we observe that the non-linear expectations are also monotone with respect to parameter q, with higher prices obtained for smaller values of q.

Remark 5.3. As we previously mentioned, the case $q = 1$ corresponds to the indifference price defined and constructed herein (cf. (5.3)). On the other hand, one can show that the case of $q = 0$ corresponds to an indifference price that is built on a new kind of risk preferences. These are the so-called forward performance processes (see, for example, [22]). A complete study of the forward indifference prices can be found in [21]. Interesting questions related to qualitative and quantitative studies of the traditional and the forward prices are left for future research.

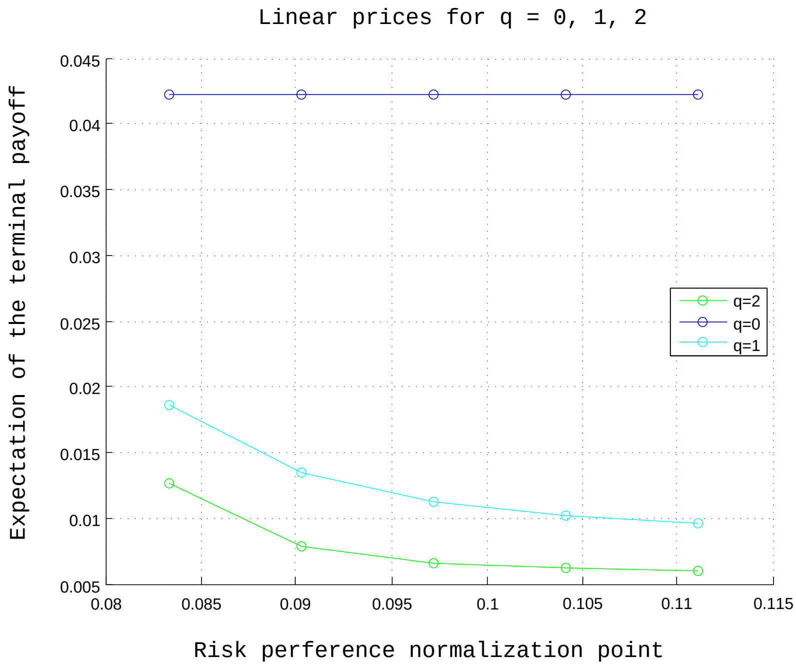

FIGURE 4. Dependence of the linear prices on the risk preference normalization point when the parameter q varies

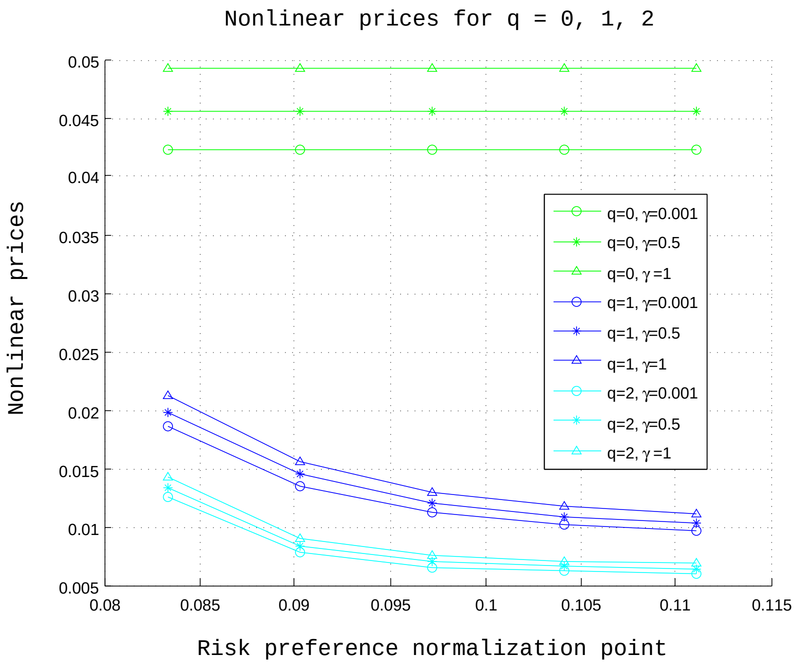

FIGURE 5. Dependence of the nonlinear prices on the risk preference normalization point when the parameters q and γ vary

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6. Glossary

- $V_t(x)$: value function (maximal expected utility) process
- T : terminal horizon, risk preference normalization point
- $\mathbb{Q}^{mm}(\cdot|\mathcal{F}_t)$, $t = 1, ..., T$, minimal martingale measure defined on $\mathcal{F}_T$
- $\mathbb{Q}^{mm}$: abbreviated notation for the minimal martingale measure
- $\mathbb{Q}^{me}(\cdot|\mathcal{F}_t)$, $t = 1, ..., T$, minimal entropy measure defined on $\mathcal{F}_T$
- $\mathbb{Q}^{me}$: abbreviated notation for the minimal entropy measure
- $\mathcal{H}_{t,T}^{me}$: minimal aggregate entropy with respect to the minimal entropy measure
- $\mathcal{J}_{\mathbb{Q}}^{(s,s+1)}$and $\mathcal{J}_{\mathbb{Q}}^{(s,s')}$: single- and multi-step entropic nonlinear functionals evaluated at a generic martingale measure measure $\mathbb{Q}$
- $\nu_s(C_t)$: indifference price, evaluated at s, of a claim maturing at $t = 0, 1, ..., T$
- $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s')}$: single- and multi-step price functionals evaluated at the minimal martingale measure
- $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s')}$: single- and multi-step price functionals evaluated at the minimal entropy measure
- $\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}$: conditional certainty equivalent evaluated at a generic martingale measure $\mathbb{Q}$

  $\nu_s\left(C_t; \hat{T}\right)$: indifference price, evaluated at s, of a claim maturing at t with terminal utility set at time $\hat{T}$, $t < \hat{T}$.

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