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The Transfer Dilemma[1]

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Abstract

In this paper we provide an overview of research on transfer, highlighting its main tenets. Then we look at interviews of two fifth grade students learning about mathematical concepts regarding operations on positive and negative quantities. We attempt to focus on how their learning is influenced by their prior knowledge and experience. We take the position that transfer is a theory of learning and we attempt to show that it cannot provide a solid foundation for explaining such examples of learning.

[1] This paper evolved from a presentation at the symposium "If Not Transfer, Then What?", 1998 Annual Meeting of the American Educational Research Association, San Diego, CA, April 17, 1998. Thanks to Giyoo Hatano and King Beach for their comments.

Transfer: Problem To Be Explained Or Theory Of Learning?

Transfer, the issue of "how knowledge acquired in one situation applies (or fails to apply) in other situations (Singley & Anderson, 1989, p. 1)", has long been regarded as one of the most important problems in learning. Anyone hoping to account for learning would have to come to grips with the problem of transfer.

We take issue with the widely accepted view that learning generalizes through transfer even though we recognize that prior knowledge and experience play fundamental roles in learning. This may seem contradictory. But we urge the reader to keep an open mind over the matter. Views of transfer have become so deeply woven into the collective thinking of educators and learning scientists that it is difficult to put them aside.

Transfer is not usually treated as a theory. Traditional wisdom views transfer as a *problem* to be explained, a topic for the textbook on learning. We believe, however, that transfer contains theoretical assumptions and should be thought of as a theory about learning. If transfer is not an uncontested fact but rather a theoretical approach to learning then it makes sense to ask how well transfer has stood up to the task. We can ask how well it has fulfilled its role and what issues lie beyond its purview. In addition we can consider alternatives to transfer for explaining how learning depends upon prior knowledge, learning, and experience. This is what we will attempt to do in this paper.

We will first provide an overview of research by transfer theorists, highlighting their shared tenets. Then we will look at interviews of middle school students learning about mathematical concepts. We will attempt to point out instances where their learning is influenced by what they already know. We hope to show why a theory of transfer cannot provide a solid foundation for explaining such examples of learning. We conclude by describing some of the necessary features an alternative theory of learning will need to address and we explore Piaget's notion of assimilation and accommodation as seeds for a theory of learning that takes into account the complexity of the learning phenomenon.

A Rereading of Transfer Research

Transfer theory arose out of an empiricist position in which the learner was viewed as a passive agent whose learning depended upon the similarity between a presumed "original learning situation" and the present situation. Thorndike and Woodworth (1901) expressed this view in terms of their hypothesis of "identical elements". For them, identical elements were objective, physical features common to situations. Learning was conceived of as a kind of generalization of response based upon an inherent similarity between stimuli. Thorndike's views about mathematics education were consistent with his notions about identical elements. He proposed that students should face problems as similar as possible to those they will later on encounter in the workplace (Thorndike, 1913); mathematical problems with Roman numerals or atypical values, for example, were not to be used since they differ significantly from the problems a future worker or citizen might be asked to solve[2].

Other researchers relaxed the requirement of surface similarity, focusing instead on structural or conceptual similarities across tasks. This is indeed a major step beyond stimulus response psychology. Judd (1908), for example, argues that transfer occurs when learners have previous conceptual understanding of the original problem. In the 1940's, Gestalt theorists revived the focus on understanding, making the distinction between blind transfer and transfer based on an understanding of the structural features of the task (Wertheimer, 1961). In Wertheimer's experiments, original and target tasks were still rather similar: often the new task was merely a slight variation of the original task. For instance, after learning how to compute the area of a parallelogram with parallel sides along the horizonal, students were asked to compute the area of parallelograms with parallel sides running vertically. Subjects who received instructions stressing the structural features of the task were more successful (presumably "more prone to show transfer") than those who learned the procedure without benefit of clarifying instructions.

During the 70s and 80s, transfer researchers focused on rules for solving puzzle-like problems such as the Missionaries and Cannibals (Reed, Ernst, & Banerji, 1974), the Tower of Hanoi problems (Hayes & Simon, 1977), and Duncker's radiation problem (Gick & Holyoak, 1980, 1983). In these laboratory studies participants first learned to solve a highly circumscribed problem. They then attempted to solve problems about topics totally unrelated to the original problem but which, from the point of view of the experimenter, embodied the same logical structure and could presumably be solved by applying similar procedures. These studies generally showed that subjects failed to solve the target tasks unless prodded to refer to the previously learned procedures or problem situation.

For many years transfer research was a study in frustration: many studies failed to provide evidence that transfer occurs at all. The problem with taking such results at face value lies in the fact that researchers had unwittingly put blinders on when looking for evidence of transfer. For example, when a psychological subject failed to make use of information about infantry attacks in devising a medical procedure for destroying a tumor, Dunker (1945) concluded that students did not exhibit transfer. Dunker's results did not actually show that students' answers were unrelated to their prior knowledge and experience; they merely showed that the students did not draw upon the particular learning situations he had hoped they would draw upon. So the failure to detect transfer was almost certainly a reflection of the tunnel vision of researchers.

Lave (1988) confronted theoretical and methodological suppositions of studies which fail to show transfer, pointing out that in laboratory studies of transfer "... the target analogy is a pre-formulated, static object, and its unmodified use by the subject is the object of exercise" (p. 37). Lave argued that learning and thinking take place in specific

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[2] As reasonable as this may position may seem at first glance, it leads to a peculiar view about how mathematics should be taught and how it is best learned. Essentially it presumes that all curriculum activities must pass a litmus test for realism. As we have argued elsewhere (Carraher & Schliemann, in press) flexible mathematical reasoning requires being able to suppress realistic concerns.

contexts that, far from being incidental, are essential to what is learned and thought. Situated learning approaches confronted the theoretical and methodological premises of transfer studies, pointing to the contextualized nature of thought. They rightly questioned the premise of transfer that knowledge is ready made. To the extent that particular features of the new situation at hand play a role in learning, knowledge is not prêt a porter.

However, as much as situated analyses represent progress, they do not provide an account of how an individual's understandings are generalized beyond the specificity of their originating contexts, nor how previous understandings are brought into play in new situations. We would argue, unlike situated learning approaches, that in many cases there is an outright struggle to reconcile former experience with the information at hand. As interest in the study of cognition in natural, less-constraining contexts grew, researchers began to find evidence of how learners used previous knowledge to solve new problems. Brown (1989) found that toddlers use previous knowledge to approach new situations if they involve familiar contents. Bassok & Holyoak (1989) show that students who learn arithmetical progressions as a general model spontaneously recognize that the same equations could be used to solve physics problems related to their classroom curriculum. Everyday cognition studies also show that individuals need to be familiar with new situations to recognize that previously learned strategies can play a useful role (see studies by Schliemann & Nunes, 1990 and Schliemann and Magalhães, 1990, and the review of transfer of everyday knowledge by Guberman & Greenfield, 1991).

It stands to reason that learners rely upon former knowledge and experience. To capture evidence of this, however, researchers would need to broaden their outlooks. They had initiated a shift in this direction when they removed the requirement that the same stimuli (identical elements) be present in the original and the transfer situation. Over the years, theorists embraced more abstract, structural accounts of similarity across situations. They would need to consider larger time frames than those associated with laboratory training studies. Even more importantly, they would need to be open to the influence of forms of prior knowledge and experience that they had not necessarily anticipated.

In all of the above views of transfer, from Thorndike's associationism to situated cognition analyses, transfer is characterized as the application, to a new problem situation or content, of previously learned solution strategies, without structural transformation of what was learned. However, as we looked at the complexity of mathematical learning, we came to believe that (1) former knowledge and experience may contribute to learning in subtle ways and (2) learning only exceptionally relies upon the direct borrowing of procedures from one, easily identifiable "prior situation".

Learning entails the building up of structures, schemes and intuitions that have developed over long periods of time. This shift in our argument happens before the view expressed in the first sentence is given proper airing. Yet mathematical learning also has a distinctly situated character. Children's representations of mathematics draw from the particular social and physical activities in which they engage (buying and selling, comparing, measuring, and so on). Their ways of conceiving and doing mathematics owe much to the specific representations and tools they learn to use such as abacus, weights and measures, notational systems, etc. (Nunes, Schliemann, & Carraher, 1993; Hatano, 1982). Greeno, Moore, & Smith (1993) argue that transfer depends on transformations of activity and is characterized by the adoption of previous interactional roles in new situations. Beach (1992), following Lave (1988), analyzed transfer not only as continuities between activities but as the transformation and construction of new knowledge along a period of time. He found that Nepalese students working as shopkeepers used previously learned school procedures at work and that shopkeepers participating in an adult education class used non-written procedures developed at work in combination with written school procedures. We proposed the concept of "situated generalization" (Carraher, Nemirovsky, and Schliemann, 1995) to highlight the paradoxical interpenetration and inseparability of concrete (specific and contextual) and abstract (general) in learning. This position stands in general accordance with Kant's ‘schemes’, with Cassirer's critique of Aristotelian concepts, and with Piaget's concepts of assimilation-accommodation. This approach characterizes learning as crafted from a wealth of previous learning experiences. This is very different from the direct borrowing of transfer theory.

We will exemplify our points by looking carefully at episodes of two fifth graders trying to solve an open-ended problems involving directed numbers in a software environment (the Visual Calculator™) we designed. We say “exemplify” because we realize that one cannot dislodge, in one fell swoop, a concept rooted so deeply in the history of learning theory. The interview offer numerous instances of the students attempting to reconcile old knowledge with new knowledge and are thus well suited to the present discussion about transfer.

Two Examples

One of the major challenges students face in middle school mathematics consists in trying to understand how operations on numbers correspond to changes in physical quantities. The beliefs that “multiplying makes bigger” and “dividing makes smaller” are cases in point. Students are regularly introduced to multiplication as repeated addition. They are encouraged to think of the multiplication of one number by another as adding one number a certain number of times. A similar reasoning is encouraged for scalar multiplication—multiplication of a physical quantity or amount by a number. The student is asked to think of the process as a joining of *n* instances of a particular quantity. Given that these beliefs are consistent with the results of multiplication by natural multipliers (integers greater than zero), they need not be considered misconceptions. But our focus in the present discussion is not whether such beliefs are right or wrong. Rather we wish to know how former experience with physical quantities and natural numbers comes to bear on new situations.

This is easier said than done. We need to create a situation in which we can observe students making predictions about arithmetical operations on quantities. But we also need to have a sense of what their reasoning is relying upon. Our approach was to conduct open-ended clinical interviews in a specially staged setting. We employed a software environment called the *Visual Calculator* to provide an environment in which quantities are embodied as magnitudes (directed line segments or sticks) and the operations of addition, subtraction, multiplication, and division are represented through joining, removing, n-folding and its inverse (see Carraher, 1993, 1996, and Carraher & Schliemann, 1993).

Our goal in designing the software was to investigate student's understandings of the relationships among physical magnitudes and arithmetical operations. We were familiar with the extensive literature showing that middle school students commonly had difficulty reconciling the effects of arithmetical operations with effects upon quantities. And we felt that the software we had developed would provide an appropriate framework for such an interview since it displayed visual representations of magnitudes and how they changed under arithmetical operations. We originally expected that the setting would allow us to explore how division, multiplication affected quantities and how fractional quantities could be understood as resulting from multiplying and dividing a quantity by integers. In introducing the students to the environment, we found it necessary to go over several additional ideas, including the meaning of negative numbers and their roles in multiplication and addition. In this paper we will focus on a segment in the interview during which we explored such relations.

We view the software as merely providing a context for discussing mathematical concepts. Although the software renders quantities as crisp visual images on the monitor, there is no reason to expect that the students will understand the concepts merely by inspecting the diagrams and notation. On the contrary, every configuration of objects on the computer screen was open to multiple interpretations. One of our main points is that, at almost every turn, the students are trying to reconcile what they observe happening on screen with rich interpretive schemes. In fact, the degree to which the students adjust their understanding to the cases at hand proves to be central to the present argument.

The first example concerns how the students interpret the "multiplication of a physical magnitude by a negative integer". Here they use what they know about "taking away" and "multiplication by integers" to predict and explain what takes place in the new situation. In the second example they predict and explain the result of repeatedly adding positive quantities to a(n initially) negative quantity. A tension arises between "absolute magnitudes" and "directed magnitudes" as well as their respective relations to natural numbers and integers.

Understanding negative numbers and operations on negative numbers is no trivial issue. Although computation with negative quantities appear relatively early in the history of mathematics it is only in the 19th century that present day conceptions and representations of negative numbers became fully accepted (Boyer, 1985). However, it appears that even 4 and 5 year-olds can deal with negative quantities in activities where positive and negative numbers represent, respectively, forward and negative movements along a strait path (see Davidson, 1987). Moreover, Santos (1990) and Carraher (1990) found understanding and use of negative numbers in everyday situations in the absence of formal written representation. Students difficulties start to appear when formal representation for operations with negative numbers comes into play. Mukhopadhyay, Peled e Resnick (1989) show that third to ninth grade American children can operate with negative quantities on a number line but do not spontaneously relate their mathematical computations to contexts referring to money transactions. Borba (1998) found that fourth-grade Brazilian children could solve problems involving negative quantities (stories about money transactions and variation in temperature) but could not deal with written expressions involving similar relations. Written representations, as they do not correspond to these children's intuitive methods, seem to be a source of students' difficulties in operating with directed numbers in school contexts (Nunes, 1993). Absence of good contextual models to represent subtraction, multiplication, and division involving negative numbers make these operations particularly difficult for students (Bell, Costello, and Küchemann, 1985).

1. Negative Multipliers

We will first look at how the two fifth grade students, Tallulah and Emily, predict and explain what happens when they multiply a quantity by a negative number. Figure 1 outlines the steps in this interview segment. The uppermost segment, U, was introduced by clicking the "+ U" button at the top left of the screen. The next result, -U (or U times minus one) resulted from multiplying U by minus one (see the second command button from the left on the screen). The interview proceeds slowly, with the interviewer asking the students to predict what will happen before an operation is carried out. After they see the results, they try to explain what took place and then attempt to use what they have learned when they move to the next item in the interview. We will review now the part of the interview that corresponds to the five line segments and notations on the screen.

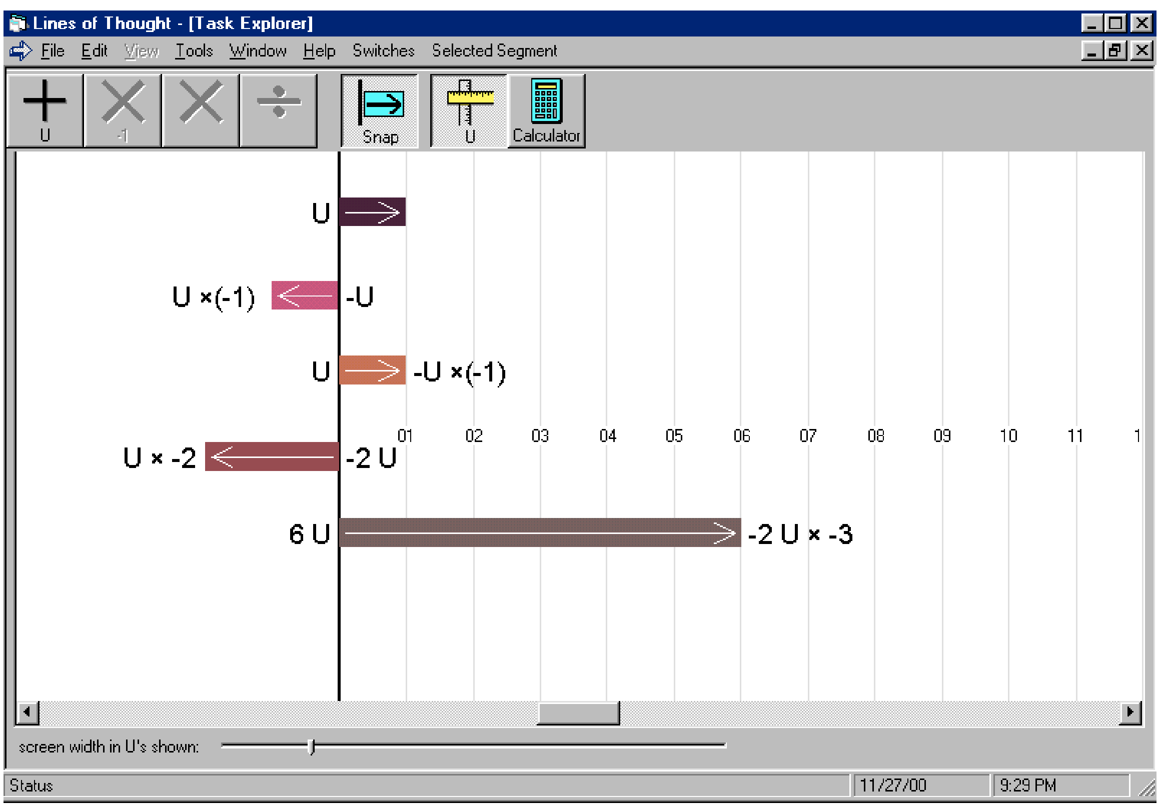

Fig. 1: Screen shot showing, from top to bottom, an overview of steps in the present part of the interview.

A Negative Multiplier As An Obliterator

The interviewer, David, starts by asking the girls to describe what would happen to the segment U (first line segment shown on the screen) if it is multiplied by minus 1:

David: Here's another thing that you might want to think about. If I put another U there and multiply it by a minus 1 (I have a special key just for that). All right, who's got a prediction there?

Emily: **It won't be anything**.

David: It won't be anything? It's gonna kinda wipe out? OK… Emily thinks it won't be anything if you multiply this U here by minus 1. What about you? (*David realizes the possible confusion between "you" and "U".)* I mean Tallulah. That was a bad choice of unit name, wasn't it?

Tallulah: Umm, I'm not exactly sure. I guess (*inaudible*...) I'm not really sure.

David (to Emily): OK, so you're saying it will just go away. Negative… sort of like zapping it.

Tallulah: Yes.

At the start Tallulah and Emily show that they are not familiar with the convention that multiplication by negative one merely changes the direction of a quantity. Emily predicts that "it won't be anything", as if the minus one conveyed a sense of negation, annihilation. It is possible that she understands the operation as equivalent to joining the initial value, U, with its additive inverse, -1U. Her answer could be an attempt to use what she knows about subtraction of negative numbers. Tallulah is not sure what to expect.

"Small Like A Fraction" Versus "Small Like A Negative Number"

David then asks Emily to perform the operation on the screen by clicking on the "□–1" button. The second line segment on Figure 1 shows the result of that action; namely, the result is a directed line segment beginning at zero and ending at –1. After observing the result they attempt to explain it. The following dialogue is a continuation of the dialogue begun above.

David: Well, go ahead Emily, you click on the minus 1 button there. What happened there?

Emily: It's minus 1.

David (*extending her phrase*): It's minus 1U.

Emily: OK.

David: And did it get bigger or smaller, or is it the same?

Emily: **Smaller, because it's less than here, if this is zero**...

David: Yes, that's what it is, That's what we're going to use it as.

Emily: **So everything to the right of it is a whole number and everything to the left of it is like a fractional number sort of, so it's small**.

David: So this is a fraction of a number?

Tallulah: Well, yeah because it is smaller than a whole number.

Emily correctly responds that the resulting segment is less than zero; she seems to be making this judgment based on what she has learned about the order of numbers on the number line. However, shortly afterwards, Emily hints that she is understands the order of numbers and quantities in a peculiar way. She states that numbers to the left of zero are fractions and Tallulah agrees with her stating that this is so “because it is smaller than a whole number”. From their previous understandings about directed numbers and fractions, they may be considering that both, negative numbers and (common) fractions are smaller than one (and hence smaller than all the positive integers). However, when the interviewer asks whether the result lies between zero and one (see below) Tallulah reviews her position and states that the result corresponds to a “negative number, below zero”, rather than a fractional number (note, however, that these are not mutually exclusive classifications).

David: Is it between zero and one, this thing here?

Girls: No.

Tallulah: **It's a negative number, below zero**.

David: OK, so you’re revising your opinion though. You first thought it was a fraction, but now you say it's a negative number.

Tallulah: Yes.

David: It's hard to know how to keep those apart. A fraction seems like less than a whole number, doesn't it?

Tallulah: Yes, part of a whole number.

This simple example reveals how unsatisfactory it is to characterize the students’ reasoning as “detecting similarity across situations”. Instead, they are wrestling with multiple pieces of evidence and trying to interpret the new situation on the basis of concepts and schemes that are part of their previous experience such as the number line, fractions, and subtraction of negative numbers. The values shown to the right of the vertical original line are whole numbers; those to the left of the origin presumably are not like the ones to the right. Further, numbers get smaller as one proceeds from right to left on the number line. The two features “not whole number” and “smaller” suggest the idea that the resultant segment represents a fraction. However negative numbers satisfy the present requirements more than fractions do since the former lie to the “left of zero”.

At this point the students tend to correctly view the result as corresponding to a negative number or measure. Let’s now see how they approach the issue of multiplying a quantity by a negative number.

Negative Multipliers Push Quantities In a Negative Direction

In the next step the result of the first operation (-1U in Figure 1) will serve as the input for the inverse operation, times –1. David asks them to predict what will happen if one multiplies the quantity –1U by minus 1. The students will only see the result of this second operation (Third line segment in Figure 1) after they have made and explained their predictions.

David: OK, Here's another prediction. You'll notice I have you making predictions all the time, because I want to see if these things surprise you or kind of go along with the way you're thinking about operations. We have this one highlighted. What if you hit minus 1 again, times minus 1?

Tallulah: **I think it would be minus 2 U's**.

David: Minus 2 U's? OK, let me just make some space over here. OK, what about you? What about Emily, what do you think?

Emily: Well, I thought that **if you took two negative numbers and added them**... well....

Tallulah: **We're learning all this stuff in math, but it's kind of confusing. We're learning in math class how to add and subtract negative numbers, and it's something... It's really confusing**. **If you're adding, you're really subtracting** ...

David: Yes, that sounds like it would be really confusing.

Emily: **It might be zero or a positive…**

The discussion from the above interview segment makes it clear that the students are unsure what to predict. Tallulah first states that multiplying –U by –1 will yield –2U. Her explanation appeals to the idea that she is adding two negative quantities: "Well, I thought that if you took two negative numbers and added them…". She knows that the present problem has some relationship to the subject of recent mathematics lessons, but admits that it is confusing ("It's really confusing. If you're adding, you're really subtracting"). Since operations involving negative numbers may produce counter-intuitive results, she reckons that "it might be zero or a positive (number)…".

Are We Adding Or Multiplying?

Sensing that the students are wavering between viewing the operation as adding and multiplying, David tries to get them to commit themselves to one interpretation.

David: OK, well, you've both been talking about adding and subtracting, but this thing up here that we've been clicking on says times minus 1. **Is that adding, subtracting, multiplying or dividing... or none of those**?

Tallulah & Emily: **Multiplying**.

David: Multiplying? You said multiplying, and you said multiplying by a negative number. **Does that minus 1 means "take away"?**

Tallulah: No, wait... you mean the minus 1?

David: The minus 1 on the block that says *times* minus 1. Does this mean take away?

…

Tallulah: OK. **I think it means "minus 1", like the negative number 1**.

David: OK. That's different?

Tallulah: (Different) **than take away 1, yes**.

When asked to clarify the operation both choose multiplying. David wonders whether the minus sign in "□ (-1)" may be a source of confusion and asks whether the minus 1 would mean "take away". They differentiate, with the interviewer, between two meanings of the negative sign but opt for "take away" interpretation over the directional interpretation. He then asks once again for a prediction for the result of multiplying -1U by minus 1.

David: So, let's go back to the prediction. We've got this minus 1 highlighted, we're going to hit the times minus 1, what's going to happen?

Emily: It's going to get higher.

David: It might get higher? Like *that* way (*pointing upwards*)?

Emily: um

David: I don't think you mean that

The interviewer is playing with the fact that increasing values are sometimes represented through vertical displacement, and sometimes, as in the case of a number line, through a left-to-right displacement. Emily appropriately answers:

Emily: Like where the positive numbers are.

…

Tallulah: **I'm kind of confused, too, but I agree with Emily that it will go toward where the positive numbers are**.

David: How far will it go? Where will it end up?

Tallulah & Emily**: zero... or one.**

David: OK, zero or one. Try it... (*They carry out the multiplication on screen*.) It went to the one. Does that make any sense?... Actually That's what we're going to mean in this environment. What we're going to mean by multiplying by minus 1. OK, I mean that's just kind of the rule, so if you keep doing that you'll flip back and forth. OK?

Tallulah & Emily: Yes.

As Tallulah and Emily agree that the result changes direction, David performs the operation obtaining +1U as the result (third line segment in Figure 1) and asserts that this is merely the convention followed: times negative one "flips" a quantity.

Does The Result Equal The Multiplier?

Now, for the first time the interviewer proposes to vary the absolute magnitude of the multiplier. The question is: what effect should this have on the result? Tallulah predicts that the result will be "minus 2 units" and Emily chooses "minus 1 or minus 2":

David: Now if you multiply (*U*) by minus 2, then you'll still flip it, but it will be a little different. What do you think is going to happen there?

Tallulah: **I think it's going to be minus 2 units**.

David: Uh huh, Emily's not so sure. What do you think?

Emily: Umm, **minus 1 or minus 2**.

David: Well, we multiplied it before by minus 1 and it became minus 1... when it was just like this, didn't we?

Emily: Yes.

David: Will it always go to minus 1 when you multiply it by a negative number?

Emily: No, umm... **negative 2**.

David reminds them that the prediction -1U would be precisely the same result as that previously observed when 1U was multiplied by –1. Faced with the question of whether minus 1 should be expected for any multiplication by a negative number, Emily eliminates the first prediction and correctly claims that the result will be –2(U).

David: Negative 2? Do you want to click on it?

Emily clicks on the appropriate button, thereby multiplying the operand segment by –2. The result begins at zero and continues to minus 2 (fourth line segment in Figure 1).

Flipping And Stretching Together

The next step involves the multiplication of a negative quantity times a negative number different from one.

David: Yeah. ... one more prediction... what if we multiply it by minus 3 now? What do you think is going to happen?

Tallulah: **I think it's going to flip back over to be plus 3 units**.

David: OK, so it will come out to be about over here. OK, click on it.

Tallulah initially predicts that multiplying the –2U segment by minus 3 will produce a result of minus three. Although this may appear strange to us, it is totally consistent with what has been observed to this point: that is, since the operand segments were always unit segments (and their inverses), the results have always had a length equal to the operator! As the operation is performed on the screen, for the first time a result (6U) appears (fifth and last line segment in Figure 1) that differs in magnitude from the operand segment (-2U) and the value of the operator (-3):

Tallulah & Emily: (*Laughs as they see the result, 6U*).

David: What happened?

Tallulah: um...

David: So what happened here? **It looked like you two were both surprised because you both had predicted that it would stop here (***at 3U***), didn't you**?

Girls: Yes.

Emily: **It kept going**.

David: It kept going?

Tallulah: Well, it was at minus 2, right?

David: Yes.

Tallulah: And then we did times minus 3.

David: That's right.

Tallulah: So, **like Emily said before, she said that minus or negative number is kind of like adding, or something. I think it's kind of the same with**

multiplying, because it was at 2 and 2 times 3 is 6, and then it ended up on six, so... I don't know, it's kind of...

David: Were we adding or multiplying?

Girls: Multiplying.

David: OK, Look at this over here, what it this?

Emily: 2 times 3...

David: 2 times 3 is 6.

Emily: They're both negative numbers so...

David: Where did the 6 come from?

Emily: **2 times 3 is 6, so I guess minus 2 times minus 3 is 6**.

When they note that the result is 6U, they realize that the 6 must have arisen as the product of 2 x 3 (as it happens with positive numbers) and that, more precisely, 6U must be the product of (-2U) x –3. So at this point they recognize that the magnitude of both the operand quantity and the multiplier determine the magnitude of the result.

2. Negative Addends

As the interview continues, a few minutes later the students attempt to predict and explain the result of repeatedly adding positive quantities to a(n initially) negative quantity. Here a tension arises between "absolute magnitudes" and "directed magnitudes" as well as their respective relations to natural numbers and integers.

Adding Should Make It Longer (But It Doesn't)

The first task is to decide what will happen when one adds a unit, U, to –2U. Figure 2 shows the starting quantity –2U and the result the students will later observe of adding to it a unit U.

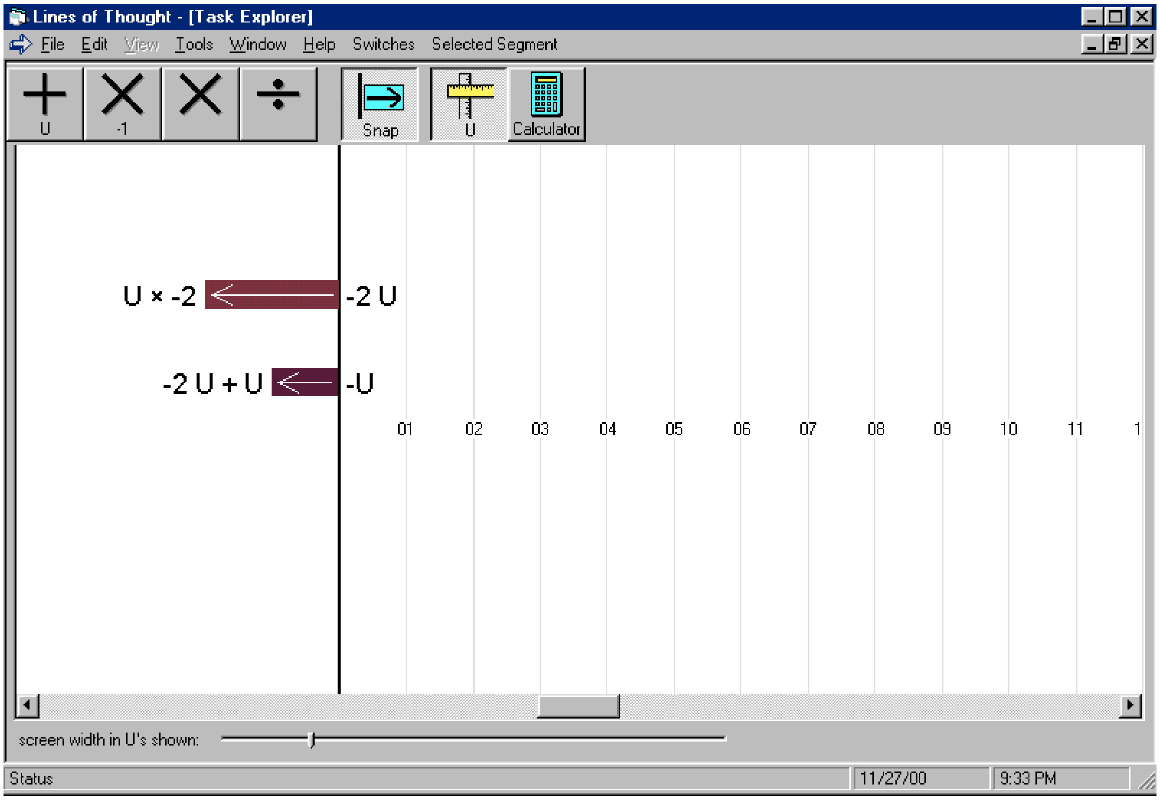

Fig. 2: Screen shot showing, the first steps in this part of the interview.

As we begin, Tallulah and Emily are trying to predict the outcome before they witness the result.

David: … I'll call this by its decimal name, minus 2U, OK?. When you start adding Us to it. **What do you think is gonna happen when you start adding a U, another U, another U**?

Tallulah: **I think it will be negative 3, negative 4, negative 5**.

David: Emily?

Emily: Yeah.

David: So, is it minus 2 now, if you click on the U. You can click on the U… We are gonna add one unit. (*The computer produces the "unexpected result", -1U.)* What happened?

Tallulah: (comparing the obtained segment -U to the previous segment minus 2U on the screen): **It's smaller**.

David: That's (-*U*) smaller?… **Is negative 1 smaller than negative 2**?

Emily (affirmatively): Uhnhum.

Tallulah's first idea is that successively adding U to the negative quantity –2U would lead to negative 3, negative 4, negative 5. Emily agrees with her suggestion. This seems consistent with what happens when we repeatedly add equal amounts to a previous amount of a certain quantity. "Negative 2" seems to be taken as a quantity or a line that would increase as more units are added to it, regardless of its relative value. They are surprised when a smaller segment, minus 1, appear on the screen as the result of the operation. If one considers only the length, that result appears to be smaller than the original amount; and yet, even though the length of –1U is shorter than the length of –2U, the value of the former is greater than the value of the latter. Much of the following discussion centers on this paradox, that what became shorter actually represents more, and shows how the students attempt to modify their previous idea or scheme, to take into account the new situation.

David: Well, so we had minus 2U, and you added a U and it became how much?

Emily: One U.

David: One U. Does that make sense?

Tallulah: I think so. Like, say, going back to what Emily said, **we are adding a negative number with another negative number.**
David: Wait a minute, did we add minus U or plus U?

Tallulah: I think we added a minus U, right?

David: I thought we hit *that* button (*pointing to the +U button*).

It is possible that Tallulah mistakenly mentioned adding a negative number because she witnessed a diminishment of the magnitude of the line segment. David calls attention to the computer button that was pushed lest the girls move forward with a mistaken impression about which button was pushed; actually there was no "-U" button on screen.

Two Senses Of More

As Tallulah accepts that they had added plus U, not -U, to minus 2, she shifts her attention to the number line for comparing the magnitudes of –2U and -U:

Tallulah: **You are adding a whole number, so, it's gonna get closer to the positive end**.

David: So, do we have more U's now, **do we have more of U than we had before**?

Emily: (*Smiles*) **Yup**.

Tallulah: **We have <u>less</u>, like, this little space** (*Tallulah points to the line segment, –1U, on the screen.)* **is less but**…

David: That little space <u>is</u> less.

Tallulah: **But I think it is…**

Emily: **<u>More</u>!**

Tallulah: I think. But I think, **you <u>see</u> less, but,** I think, .. **negative 1 U is** …

David: Think it through because I think this is a very difficult idea you are working on. But it is worth it.

Emily then spontaneously brings her previous understanding about money contexts (debt and assets) to bear on the problem.

Emily: **Negative 1U is closer to nothing…zero**. **And negative 2U… if I have minus two dollars**.

David: **What would that mean, to have minus two dollars?**

Emily: **I would be in debt**.

David: Yeah.

Tallulah: I think *that* number…

Emily: **I would be in debt… I think negative 2U's is less** [than negative 1U].

David: Less? Less money or less debt?[*Laughs*]

Tallulah: I think, I think negative…

David: Let's think about this for a moment. Emily, if you said less, I just want you to complete the thought. **Would you have less money if you owed two dollars, than if you owned one dollar?**

Emily was imprecise about the sense of less here, so the interviewer presses for a distinction between "less money" and "less debt". From this point on, substantial progress is made. The students understanding of relationships between amounts of money and debts becomes a resource to explain the apparent paradox consisting in "adding more but having less". When a person who owes two dollars receives a dollar, she has *less debt*, but *more mo*ney.

Emily: Yes … **here** (at –2U) **if I got a dollar, I would be less in debt**.

David: That's right, you would be less in debt. Is that kind of what we did here? You were owing two dollars. You have more money now?

David: Do you have more money now?

Emily: Yeah.

David: You have no money but you have <u>more</u> money in some sense.… [Do] you have more debt?

Emily: No.

David: You are sure of that. I noticed that you were very sure, very confident in your answer.

Tallulah then provides a clear description of how the quantities on the screen are interrelated, dealing with the apparent paradox as it relates to the number line. From then on they correctly predict and explain the results of successive additions of 1 to the quantity displayed on the screen.

Tallulah: I was going to say. That I think it is more. It's, yeah… Negative 1U is more than negative 2U's. Even though, when you look at it, negative 2 becomes (i.e. looks) bigger. Two spaces is bigger. But it is closer to a positive number and positive numbers are bigger than negative numbers.

David: Let's see if we can make a better prediction as to what's going to happen. If you add a plus U, that is, if you add a U to that, to the minus U, what is gonna happen now?

Tallulah & Emily: I think it will go to zero.

David: You are both quick on that. Go ahead, try. (They try)… Yeah. Zero U, minus 1 U plus U. Alright, and, now do we have more than we had before?

Tallulah: Yes, even if you can't see anything, you have more, because… if… what I was going to say, now you don't have any money, but you are out of debt.

David: Do we feel better now than in the situation a moment ago?

Tallulah: Yeah.

David: At least you don't have any debt. Now let's add one more U and see what happens.

Tallulah: It is going to be one, if we add one, two, three. Because it is kind of going up.

David: Going up.

Tallulah: Because, it's just adding on.

So, You were saying if we just do this it will go up.

David: When did we start going up?

Emily: We had a negative number, we added.

David: We started at negative 2, did we immediately started going up?

Emily: We were getting out of debt.

David: We were getting out of debt?

David: Or did we just start going up when we got to the zero?

Tallulah: We were coming up already, because we are moving this way (waves her hand from left to right).

Tallulah and Emily made progress during the course of the interview. At first, they were hard pressed to make predictions about the result of multiplying and adding negative quantities. And at several points they seem to confuse multiplication by a negative number with addition of a negative number and addition of negative numbers with addition of positive numbers. But by the end of the dialogue Tallulah has herself brought her knowledge about monetary debt into the discussion and used this to help eliminate what were initially very puzzling results: that adding to –2U has can produce a shorter line segment that nonetheless represents more money.

Final Remarks

It is important to realize that the interview does not simply elicit preformed beliefs. The representational conventions the software is built upon have relations to the mathematics the students have been learning in school (e.g. the number line representation), but there are novel features as well (for example, the representation of multiplication of a quantity by a negative multiplier as "flipping") and consequently, one has to regard the interview as a situation in which students are making considerable effort to understand the particular rules of the game, so to speak. Furthermore, the interviewer's role is hardly that of a mere elicitor of remarks from the students. He was trying to help the students work through some of the issues before them and hence was an active participant in the learning situation, aware that he was providing a scaffolding for students to explore relations that they very well might not have done had they been left to work with the software on their own.

There is little doubt that the students have made use of former knowledge during the interview. They drew upon what they understood about number lines, about the arithmetical operations of addition and multiplication, about the meaning and use of the minus sign, about complex situations involving debt, and about fractions. They allude to the fact that their classroom experience about negative numbers had convinced them that sometimes the results of operations with negative numbers lead to counter-intuitive results—for example, adding two negative numbers can appear to be subtracting:

> We're learning in math class how to add and subtract negative numbers, and it's something... it's really confusing. If you're adding, you're really subtracting ...

However, it would be misleading to characterize the students' learning as transfer. Transfer suggests a relatively passive "carrying-over" and deployment of learning from one situation to another once learners recognize the "similarity" between those situations. But when we scrutinize the present examples, there is little evidence for some monolithic skill or piece of knowledge being carried over intact from a unique prior situation to the present one. On the contrary, the students are wrestling with multiple, competing ideas. Although they do make reference to specific classroom experience about negative numbers, cited just above, there are far more signs in the discussion that they are drawing upon a broad history of experience regarding numbers, general arithmetical operations, money, notation and diagrams, and so forth. They do far more than deploy this knowledge. They draw upon it selectively to deal with the unique predicaments at hand. The have not simply unloaded a prior solution from their storehouse of knowledge. They have crafted it on the spot, adjusting and adapting their prior knowledge in the process.

It is precisely this active accommodation of knowledge to the demands of the situation (as understood by them) that so notably lacks in transfer accounts of learning. Were we to restrict ourselves to training across starkly similar situations (e.g., how students use what they have learned about one text editor to another) the adjustments required are so minor that we may feel content with a theory (of transfer) that stresses continuity and assimilation of existing knowledge over adaptations made by the learner. But the more one attempts to come to grips with major advances in knowledge—for example, in learning about negative numbers, or any other broad mathematical concept—the more one needs to recognize the role of conflict, of adaptation, adjustment, and reorganization of existing knowledge. The picture in this latter case looks very little like that painted by transfer theorists. The two cases document the sort of struggle that often occurs when former knowledge is brought to bear on new phenomena. To characterize such situations as matters of transfer would blindside us to some of the most important processes of these learning.

The students' struggle to reconcile old knowledge with new phenomena is generally consistent with the Piagetian description of learning as a result of an evolving process of assimilation and accommodation (transfer theory, instead, favors assimilation over accommodation). In our examples, however, the tension does not seem to arise at the level of general developmental-psychological structures. The examples highlight how the students are struggling with particular forms of conventional notation—something Piagetian theory has consistently failed to consider in its search for universals. Although Piagetian theory refers to general developmental issues, we believe that his approach can help understand the intricate questions of how conventional symbolic systems, including linguistic representations, worked their way into the child's thought. Ferreiro (1996) and Ferreiro & Teberosky (1979) provide an example of how such adaptations might be made in the domain of literacy. Brizuela (2001) takes a similar approach to investigating how young children learn to write and interpret written numbers.

In recent years some learning theorists have moved gradually away from the assumption that prior knowledge is straightforwardly applied to new learning situations. Some have likewise recognized that knowledge only rarely consists of ready-made solutions to problems and that the cases where it does may not be particularly interesting from the standpoint of conceptual change. Furthermore, they have understood that there need not be an apparent, or even underlying, structural, similarity across situations for former knowledge to assume a present role. More and more they have realized that learning involves substantial adjustments and adaptations to the new cases at hand.

Researchers have tried to maintain the concept of transfer by expanding its scope to address issues that have become central to cognitive research and theories of learning over the past century For example, in their review of learning and transfer, a 16 member Committee convened by the National Research Council to focus on Developments in the Science of Learning (Bransford et al, 1999) gave considerable attention to issues of context, problem representations, active transfer, metacognitive issues, conceptual change, cultural practices, and the relations between human activity in and out of school. In a nutshell, they have tried to give transfer a new look and flavor.

This evokes a culinary comparison. Imagine a chef who makes a fish chowder that receives a bad review for smelling too much of fish. The chef adjusts his recipe—adding onions, potatoes, beef, vegetables, Creole seasoning, and significantly reducing the amount of fish—so that the fishiness is no longer an issue. Some days later the author of the review visits the restaurant and once again orders the fish chowder. He finds the concoction an edible but a peculiar mix of ingredients that do not work well together. Even more strikingly, he wonders why the chef still refers to the plate as fish chowder!

We believe it is now time for learning theory to abandon transfer as an approach to how prior knowledge and experience contribute to learning. Transfer encourages educators and theorists to continue to view learning as a direct carrying over of procedures from one situation to another. When one looks carefully at people learning rich concepts, there is evidence that learners characteristically make adjustments in knowledge, that they attempt to reconcile conflicting interpretations, and they work with schematized understandings that stand at odds with a theory of transfer.

Learning theory must move beyond transfer as an account of how prior knowledge and experience contribute to learning. It needs to address examples such as those presented here that show the continual tension between former knowledge and new phenomena. In some respects this tension was anticipated by Piaget's notions of assimilation and accommodation. "… No behavior, even if it is new to the individual, constitutes an absolute beginning. It is always grafted onto previous schemes and therefore amounts to assimilating new elements to already constructed structures." (Piaget, 1970, pp. 7-8). Assimilation assures the continuity of learning and the integration of new elements to previous schemes. However, construction of new knowledge requires that the assimilatory scheme be modified by the elements it assimilates, i.e, it requires accommodation (Piaget, 1970). This broad conception of how learning occurs allows one to recognize the role of prior learning without subscribing to the doctrine of transfer.

Several decades ago Noam Chomsky (1959) subjected Skinner's theory of learning to a searing analysis. The heart of Chomsky's critique rested on the following dilemma. Concepts such as *reinforcement*, if used as they are in a laboratory setting, are too limited to have important theoretical meaning and implications. However, if they are understood in a broad sense (he exemplifies with Skinner's use of the term *reinforcement* to characterize an author's anticipation of future praise by some imagined readership) it loses the precise scientific meaning for which it was created.

Chomsky almost single-handedly removed the term *reinforcement* from modern educational discourse because he showed that it was irrevocably associated with a theory that could not come to grips with the complexity of human verbal learning. In doing so, he did not eliminate the need to explain the roles of reward, incentive, and motivation. These issues continue to merit investigation. However, one need not subscribe to Skinnerian behaviorism, with its emphasis on stimulus control, in order to address such a task.

Ultimately, it was the generative nature of human language that undermined the behaviorist characterization of how people learn language. And it is the generative power of learning that undermines a transfer approach to mathematical learning.

We conclude by clarifying the title of this paper. The transfer dilemma is the following: "If we deny transfer, we seem to deny that new learning rests on former learning; if we endorse the idea of transfer, we subscribe to questionable beliefs about knowledge. Neither of these options is acceptable". This is the heart of the transfer dilemma.

We believe that the metaphor underlying transfer—namely, of transporting knowledge from one concrete situation to another—is fundamentally flawed, and leads to an impoverished caricature of how learning actually works. Situations and contexts cannot be treated exclusively as "givens" because to a large extent they are mental constructions (Carraher et. al., 2001). This leads to ramifications that will challenge any theory of learning but lie far beyond the purview of transfer theory, that relies, as it does, on the assumption that contexts are tantamount to physical settings. Our goal is to recommend not an "improved version" of transfer, but rather the abandonment altogether of "transfer" as a view of how learning takes place. This is by no means a mere terminological battle; the term *transfer* is infused with theoretical views of learning and cognition. We realize that transfer theory has been developed to come to grips with the issue of how prior knowledge and experience play a role in learning. This goal continues to deserve our attention. But we need to frame our questions about the continuity and discontinuity of knowledge in more suitable ways. The way out of the dilemma consists in recognizing that there are other ways to frame the way prior knowledge and experience contribute to learning.

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