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ON A FUNCTIONAL EQUATION RELATED TO THE DETERMINANT OF SYMMETRIC TWO-BY-TWO MATRICES

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Abstract. The present work aims to find the solutions $f, g, h, \ell, m : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation $f(ux - vy,\, uy - vx) = g(x, y) + h(u, v) + \ell(x, y)\, m(u, v)$ for all $x, y, u, v \in \mathbb{R}$ without any regularity assumptions on the unknown functions. This equation is a generalization of a functional equation which arises from the characterization of the determinant of symmetric matrices.

1. Introduction

Let us define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = det \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

for all $x, y \in \mathbb{R}$. Then, since

$$det \begin{pmatrix} ux - vy & uy - vx \\ uy - vx & ux - vy \end{pmatrix} = det \begin{pmatrix} x & y \\ y & x \end{pmatrix} \; det \begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

we have the functional equation

$$f(ux - vy,\, uy - vx) = f(x, y)\, f(u, v) \tag{1}$$

for all $x, y, u, v \in \mathbb{R}$. Obviously, $f(x, y) = x^2 - y^2$ is a solution of the functional equation (1). A functional equation similar to (1) was studied in [2]. In the solved and unsolved problems column of the *News Letter* of the European Mathematical Society, the second author [6] posed the following problem: Determine the general solutions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation

$$f(ux - vy,\, uy - vx) = f(x, y) + f(u, v) + f(x, y)\, f(u, v) \tag{2}$$

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for all $x, y, u, v \in \mathbb{R}$. In [4], among others we gave the general solution of the functional equation (2) without any regularity assumptions on f. A generalization of the functional equations (1) and (2) is the following:

$$f(ux - vy,\, uy - vx) = g(x, y) + h(u, v) + \ell(x, y)\, m(u, v) \tag{3}$$

for all $x, y, u, v \in \mathbb{R}$. Here $f, g, h, \ell, m : \mathbb{R}^2 \to \mathbb{R}$ are unknown functions to be determined. For an account on the subject of functional equations the interested reader should refer to [1], [3], [5], [7], [8], and [9].

In this paper, we determine the general solution of the functional equation (3) without any regularity assumptions on the unknown functions f,g,h, ℓ,m : $\mathbb{R}^2 \to \mathbb{R}$. Our method of solution is elementary and direct.

2. Preliminary results

A function $M : \mathbb{R} \to \mathbb{R}$ is said to be a *multiplicative function* if and only if it satisfies $M(xy) = M(x)\, M(y)$ for all $x, y \in \mathbb{R}$. An identically constant multiplicative function M is either $M = 0$ or $M = 1$.

Let D be an interval in $\mathbb{R}$ such that whenever $x, y \in D$, then $xy \in D$. A function $L : D \to \mathbb{R}$ is said to be a *logarithmic function* if and only if $L(xy) = L(x) + L(y)$ for all $x, y \in D$. Note that if $0 \in D$, then $L = 0$.

Lemma 1. *Let $D \subseteq \mathbb{R}$ be an interval such that if $x, y \in D$, then $xy \in D$. The general solution $f : D^2 \to \mathbb{R}$ of the functional equation*

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) + f(y_2, x_2) \tag{4}$$

holding for all $x_1, x_2, y_1, y_2 \in D$ is given by

$$f(x, y) = L_1(x) + L_2(y), \tag{5}$$

where $L_1, L_2 : D \to \mathbb{R}$ are logarithmic functions. If $D = \mathbb{R}$, then $f(x, y) \equiv 0$ is the only solution to functional equation (4).

Proof. It is easy to check that the solution enumerated in (5) satisfies functional equation (4). Next, we show that (5) is the only solution of (4).

Suppose f is identically a constant, say $f \equiv c$. Then from (4) we have $c = 0$. Hence the identically constant solution of (4) is $f(x, y) = 0$ for all $x, y \in D$, which is a solution included in (5).

From now on we assume that f is not identically constant. Let $a \in D$ be a fixed element and $f : D^2 \to \mathbb{R}$ be such that it satisfies (4). Then

$$\begin{aligned} f(x, y) &= f(x, y) + f(a, a) + f(a, a) - 2f(a, a) \\ &= f(xaa, aay) - 2f(a, a) \\ &= f((xa)a, (ya)a) - 2f(a, a) \\ &= f(xa, a) + f(a, ya) - 2f(a, a) \\ &= f(xa, a) - f(a, a) + f(a, ya) - f(a, a) \end{aligned}$$

$$= L_1(x) + L_2(y)$$

where

$$L_1(x) := f(xa, a) - f(a, a)$$

and

$$L_2(y) := f(a, ya) - f(a, a).$$

Now we show that L_1 and L_2 are logarithmic functions in D. Consider

$$\begin{aligned} L_1(xy) &= f(xya, a) - f(a, a) \\ &= f(xya, a) + f(a, a) - 2f(a, a) \\ &= f(xyaa, aa) - 2f(a, a) \\ &= f(xa, a) + f(ya, a) - 2f(a, a) \\ &= f(xa, a) - f(a, a) + f(ya, a) - f(a, a) \\ &= L_1(x) + L_1(y). \end{aligned}$$

Hence L_1 is logarithmic. Similarly, one can show that

$$L_2(xy) = L_2(x) + L_2(y)$$

and hence L_2 is logarithmic. Thus

$$f(x, y) = L_1(x) + L_2(y)$$

where L_1 and L_2 are logarithmic functions. □

Lemma 2. *Let $D \subseteq \mathbb{R}$ be an interval such that if $x, y \in D$, then $xy \in D$. The general solution $f : D^2 \to \mathbb{R}$ of the functional equation*

$$f(x_1y_2, x_2y_1) = f(x_1, y_1)\, f(y_2, x_2) \tag{6}$$

holding for all $x_1, x_2, y_1, y_2 \in D$ is given by

$$f(x, y) = M_1(x)\, M_2(y), \tag{7}$$

where $M_1, M_2 : D \to \mathbb{R}$ are multiplicative functions.

Proof. It is easy to check that the solution enumerated in (7) satisfies functional equation (6). Next, we will show that (7) is the only solution of (6).

Suppose f is identically a constant, say $f \equiv c$. Then from (6) we have $c^2 - c = 0$ for all $x, y \in D$. Hence $c = 1$ or $c = 0$. Thus the only identically constant solutions of (6) are $f(x, y) = 1$ and $f(x, y) = 0$ for all $x, y \in D$, which are solutions included in (7).

From now on we assume that f is not identically constant. Let $a \in D$ be a fixed element and $f : D^2 \to \mathbb{R}$ be such that it satisfies (6) with $f(a, a) \neq 0$. Then

$$\begin{aligned} f(x, y) &= f(x, y)\, f(a, a)\, f(a, a)\, f(a, a)^{-2} \\ &= f(xaa, aay)\, f(a, a)^{-2} \end{aligned}$$

$$
\begin{aligned}
&= f((xa)a,(ya)a)\, f(a,a)^{-2}\\
&= f(xa,a)\, f(a,ya)\, f(a,a)^{-2}\\
&= f(xa,a)\, f(a,a)^{-1}\, f(a,ya)\, f(a,a)^{-1}\\
&= M_1(x)\, M_2(y)
\end{aligned}
$$

where

$$M_1(x) := f(xa,a)\, f(a,a)^{-1}$$

and

$$M_2(y) := f(a,ya)\, f(a,a)^{-1}.$$

Now we show that M_1 and M_2 are multiplicative functions in D. Consider

$$
\begin{aligned}
M_1(xy) &= f(xya,a)\, f(a,a)^{-1}\\
&= f(xya,a)\, f(a,a)\, f(a,a)^{-2}\\
&= f(xyaa,aa)\, f(a,a)^{-2}\\
&= f(xa,a)\, f(ya,a)\, f(a,a)^{-2}\\
&= f(xa,a)\, f(a,a)^{-1}\, f(ya,a)\, f(a,a)^{-1}\\
&= M_1(x)\, M_1(y).
\end{aligned}
$$

Hence M_1 is multiplicative. Similarly, one can show that

$$M_2(xy) = M_2(x)\, M_2(y)$$

and hence M_2 is multiplicative. Thus

$$f(x,y) = M_1(x)\, M_2(y)$$

where M_1 and M_2 are multiplicative functions. □

Remark 1. Note that if we replace D by a commutative semigroup in Lemma 1 and Lemma 2, then both of the lemmas are still valid.

Lemma 3. *The general solution $f, g : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation*

$$f(x_1y_2, x_2y_1) = f(x_1,y_1) + f(y_2,x_2) + g(x_1,y_1)\, g(x_2,y_2) \tag{8}$$

holding for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ is given by

$$\begin{cases} f(x,y) = \delta^2\, [M(xy) - 1] \\ g(x,y) = \delta\, [M(xy) - 1], \end{cases} \tag{9}$$

where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function and δ is an arbitrary constant.

Proof. It is easy to check that the solution enumerated in (9) satisfies functional equation (8). Next, we will show that (9) is the only solution of (8).

Suppose g is identically constant, say $g \equiv -\delta$. Then from (8) we have that

$$f(x_1y_2, x_2y_1) = f(x_1,y_1) + f(y_2,x_2) + \delta^2 \tag{10}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Now we define a function $F: \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) := f(x,y) + \delta^2 \tag{11}$$

for all $x, y \in \mathbb{R}$. Then (10) becomes

$$F(x_1y_2, x_2y_1) = F(x_1, y_1) + F(y_2, x_2) \tag{12}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. From Lemma 1, we see that the solution of (12) is

$$F(x,y) = 0 \tag{13}$$

for all $x, y \in \mathbb{R}$. Hence

$$f(x,y) = -\delta^2 \tag{14}$$

for all $x, y \in \mathbb{R}$, which is a solution included in (9).

From now on we assume that g is not identically constant. Interchanging x_1 with y_2 and x_2 with y_1 in (8) and comparing the resulting equations with (8) we see that

$$g(x_1, y_1)\, g(x_2, y_2) = g(y_1, x_1)\, g(y_2, x_2) \tag{15}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Since g is non-constant, there exists $x_0, y_0 \in \mathbb{R}$ such that $g(y_0, x_0) \neq 0$. Letting $x_2 = x_0$ and $y_2 = y_0$ in (15) we obtain

$$g(y,x) = \alpha\, g(x,y) \tag{16}$$

for all $x, y \in \mathbb{R}$ where α is a constant and $\alpha \neq 0$ (since otherwise $g \equiv 0$).

Setting $x_2 = y_2 = 0$ in (8) we obtain

$$f(0,0) = f(x_1, y_1) + f(0,0) + g(x_1, y_1)\, g(0,0) \tag{17}$$

for all $x_1, y_1 \in \mathbb{R}$. Thus,

$$f(x,y) = \alpha_0\, g(x,y) \tag{18}$$

for all $x, y \in \mathbb{R}$. Again note that $\alpha_0 \neq 0$ since otherwise $f \equiv 0$ and hence $g \equiv 0$ by (8). Therefore,

$$g(x,y) = k\, f(x,y) \tag{19}$$

for all $x, y \in \mathbb{R}$ where the constant $k \neq 0$.

Now, using (19) in (8) we see that

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) + f(y_2, x_2) + k^2\, f(x_1, y_1)\, f(x_2, y_2) \tag{20}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Using (16) and (19) we see that

$$k\, f(x,y) = g(x,y) = \alpha\, g(y,x) = \alpha\, k\, f(y,x)$$

which gives us

$$f(x,y) = \alpha\, f(y,x) \tag{21}$$

for all $x, y \in \mathbb{R}$.

Next, using (20) and (21) we have

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) + f(y_2, x_2) + \alpha\, k^2\, f(x_1, y_1)\, f(y_2, x_2) \tag{22}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Define $F: \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) := \alpha\, k^2\, f(x,y) + 1 \tag{23}$$

for all $x, y \in \mathbb{R}$. Then using (22) and (23) we have

$$F(x_1y_2, x_2y_1) = F(x_1, y_1)\, F(y_2, x_2) \tag{24}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

From Lemma 2 we obtain

$$F(x,y) = M_1(x)\, M_2(y) \tag{25}$$

for all $x, y \in \mathbb{R}$. Thus from (19) and (23) we have

$$\begin{cases} f(x,y) = \frac{1}{\alpha\, k^2}\,[M_1(x)\, M_2(y) - 1] \\ g(x,y) = \frac{1}{\alpha\, k}\,[M_1(x)\, M_2(y) - 1] \end{cases} \tag{26}$$

for all $x, y \in \mathbb{R}$.

Substituting (26) back into (8) we see that $\alpha = 1$ and $M_1 = M_2 = M$ for all $x \in \mathbb{R}$. Thus if we write $\delta = 1/k$, (26) becomes

$$\begin{cases} f(x,y) = \delta^2\,[M(xy) - 1] \\ g(x,y) = \delta\,[M(xy) - 1] \end{cases} \tag{27}$$

for all $x, y \in \mathbb{R}$, which is a solution to (8). This completes the proof of the lemma. □

Lemma 4. *The general solutions $f, \ell, m : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation*

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) + f(y_2, x_2) + \ell(x_1, y_1)\, m(x_2, y_2) \tag{28}$$

holding for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ are given by

$$\begin{cases} f(x,y) \equiv 0 \\ \ell(x,y)\, m(u,v) = 0 \end{cases} \tag{29}$$

and

$$\begin{cases} f(x,y) = \frac{1}{k_1\, k_2}\,[M_1(x)\, M_2(y) - 1] \\ \ell(x,y) = \frac{1}{k_2}\,[M_1(x)\, M_2(y) - 1] \\ m(x,y) = \frac{1}{k_1}\,[M_1(y)\, M_2(x) - 1], \end{cases} \tag{30}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1 and k_2 are arbitrary nonzero constants.

Proof. It is easy to check that the solutions enumerated in (29)-(30) satisfy functional equation (28). Next, we will show that (29)-(30) are the only solutions of (28).

Suppose $\ell(x,y)\, m(u,v) = 0$. Then (28) becomes

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) + f(y_2, x_2) \tag{31}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. From (31) and Lemma 1 we see that

$$f(x, y) = 0 \tag{32}$$

for all $x, y \in \mathbb{R}$, which is a solution to (28).

Now, suppose $\ell(x, y) = m(x, y)$ for all $x, y \in \mathbb{R}$. Then, (28) becomes

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + \ell(x_1, y_1)\, \ell(x_2, y_2) \tag{33}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. From (33) and Lemma 3 we see that

$$\begin{cases} f(x, y) = \delta^2 \left[M(xy) - 1\right] \\ \ell(x, y) = \delta \left[M(xy) - 1\right] \\ m(x, y) = \delta \left[M(xy) - 1\right] \end{cases} \tag{34}$$

for all $x, y \in \mathbb{R}$ where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function, which is a solution to (28) and a special case of (30).

Finally, suppose $\ell(x, y) \neq m(x, y)$ and neither is identically zero. Setting $y_2 = x_2 = 0$ in (28) we see that

$$f(0, 0) = f(x_1, y_1) + f(0, 0) + \ell(x_1, y_1)\, m(0, 0) \tag{35}$$

for all $x_1, y_1 \in \mathbb{R}$. Thus,

$$\ell(x, y) = k_1\, f(x, y) \tag{36}$$

for all $x, y \in \mathbb{R}$ where $k_1 \neq 0$ (since otherwise $\ell \equiv 0$).

Similarly, setting $y_1 = x_1 = 0$ in (28) we see that

$$f(0, 0) = f(0, 0) + f(y_2, x_2) + \ell(0, 0)\, m(x_2, y_2) \tag{37}$$

for all $x_2, y_2 \in \mathbb{R}$. Thus,

$$m(x, y) = k_2\, f(y, x) \tag{38}$$

for all $x, y \in \mathbb{R}$ where $k_2 \neq 0$ (since otherwise $m \equiv 0$).

Now using (28), (36), and (38) we have

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + k_1\, k_2\, f(x_1, y_1)\, f(y_2, x_2) \tag{39}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Define a function $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(x, y) := k_1\, k_2\, f(x, y) + 1 \tag{40}$$

for all $x, y \in \mathbb{R}$. Then (39) becomes

$$H(x_1 y_2, x_2 y_1) = H(x_1, y_1)\, H(y_2, x_2) \tag{41}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

From Lemma 2 we get

$$H(x, y) = M_1(x)\, M_2(y) \tag{42}$$

for all $x, y \in \mathbb{R}$. Thus using (36), (38), (40), and (42) we see that

$$\begin{cases} f(x,y) = \frac{1}{k_1\,k_2}\,[M_1(x)\,M_2(y)-1] \\ \ell(x,y) = \frac{1}{k_2}\,[M_1(x)\,M_2(y)-1] \\ m(x,y) = \frac{1}{k_1}\,[M_1(y)\,M_2(x)-1], \end{cases} \tag{43}$$

for all $x, y \in \mathbb{R}$ where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions, which is a solution to (28). □

3. Main result

Now we are ready to prove our main result.

Theorem 1. *The general solutions $f, g, h, \ell, m : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation*

$$f(ux - vy, uy - vx) = g(x,y) + h(u,v) + \ell(x,y)\,m(u,v) \tag{44}$$

holding for all $x, y, u, v \in \mathbb{R}$ are given by

$$\begin{cases} f(x,y) \equiv \alpha_1\,\alpha_2 - \beta_1 - \beta_2 \\ g(x,y) = \alpha_1\,\ell(x,y) - \beta_2 + \alpha_1\,\alpha_2 \\ h(x,y) = -\beta_1 \\ \ell(x,y)\,is\,arbitrary \\ m(x,y) \equiv -\alpha_1 \end{cases} \tag{45}$$

$$\begin{cases} f(x,y) \equiv \alpha_1\,\alpha_2 - \beta_1 - \beta_2 \\ g(x,y) \equiv -\beta_2 \\ h(x,y) = \alpha_2\,m(x,y) - \beta_1 + \alpha_1\,\alpha_2 \\ \ell(x,y) \equiv -\alpha_2 \\ m(x,y)\,is\,arbitrary \end{cases} \tag{46}$$

and

$$\begin{cases} f(x,y) = \frac{1}{k_1 k_2}\,[M_1(x+y)M_2(x-y)-1] + \alpha_1\,\alpha_2 - \beta_1 - \beta_2 \\ g(x,y) = \frac{1}{k_1 k_2}\,[M_1(x+y)M_2(x-y)-1] \\ \qquad\qquad + \frac{\alpha_1}{k_2}\,[M_1(x+y)M_2(x-y)-1] - \beta_2 \\ h(x,y) = \frac{1}{k_1 k_2}\,[M_1(x+y)M_2(x-y)-1] \\ \qquad\qquad + \frac{\alpha_2}{k_1}\,[M_1(x-y)M_2(x+y)-1] - \beta_1 \\ \ell(x,y) = \frac{1}{k_2}\,[M_1(x+y)M_2(x-y)-1] - \alpha_2 \\ m(x,y) = \frac{1}{k_1}\,[M_1(x-y)M_2(x+y)-1] - \alpha_1, \end{cases} \tag{47}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions, $\alpha_1, \alpha_2, \beta_2$ and β_1 are arbitrary constants, and k_1 and k_2 are nonzero arbitrary constants.

Proof. It is easy to check that the solutions enumerated in (45)-(47) satisfy functional equation (44). Next, we will show that (45)-(47) are the only solutions of (44).

First, we define functions $F, G, H, L, N : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{cases} F(x,y) := f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ G(x,y) := g\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ H(x,y) := h\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ L(x,y) := \ell\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ N(x,y) := m\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \end{cases} \tag{48}$$

for all $x, y \in \mathbb{R}$. Using (48) in (44) we see that

$$\begin{aligned} &F((x+y)(u-v), (x-y)(u+v)) \\ &\quad = G(x+y, x-y) + H(u+v, u-v) + L(x+y, x-y)\, N(u+v, u-v) \end{aligned} \tag{49}$$

for all $x, y, u, v \in \mathbb{R}$. Letting $x_1 = x+y,\ y_1 = x-y,\ x_2 = u+v,\ \textit{and}\ y_2 = u-v$ we have

$$F(x_1y_2, x_2y_1) = G(x_1, y_1) + H(x_2, y_2) + L(x_1, y_1)\, N(x_2, y_2) \tag{50}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Setting $x_2 = y_2 = 1$ in (50) we have

$$F(x_1, y_1) = G(x_1, y_1) + H(1,1) + L(x_1, y_1)\, N(1,1) \tag{51}$$

for all $x_1, y_1 \in \mathbb{R}$. Simplifying we get

$$F(x_1, y_1) = G(x_1, y_1) - \alpha_1\, L(x_1, y_1) - \beta_1 \tag{52}$$

for all $x_1, y_1 \in \mathbb{R}$ where α_1 and β_1 are arbitrary constants.

Similarly, setting $x_1 = y_1 = 1$ in (50) we have

$$F(y_2, x_2) = G(1,1) + H(x_2, y_2) + L(1,1)\, N(x_2, y_2) \tag{53}$$

for all $x_2, y_2 \in \mathbb{R}$. Simplifying we get

$$F(y_2, x_2) = H(x_2, y_2) - \alpha_2\, N(x_2, y_2) - \beta_2 \tag{54}$$

for all $x_2, y_2 \in \mathbb{R}$ where α_2 and β_2 are arbitrary constants.

Then using (50), (52), and (54) we have

$$\begin{aligned} F(x_1y_2, x_2y_1) - \beta = F(x_1, y_1) + F(y_2, x_2) + \alpha_1\, L(x_1, y_1) \\ + \alpha_2\, N(x_2, y_2) + L(x_1, y_1)\, N(x_2, y_2) \end{aligned} \tag{55}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ where $\beta = \beta_1 + \beta_2$. Next, we define functions $F_1, L_1, N_1 : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{cases} F_1(x,y) := F(x,y) + \beta - \alpha_1\,\alpha_2 \\ L_1(x,y) := L(x,y) + \alpha_2 \\ N_1(x,y) := N(x,y) + \alpha_1 \end{cases} \tag{56}$$

for all $x, y \in \mathbb{R}$. Thus using (55) and (56) we see that

$$F_1(x_1y_2, x_2y_1) = F_1(x_1, y_1) + F_1(y_2, x_2) + L_1(x_1, y_1)\,N_1(x_2, y_2) \tag{57}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

By Lemma 4, (48), (52), (54), and (56) we see that

$$\begin{cases} f(x,y) \equiv \alpha_1\,\alpha_2 - \beta \\ g(x,y) = \alpha_1\,\ell(x,y) + \beta_1 + \alpha_1\,\alpha_2 - \beta \\ h(x,y) \equiv \beta_2 - \beta \\ \ell(x,y)\,is\,arbitrary \\ m(x,y) \equiv -\alpha_1 \end{cases} \tag{58}$$

$$\begin{cases} f(x,y) \equiv \alpha_1\,\alpha_2 - \beta \\ g(x,y) \equiv \beta_1 - \beta \\ h(x,y) = \alpha_2\,m(x,y) - \beta + \beta_2 + \alpha_1\,\alpha_2 \\ \ell(x,y) \equiv -\alpha_2 \\ m(x,y)\,is\,arbitrary \end{cases} \tag{59}$$

and

$$\begin{cases} f(x,y) = \frac{1}{k_1\,k_2}\,[M_1(x+y)\,M_2(x-y) - 1] + \alpha_1\,\alpha_2 - \beta \\ g(x,y) = \frac{1}{k_1\,k_2}\,[M_1(x+y)\,M_2(x-y) - 1] \\ \qquad\qquad + \frac{\alpha_1}{k_2}\,[M_1(x+y)\,M_2(x-y) - 1] + \beta_1 - \beta \\ h(x,y) = \frac{1}{k_1\,k_2}\,[M_1(x+y)\,M_2(x-y) - 1] \\ \qquad\qquad + \frac{\alpha_2}{k_1}\,[M_1(x-y)\,M_2(x+y) - 1] - \beta + \beta_2 \\ \ell(x,y) = \frac{1}{k_2}\,[M_1(x+y)\,M_2(x-y) - 1] - \alpha_2 \\ m(x,y) = \frac{1}{k_1}\,[M_1(x-y)\,M_2(x+y) - 1] - \alpha_1 \end{cases} \tag{60}$$

for all $x, y \in \mathbb{R}$ where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions. Note that $\beta = \beta_1 + \beta_2$. Hence (58) - (60) yield the asserted solutions (45) - (47). The proof of the theorem is now complete. □

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