How Generic are Dipolar Jet EOFs?

Adam H. Monahan[1,2] and John C. Fyfe[3]

[1] School of Earth and Ocean Sciences
University of Victoria P.O. Box 3055 STN CSC
Victoria, BC, Canada, V8W 3P6
(email: monahana@uvic.ca)

[2] Earth System Evolution Program
Canadian Institute for Advanced Research, Toronto, ON, Canada

[3] Canadian Centre for Climate Modelling and Analysis
Meteorological Service of Canada
University of Victoria P.O. Box 1700, STN CSC
Victoria, BC, Canada, V8W 2Y2
(email: John.Fyfe@ec.gc.ca)

Submitted to
Journal of the Atmospheric Sciences

April 11, 2008

Abstract

Dipolar structures arise as Empirical Orthogonal Functions (EOFs) of extratropical tropospheric zonal-mean zonal wind in observations, in idealised dynamical models, and in complex general circulation models. This study characterises the conditions under which dipoles emerge as EOFs of a jet of fixed shape $f(x)$ which takes a unique localised extremum but is otherwise arbitrary, characterised by fluctuations in strength, position, and width of arbitrary distribution. It is shown that the factors which influence the extent to which a dipole-like structure will arise as an EOF are: (i) the skewness of position fluctuations, (ii) the dependence of position fluctuations on strength and width fluctuations, and (iii) the relative strength of position and width fluctuations. In particular, the leading EOF will be a dipole if jet position fluctuations are not strongly skewed, not strongly dependent on strength and width fluctuations, and sufficiently large relative to strength and width fluctuations. As these conditions are generally satisfied to a good approximation by observed and simulated tropospheric eddy-driven jets, this analysis provides a simple explanation of the ubiquity of dipolar jet EOFs.

1 Introduction

The leading Principal Component Analysis (PCA) mode of extratropical zonal-mean zonal wind variability is known as the zonal index (e.g. Lorenz and Hartmann, 2001). The spatial structure corresponding to this mode (that is, the Empirical Orthogonal Function (EOF)) is found to be a dipole in observations and in a range of models from randomly-forced barotropic β-plane dynamics (e.g. Vallis et al., 2004) through dry dynamical cores (e.g. Fyfe and Lorenz, 2005) to complex general circulation models (e.g. Fyfe et al., 1999), and is related to (but not identical with; cf. Monahan and Fyfe, 2008) the leading mode of zonal-mean geopotential height variability (the annular mode). As noted in Wittman et al. (2005), the ubiquity of this dipolar structure suggests that it reflects some generic feature of variability of the extratropical atmosphere - in particular, the existence of a jet in zonal-mean zonal winds characterised by fluctuations in position. The numerical simulation results presented in Wittman et al. (2005) were confirmed analytically and extended in Monahan and Fyfe (2006), the central conclusions of which were:

1. A small number of basic shapes contribute to the leading order EOFs, corresponding to monopole, dipole, and tripole structures. As noted in Monahan and Fyfe (2008), these shapes are successive derivatives of the jet shape function. All of these basis functions and EOFs are either symmetric or antisymmetric around the jet axis. Symmetric and antisymmetric basis functions are mutually orthogonal, but the symmetric basis functions are not orthogonal to other symmetric basis functions. Similarly, antisymmetric basis functions are not mutually orthogonal.

2. The leading EOF structures corresponding to kinematic degrees of freedom representing jet fluctuations in strength, position, or width individually can be computed and

correspond respectively to monopole, dipole, and coupled monopole/tripole structures.

3. If the jet fluctuates in more than one of these kinematic degrees of freedom, the dipole arises as a distinct EOF mode as a result of fluctuations in jet position (as the leading EOF if fluctuations in position are sufficiently large compared to those in strength and width). However, the associated Principal Component (PC) time series mixes together variability in strength, position, and width: the "zonal index" mode cannot be uniquely associated with a single kinematic jet degree of freedom.

4. The EOFs associated with the individual kinematic degrees of freedom are not generally orthogonal, and when more than one degree of freedom is active the EOFs other than the dipole will generally consist of a mixture of monopole, dipole, and tripole structures.

These conclusions were obtained through a perturbation analysis of the covariance structure of a jet in zonal-mean zonal wind with idealised spatial structure (Gaussian profile) and fluctuations in strength, position, and width (all Gaussian distributed). Although these are reasonable first-order approximations, the observed tropospheric zonal-mean jet is not exactly Gaussian in profile and the statistics of its fluctuations are not exactly Gaussian. The present study generalises the results of Monahan and Fyfe (2006) for the case of a jet of arbitrary (sufficiently smooth) profile with fluctuations of arbitrary distribution (for which a sufficiently large number of moments exist). The fundamental conclusions of Monahan and Fyfe (2006) are recovered in generalised form, and new results associated with asymmetric jet shape and non-Gaussian fluctuations are obtained. In particular, the conditions under which a dipole-like structure arises as the leading EOF of the fluctuating jet are characterised. The generalised model is presented in Section 2. Analytic computations of the EOFs of this model for a number of illustrative special cases are presented in Sections 3 through 7 before the general case is considered in Section 8. Conclusions follow in Section 9.

2 Idealised Jet Model

Consider a jet in zonal-mean zonal wind $u(x,t)$ of strength $U(t)$, central position $x_c(t)$, and width $\sigma(t)$:

$$u(x,t) = U(t) f\left(\frac{x - x_c(t)}{\sigma(t)}\right), \qquad (1)$$

where f is a $\mathcal{C}^K$ (that is, K times continuously differentiable) and localised (that is, with substantially nonzero values over only part of the domain $\mathcal{D} = (x_1, x_2)$) but otherwise arbitrary function. We assume that strength, position, and width are fluctuating quantities:

$$U(t) = U_0(1 + l\xi(t)) \qquad (2)$$

$$x_c(t) = h\lambda(t) \qquad (3)$$

$$\sigma^{-1}(t) = 1 + v\eta(t), \qquad (4)$$

where ξ, λ, and η are random variables of mean zero and unit variance. Other than assuming that sufficiently many moments of these random variables exist for the following series expansions to be meaningful, their joint distribution $p(\xi, \lambda, \eta)$ is arbitrary. Note that without any loss of generality we have defined our coordinate system so that the average jet central position $\mathsf{E}\{x_c\}$ is zero and the average jet width $\mathsf{E}\{\sigma\}$ is one, where the expectation of any function $q(\xi, \lambda, \eta)$ is defined as

$$\mathsf{E}\{q\} = \int q(\xi, \lambda, \eta) p(\xi, \lambda, \eta)\ d\xi d\lambda d\eta. \qquad (5)$$

We will assume that l, h, and v are all $<< 1$; for the Southern Hemisphere eddy-driven jet Monahan and Fyfe (2006) obtained the estimates $l \sim 0.1$, $h \sim 0.3$ and $v \sim 0.2$.

In the following discussion, it will be useful to define the vectors (in "bra and ket notation", as discussed in the Appendix)

$$|f_j\rangle = \frac{1}{N_j}\frac{d^j f}{dx^j} \tag{6}$$

and

$$|\mathcal{F}_j\rangle = \frac{1}{\mathcal{N}_j} x^j \frac{d^j f}{dx^j}, \tag{7}$$

where

$$N_j^2 = \int_{x_1}^{x_2} \left(\frac{d^j f}{dx^j}\right)^2 \, dx \tag{8}$$

and

$$\mathcal{N}_j^2 = \int_{x_1}^{x_2} x^{2j} \left(\frac{d^j f}{dx^j}\right)^2 \, dx. \tag{9}$$

We will assume that the function $f(x)$ is sufficiently smooth that enough of the vectors $|f_k\rangle$ and $|\mathcal{F}_k\rangle$ exist for the series expansions in the following discussion to be meaningful. Note that if $f(x)$ is a symmetric function, then the $|f_j\rangle$ will be alternately odd and even functions, while $|\mathcal{F}_j\rangle$ will all be even.

It will be assumed that $f(x)$ is a "jet" characterised by a unique extremum, so its derivative changes sign only once in the domain and $|f_1\rangle$ is "dipolar" in structure. Furthermore, we assume that $f(x)$ is sufficiently localised that the function and its derivatives vanish at the boundaries of the domain $\mathcal{D}$. It then follows from repeated integration by parts that

$$\int_{x_1}^{x_2} \left(\frac{d^j f}{dx^j}\right)\left(\frac{d^k f}{dx^k}\right) \, dx = (-1)^{j-k} \int_{x_1}^{x_2} \left(\frac{d^j f}{dx^j}\right)\left(\frac{d^k f}{dx^k}\right) \, dx, \tag{10}$$

from which we obtain the selection rule that even and odd indexed vectors $|f_j\rangle$ are orthogonal:

$$\langle f_j | f_k \rangle = 0 \text{ for } j-k \text{ odd}. \tag{11}$$

This result follows trivially by symmetry for $f(x)$ symmetric as $|f_j\rangle$ are then alternately odd and even functions; it is important for the following discussion that Eqn. (11) holds generally for localised jet shape functions $f(x)$. No such selection rule exists among the vectors $|\mathcal{F}_j\rangle$.

By definition, the EOFs $\left|E^{(k)}\right\rangle$ of $u(x,t)$ are the eigenfunctions of the covariance "matrix" (technically, operator)

$$\mathbf{C} = \mathsf{E}\left\{|u'\rangle\langle u'|\right\} \tag{12}$$

where

$$|u'\rangle = |u\rangle - \mathsf{E}\left\{|u\rangle\right\}. \tag{13}$$

That is,

$$\mathbf{C}\left|E^{(k)}\right\rangle = \mu^{(k)}\left|E^{(k)}\right\rangle. \tag{14}$$

We now proceed to obtain expressions for the leading EOFs of $u(x,t)$,considering a set of illustrative special cases before addressing the general case.

3 Fluctuations in Strength Alone

Consider first a jet which fluctuates in strength alone:

$$u(x,t) = U_0(1 + l\xi)f(x). \tag{15}$$

We can write

$$|u\rangle = U_0(1 + l\xi)N_0\,|f_0\rangle \tag{16}$$

and

$$\mathsf{E}\left\{|u\rangle\right\} = U_0 N_0\,|f_0\rangle \tag{17}$$

so

$$|u'\rangle = |u\rangle - \mathsf{E}\left\{|u\rangle\right\} = U_0 N_0 l\xi\,|f_0\rangle. \tag{18}$$

It follows that the covariance matrix

$$\mathbf{C} = \mathsf{E}\left\{|u'\rangle\langle u'|\right\} = U_0^2 N_0^2 l^2 \,|f_0\rangle\langle f_0|\,, \tag{19}$$

from which we conclude that $|f_0\rangle$ is an eigenfunction with eigenvalue $U_0^2 N_0^2 l^2$. A jet fluctuating in strength alone has a single EOF with nonzero variance, with the spatial pattern of the mean jet. This result is not surprising, as the jet fluctuating in strength alone is a standing wave (that is, separable into a product of functions of space and time alone). We also note that the structure of the EOF is independent of the pdf of ξ. This result is in contrast to the EOFs of fluctuations in position alone, to which we now turn.

4 Fluctuations in Position Alone

For the case of fluctuations in position alone, we have

$$u(x,t) = U_0 f(x - h\lambda). \tag{20}$$

We expand $u(x,t)$ as a Taylor series:

$$u(x,t) = \sum_{j=0}^{\infty} \frac{U_0(-h\lambda)^j}{j!} \frac{d^j f}{dx^j}, \tag{21}$$

or, in bra and ket notation,

$$|u\rangle = \sum_{j=0}^{\infty} \frac{U_0(-h\lambda)^j}{j!} N_j \,|f_j\rangle\,. \tag{22}$$

With

$$|u'\rangle = |u\rangle - \mathsf{E}\left\{|u\rangle\right\} = \sum_{j=1}^{\infty} \frac{U_0(-h)^j \left(\lambda^j - \mathsf{E}\left\{\lambda^j\right\}\right)}{j!} N_j \,|f_j\rangle \tag{23}$$

it follows that the covariance matrix is given by

$$\mathbf{C} = \mathsf{E}\left\{|u'\rangle\langle u'|\right\} = \sum_{j,k=0}^{\infty} \frac{U_0^2(-h)^{j+k}}{j!k!} N_j N_k m_{jk} \,|f_j\rangle\langle f_k|\,, \tag{24}$$

where

$$m_{jk} = \mathsf{E}\left\{\lambda^{j+k}\right\} - \mathsf{E}\left\{\lambda^{j}\right\}\mathsf{E}\left\{\lambda^{k}\right\}. \tag{25}$$

With $h << 1$, we have to $O(h^4)$ that

$$\begin{aligned}\mathbf{C} \quad &\simeq \quad N_1^2 h^2 U_0^2 \left|f_1\right\rangle \left\langle f_1\right| - N_1 N_2 s_\lambda U_0^2 \frac{h^3}{2}\left(\left|f_1\right\rangle \left\langle f_2\right| + \left|f_2\right\rangle \left\langle f_1\right|\right) \\ &+h^4\left(\frac{N_1N_3(\kappa_\lambda+3)U_0^2}{6}\left|f_1\right\rangle \left\langle f_3\right| + \frac{N_2^2(\kappa_\lambda+2)U_0^2}{4}\left|f_2\right\rangle \left\langle f_2\right| + \frac{N_1N_3(\kappa_\lambda+3)U_0^2}{6}\left|f_3\right\rangle \left\langle f_1\right|\right)\end{aligned} \tag{26}$$

where s_λ and κ_λ are respectively the skewness and kurtosis of λ: $s_\lambda = \mathsf{E}\{\lambda^3\}$ and $\kappa_\lambda = \mathsf{E}\{\lambda^4\} - 3$.

If the pdf of λ is symmetric so that $s_\lambda = 0$, it follows by the selection rule (11) that $|f_2\rangle$ is an eigenfunction of $\mathbf{C}$ with eigenvalue $\mu^{(2)} = N_2^2(\kappa_\lambda+2)U_0^2 h^4/4$, and that to leading order in h^2, $|f_1\rangle$ is an eigenfunction with eigenvalue $\mu^{(1)} = N_1^2 U_0^2 h^2$ (the off-diagonal terms of the covariance matrix will result in $O(h^4)$ corrections to $\mu^{(1)}$). The relative variances of these two PCA modes will depend on the size of h, on the kurtosis of λ, and on the normalisation factors N_1 and N_2. Because h is by assumption $<< 1$, $|f_1\rangle$ will be the leading EOF unless the kurtosis κ_λ is very large.

As an example of how the EOF structure is influenced by jet shape, consider a jet with profile

$$f(x) = \exp\left(\sum_{j=1}^{4} L_j x^j\right). \tag{27}$$

The coefficients L_j are determined so that the jet has spatial mean zero, unit spatial variance, and spatial skewness $\mathcal{S}$:

$$\int_{x_1}^{x_2} x f(x) = 0 \quad , \quad \int_{x_1}^{x_2} x^2 f(x) = 1 \quad , \quad \int_{x_1}^{x_2} x^3 f(x) = \mathcal{S} \tag{28}$$

(and $L_4 < 0$ ensures the jet is spatially localised). Plots of $f(x)$ for $\mathcal{S} = 0$ and $\mathcal{S} = 1$ are presented in Figure 1. The analysis above predicts that the leading EOFs for $s_\lambda = 0$ will be:

$$\left|E^{(1)}\right\rangle = \frac{1}{N_1}\left(\sum_{j=1}^{4} jL_j x^{j-1}\right)\exp\left(\sum_{j=1}^{4} L_j x^j\right) \tag{29}$$

and

$$\left|E^{(2)}\right\rangle = \frac{1}{N_2}\left[\sum_{j=2}^{4} j(j-1)L_j x^{j-2} + \left(\sum_{j=1}^{4} jL_j x^{j-1}\right)^2\right]\exp\left(\sum_{j=1}^{4} L_j x^j\right). \tag{30}$$

Plots of $\left|E^{(1)}\right\rangle$ and $\left|E^{(2)}\right\rangle$ for $\mathcal{S} = 0$ and $\mathcal{S} = 1$ for $h = 0.5$ (a relatively large value selected for illustrative purposes) as predicted from Eqns (29) and (30) are presented in Figure 2. Not surprisingly, the jet asymmetry results in EOF patterns that are themselves asymmetric, although $\left|E^{(1)}\right\rangle$ remains identifiably dipolar. The leading EOFs were also computed directly from a numerical simulation of the fluctuating jet; these EOFs (also presented in Figure 2) agree very well with the theoretically predicted structures. The envelopes of the numerically simulated EOFs are somewhat wider than those of the theoretical EOFs, reflecting the fact that $h^2 = 0.25$ is not strictly much less than one. As h is reduced, the simulated and predicted EOF patterns (not shown) become indistinguishable.

It is clear from Eqn. (26) that the effect of a skewed distribution of λ is to mix the vectors $|f_1\rangle$ and $|f_2\rangle$ in the EOFs of $u(x,t)$, although this effect is only of order $O(h)$. The effects of skewness in λ will be considered further in Section 6.1.

5 Fluctuations in Width Alone

We now turn our attention to a jet which fluctuates in width alone:

$$u(x,t) = U_0 f\left[x(1+v\eta)\right]. \tag{31}$$

We can then write

$$u(x,t) = \sum_{j=0}^{\infty} \frac{U_0 (xv)^j \eta^j}{j!}\frac{d^j f}{dx^j}, \tag{32}$$

or

$$|u\rangle = \sum_{j=0}^{\infty} \frac{U_0 v^j \eta^j}{j!} \mathcal{N}_j |\mathcal{F}_j\rangle . \tag{33}$$

It follows that

$$\mathsf{E}\{|u\rangle\} = \sum_{j=0}^{\infty} \frac{U_0 v^j \mathsf{E}\{\eta^j\}}{j!} \mathcal{N}_j |\mathcal{F}_j\rangle , \tag{34}$$

so

$$|u'\rangle = |u\rangle - \mathsf{E}\{|u\rangle\} = \sum_{j=1}^{\infty} \frac{U_0 v^j \left(\eta^j - \mathsf{E}\{\eta^j\}\right)}{j!} \mathcal{N}_j |\mathcal{F}_j\rangle \tag{35}$$

and

$$\mathbf{C} = \sum_{j,k=1}^{\infty} \frac{U_0^2 v^{j+k} m_{jk}}{j!k!} \mathcal{N}_j \mathcal{N}_k |\mathcal{F}_j\rangle \langle\mathcal{F}_k| , \tag{36}$$

where

$$m_{jk} = \mathsf{E}\left\{\eta^{j+k}\right\} - \mathsf{E}\left\{\eta^j\right\} \mathsf{E}\left\{\eta^k\right\} . \tag{37}$$

Thus, to $O(v^4)$

$$\begin{aligned} \mathbf{C} \;\simeq\; & U_0^2 v^2 \mathcal{N}_1^2 |\mathcal{F}_1\rangle\langle\mathcal{F}_1| + \frac{U_0^2 \mathcal{N}_1 \mathcal{N}_2 v^3 s_\eta}{2} \left(|\mathcal{F}_1\rangle\langle\mathcal{F}_2| + |\mathcal{F}_2\rangle\langle\mathcal{F}_1|\right) \\ & + v^4 \left(\frac{\mathcal{N}_1 \mathcal{N}_3 (\kappa_\eta + 3) U_0^2}{6} |\mathcal{F}_1\rangle\langle\mathcal{F}_3| + \frac{\mathcal{N}_2^2 (\kappa_\eta + 2) U_0^2}{4} |\mathcal{F}_2\rangle\langle\mathcal{F}_2| + \frac{\mathcal{N}_1 \mathcal{N}_3 (\kappa_\eta + 3) U_0^2}{6} |\mathcal{F}_3\rangle\langle\mathcal{F}_1| \right) , \end{aligned} \tag{38}$$

where s_η and κ_η are respectively the skewness and kurtosis of width fluctuations. To $O(v^2)$ the leading EOF structure is $|\mathcal{F}_1\rangle$, with associated eigenvalue $\mu^{(1)} = U_0^2 \mathcal{N}_1^2 v^2$. In contrast to the case of fluctuations in position alone, we cannot conclude for the case $s_\eta = 0$ that the second EOF of fluctuations in width alone is given by $|\mathcal{F}_1\rangle$, as in general $\langle\mathcal{F}_1|\mathcal{F}_2\rangle \neq 0$. Instead, the second and higher-order EOFs will necessarily be mixtures of the functions $|\mathcal{F}_j\rangle$. Only for the leading EOF can the simple statement be made that (to leading order) it is given by $|\mathcal{F}_1\rangle$. Note that for the symmetric Gaussian jet considered in Monahan and Fyfe (2006), $|\mathcal{F}_1\rangle$ could be expressed as a linear combination of $|f_0\rangle$ and $|f_2\rangle$; such a decomposition of $|\mathcal{F}_j\rangle$ into a small number of $|f_k\rangle$ will not be possible in general.

For the asymmetric jet considered in Section 4, this computation predicts that the first EOF should be

$$\left|E^{(1)}\right\rangle = \frac{1}{\mathcal{N}_1}\left(\sum_{j=1}^{4} jL_j x^j\right)\exp\left(\sum_{j=1}^{4} L_j x^j\right). \tag{39}$$

Plots of $\left|E^{(1)}\right\rangle$ for $v = 0.15$ (both predicted and numerically simulated) are shown in Figure 3 for $\mathcal{S} = 0$ and $\mathcal{S} = 1$. As was the case for fluctuations in jet position alone, fluctuations in width of the asymmetric jet lead to EOF structures that are themselves asymmetric. Agreement between the predicted and simulated EOF structures is good, with the primary difference being in the width of the envelope. As v is reduced, agreement between the predicted and simulated EOFs improves.

6 Fluctuations in Both Position and Strength

Allowing for fluctuations in both position and strength,

$$u(x,t) = U_0\left(1 + l\xi\right) f(x - h\lambda), \tag{40}$$

we have

$$|u\rangle = \sum_{j=0}^{\infty} \frac{U_0(-h)^j(1 + l\xi)\lambda^j}{j!} N_j\,|f_j\rangle \tag{41}$$

from which it follows that

$$|u'\rangle = |u\rangle - \mathsf{E}\left\{|u\rangle\right\} = \sum_{j=0}^{\infty} \frac{U_0(-h)^j\left[(\lambda^j - \mathsf{E}\left\{\lambda^j\right\}) + l\left(\xi\lambda^j - \mathsf{E}\left\{\xi\lambda^j\right\}\right)\right]}{j!} N_j\,|f_j\rangle\,. \tag{42}$$

The covariance function is then given by Eqn. (24) with

$$\begin{aligned} m_{jk} \;=\; & \mathsf{E}\left\{\lambda^{j+k}\right\} - \mathsf{E}\left\{\lambda^j\right\}\mathsf{E}\left\{\lambda^k\right\} + l\left[2\mathsf{E}\left\{\xi\lambda^{j+k}\right\} - \mathsf{E}\left\{\lambda^j\right\}\mathsf{E}\left\{\xi\lambda^k\right\} - \mathsf{E}\left\{\xi\lambda^j\right\}\mathsf{E}\left\{\lambda^k\right\}\right] \\ & + l^2\left[\mathsf{E}\left\{\xi^2\lambda^{j+k}\right\} - \mathsf{E}\left\{\xi\lambda^j\right\}\mathsf{E}\left\{\xi\lambda^k\right\}\right]. \end{aligned} \tag{43}$$

Note that moments of ξ of power higher than two do not affect the covariance structure (in contrast to those of λ).

6.1 Independent Fluctuations in Strength and Position

For the simplest case of independent fluctuations in strength and position, we have

$$m_{jk} = \begin{cases} l^2 & j = k = 0 \\ \mathsf{E}\left\{\lambda^{j+k}\right\} - \mathsf{E}\left\{\lambda^j\right\}\mathsf{E}\left\{\lambda^k\right\} & \text{otherwise.} \end{cases} \tag{44}$$

It follows that to $O(h^4)$, the covariance function is

$$\begin{aligned} \mathbf{C} \simeq\ & N_0^2 U_0^2 l^2 \left|f_0\right\rangle\left\langle f_0\right| + N_1^2 U_0^2 h^2 \left|f_1\right\rangle\left\langle f_1\right| - N_1 N_2 s_\lambda U_0^2 \frac{h^3}{2}\left(\left|f_1\right\rangle\left\langle f_2\right| + \left|f_2\right\rangle\left\langle f_1\right|\right) \\ & + U_0^2 h^4 \left(\frac{N_1 N_3(\kappa_\lambda+3)}{6}\left|f_1\right\rangle\left\langle f_3\right| + \frac{N_2^2(\kappa_\lambda+2)}{4}\left|f_2\right\rangle\left\langle f_2\right| + \frac{N_1 N_3(\kappa_\lambda+3)}{6}\left|f_3\right\rangle\left\langle f_1\right|\right). \end{aligned} \tag{45}$$

The EOFs will then be given by

$$\left|E\right\rangle = \alpha\left|f_0\right\rangle + \beta\left|f_1\right\rangle + \gamma\left|f_2\right\rangle, \tag{46}$$

where (α, β, γ) is an eigenvector of the matrix

$$C = \begin{pmatrix} N_0^2 U_0^2 l^2 & 0 & N_0^2 U_0^2 l^2 F_{02} \\ -\frac{1}{2} N_1 N_2 s_\lambda U_0^2 h^3 F_{02} & N_1^2 U_0^2 h^2 & -\frac{1}{2} N_1 N_2 s_\lambda U_0^2 h^3 \\ \frac{1}{4} N_2^2(\kappa_\lambda+2) U_0^2 h^4 F_{02} & -\frac{1}{2} N_1 N_2 s_\lambda U_0^2 h^3 & \frac{1}{4} N_2^2(\kappa_\lambda+2) U_0^2 h^4 \end{pmatrix}, \tag{47}$$

where

$$F_{02} = \left\langle f_0 | f_2\right\rangle \tag{48}$$

and the matrix (47) is not symmetric as the basis set is non-orthogonal.

In the case that the pdf of λ is symmetric so that $s_\lambda = 0$, then the even and odd sectors of this covariance matrix decouple and to leading order in h, $\left|f_1\right\rangle$ is an eigenfunction with eigenvalue $\mu^{(1)} = N_1^2 U_0^2 h^2$. The other two eigenfunctions will then take the form

$$\left|E\right\rangle = \alpha\left|f_0\right\rangle + \gamma\left|f_2\right\rangle, \tag{49}$$

where (α, γ) is an eigenvector of the matrix

$$C = \left(\begin{array}{cc} N_0^2 U_0^2 l^2 & N_0^2 U_0^2 l^2 F_{02} \\ \frac{1}{4} N_2^2 (\kappa_\lambda + 2) U_0^2 h^4 F_{02} & \frac{1}{4} N_2^2 (\kappa_\lambda + 2) U_0^2 h^4 \end{array} \right). \tag{50}$$

These other leading EOFs will therefore be hybrids of $|f_0\rangle$ and $|f_2\rangle$, with eigenvalues

$$\mu_\pm = \frac{N_0^2 U_0^2 l^2}{2} \left(1 + \delta \pm \sqrt{(1+\delta)^2 - 4(1 - F_{02}^2)\delta} \right), \tag{51}$$

where

$$\delta = \frac{(\kappa_\lambda + 2) N_2^2}{4 N_0^2} \frac{h^4}{l^2} \tag{52}$$

(for a jet with Gaussian profile and Gaussian fluctuations, $\delta = 3h^4/8l^2$; cf. Monahan and Fyfe (2006, 2008)). The degree of hybridisation will depend on the inner product F_{02} and the size of the ratio δ. Thus, the three leading EOFs will be given by $|f_1\rangle$ and the hybrids $\left|E^{(\pm)}\right\rangle$ of $|f_0\rangle$ and $|f_2\rangle$ - with ordering $\left(|f_1\rangle, \left|E^{(+)}\right\rangle, \left|E^{(-)}\right\rangle\right)$ or $\left(\left|E^{(+)}\right\rangle, |f_1\rangle, \left|E^{(-)}\right\rangle\right)$, depending on δ and F_{02}.

As an illustration of how skewness in λ influences the EOF structure, consider a jet with symmetric profile

$$f(x) = \exp\left(-\frac{x^2}{2} \right) \tag{53}$$

with independent fluctuations in position and strength. Fluctuations in λ are assumed to be skewed, with a centred Weibull distribution:

$$p(\lambda; a, b, c) = \begin{cases} \frac{b}{a} \left(\frac{\lambda - c}{a} \right)^{b-1} \exp\left(- \left(\frac{\lambda - c}{a} \right)^b \right) & \lambda > -c \\ 0 & \lambda \leq -c \end{cases} \tag{54}$$

of mean zero, unit variance, and skewness s_λ (this pdf is selected as an illustrative example of a skewed distribution). As discussed in Monahan (2006), the skewness of this pdf is a function of the shape parameter b alone; for $b \simeq 3.6$, the pdf is approximately Gaussian. For the jet profile Eqn. (53) we have

$$N_0 = \pi^{1/4} \quad , \quad N_1 = \sqrt{\frac{\sqrt{\pi}}{2}} \quad , \quad N_2 = \sqrt{\frac{3\sqrt{\pi}}{4}}, \tag{55}$$

and $F_{02} = -1/\sqrt{3}$. Thus, Eqn. (47) predicts that the leading three EOFs will have coefficients (α, β, γ) given by the eigenvectors of the matrix

$$C = \sqrt{\pi} U_0^2 \begin{pmatrix} l^2 & 0 & -\frac{1}{\sqrt{3}} l^2 \\ \frac{1}{4\sqrt{2}} s_\lambda h^3 & \frac{1}{2} h^2 & -\frac{\sqrt{3}}{4\sqrt{2}} s_\lambda h^3 \\ -\frac{\sqrt{3}}{16}(\kappa_\lambda + 2) h^4 & -\frac{\sqrt{3}}{4\sqrt{2}} s_\lambda h^3 & \frac{3}{16}(\kappa_\lambda + 2) h^4 \end{pmatrix}. \tag{56}$$

The upper and lower panels of Figure 4 show respectively the first and second EOFs obtained with $h = 0.3$ and $l = 0.185$ (values selected for illustrative purposes) and position fluctuation skewnesses $s_\lambda = 0$, $s_\lambda = 0.75$, and $s_\lambda = 1.5$. In all cases, the first and second EOFs are respectively a dipole and a monopole. As the skewness of λ increases, these structures become increasingly asymmetric around $x = 0$ as $|f_1\rangle$ mixes with $|f_0\rangle$ and $|f_2\rangle$. In particular, one lobe of the dipole shrinks while the other strengthens, and the midpoints of both the dipole and monopole move away from $x = 0$. As the skewness enters to third order in h while the diagonal terms dominating the leading-order EOF are of second order in this small parameter, the effects of skewness in position fluctuation on the leading two EOFs are relatively weak. In this example the leading EOF remains recognisably dipolar; for the mixing of $|f_1\rangle$ with $|f_0\rangle$ and $|f_2\rangle$ to obscure the dipolar structure of the EOF the skewness of λ would have to be very large.

6.2 Dependent Fluctuations in Strength and Position

To investigate the effect of statistical dependence of ξ and λ on the covariance structure of $u(x,t)$, consider the case in which strength and position fluctuations are both Gaussian and are perfectly correlated: i.e., $\lambda = \xi$. Then, to $O(h^2)$:

$$\mathcal{C} \simeq N_0^2 U_0^2 l^2 \left|f_0\right\rangle \left\langle f_0\right| - N_0 N_1 U_0^2 l h \left(\left|f_0\right\rangle \left\langle f_1\right| + \left|f_1\right\rangle \left\langle f_0\right|\right) + N_1^2 U_0^2 h^2 (1+2l^2) \left|f_1\right\rangle \left\langle f_1\right|, \quad (57)$$

so the EOFs are vectors mixing $|f_0\rangle$ and $|f_1\rangle$, with eigenvalues

$$\mu^{(\pm)} = \frac{N_0^2 U_0^2 l^2}{2} \left(1 + \epsilon(1+2l^2) \pm \sqrt{(1-\epsilon(1+2l^2))^2 + 4\epsilon}\right), \quad (58)$$

where

$$\epsilon = \frac{N_1^2 h^2}{N_0^2 l^2}. \quad (59)$$

The mixing of $|f_0\rangle$ and $|f_1\rangle$ in the EOFs will depend on the magnitude of ϵ. When this ratio is very large or very small, the eigenvalues $\mu^{(\pm)}$ will be well separated and $|f_0\rangle$ and $|f_1\rangle$ will not be strongly mixed. For intermediate values of ϵ, the two eigenvalues $\mu^{(\pm)}$ will be of the same order of magnitude and the mixing will be more pronounced. In the case of a jet which fluctuates in both strength and position, a mixing of the "dipole" structure $|f_1\rangle$ with other basis functions in the EOFs can be induced by either skewness in the position fluctuations or coupling of the strength and position fluctuations.

7 Fluctuations in Both Position and Width

For a jet that fluctuates in both position and width:

$$u(x,t) = U_0 f[(x - h\lambda)(1 + v\eta)], \quad (60)$$

we can write

$$|u\rangle = \sum_{j=0}^{\infty} \frac{U_0 (v\eta x - h\lambda - hv\lambda\eta)^j}{j!} N_j \left|f_j\right\rangle. \quad (61)$$

To leading order in h and v:

$$|u\rangle \simeq U_0 N_0 \left|f_0\right\rangle + U_0 \mathcal{N}_1 v\eta \left|\mathcal{F}_1\right\rangle - U_0 N_1 (h\lambda + hv\lambda\eta) \left|f_1\right\rangle. \quad (62)$$

It follows that

$$|u'\rangle = |u\rangle - \mathsf{E}\{|u\rangle\} \simeq U_0\mathcal{N}_1 v\eta\,|\mathcal{F}_1\rangle - U_0 N_1(h\lambda + hv[\eta\lambda - c_{\eta\lambda}])\,|f_1\rangle\,, \tag{63}$$

where $c_{\lambda\eta} = \mathsf{E}\{\lambda\eta\}$ is the covariance of η and λ. Assuming for simplicity that fluctuations in position and width are independent:

$$\mathcal{C} = \mathsf{E}\{|u'\rangle\langle u'|\} \simeq U_0^2\mathcal{N}_1^2 v^2\,|\mathcal{F}_1\rangle\langle\mathcal{F}_1| + U_0^2 N_1^2 h^2(1+v^2)\,|f_1\rangle\langle f_1|\,. \tag{64}$$

In general, the vectors $|\mathcal{F}_1\rangle$ and $|f_1\rangle$ will not be orthogonal:

$$\langle\mathcal{F}_1|f_1\rangle = G_{11} \neq 0. \tag{65}$$

It follows that EOFs will take the form $|E\rangle = \alpha\,|f_1\rangle + \beta\,|\mathcal{F}_1\rangle$ where (α,β) is an eigenvector of the matrix:

$$C = \begin{pmatrix} N_1^2 h^2(1+v^2) & N_1^2 h^2(1+v^2)G_{11} \\ \mathcal{N}_1^2 v^2 G_{11} & \mathcal{N}_1^2 v^2 \end{pmatrix}. \tag{66}$$

That is, the "dipole" $|f_1\rangle$ will not generally be an EOF of a jet that fluctuates in both position and width unless: (1) the jet shape $f(x)$ is symmetric about the jet axis, so by symmetry $G_{11} = 0$, or (2) position fluctuations are much stronger than those of width $h >> v$, so the dipole $|f_1\rangle$ emerges the leading EOF while the second EOF is some orthogonal hybrid of $|f_1\rangle$ and $|\mathcal{F}_1\rangle$.

As an example, consider the asymmetric jet given by Eqn. (27) with spatial skewness $\mathcal{S} = 1$. The leading EOFs obtained for $v = 0.1$ and $h = 0.1, 0.2$, and 0.3 are presented in Figure 5. For the smaller value of h, variability is dominated by width fluctuations and the leading EOF is $|\mathcal{F}_1\rangle$. As h increases, $|\mathcal{F}_1\rangle$ and $|f_1\rangle$ mix: the leading EOF for $h = 0.2$ is a hybrid of these two structures. As h increases further, position fluctuations dominate over width and the leading EOF becomes the dipole $|f_1\rangle$. Note once again that if the jet structure

$f(x)$ was symmetric then the width tripole and position dipole structures would not couple in the EOFs. For a jet with independent fluctuations in position and width, the degree of hybridisation of $|\mathcal{F}_1\rangle$ and $|f_1\rangle$ depends on how strongly asymmetric the jet shape is and on the relative magnitude of strength and position fluctuations.

The special case of fluctuations in both strength and width could also be considered (as in Monahan and Fyfe (2006)), but the leading EOFs will mix $|f_0\rangle$ and $|\mathcal{F}_1\rangle$ and not project strongly along the dipole $|f_1\rangle$. We thus now turn to the general case of fluctuations in all three kinematic degrees of freedom.

8 Fluctuations in Strength, Position, and Width

For the general case of fluctuations in all of strength, position, and width, the covariance matrix can be computed as in the special cases considered above. Rather than present the full (very complicated) covariance matrix, the essential results of the analysis can be obtained from a qualitative discussion making use of the results of the previous sections. The basis vectors entering the state vector to leading order in the small parameters l, h and v will be $|f_0\rangle$, $|f_1\rangle$, $|f_2\rangle$ and $|\mathcal{F}_1\rangle$; note that $|f_1\rangle$ is orthogonal to the vectors $|f_0\rangle$ and $|f_2\rangle$, but it is not orthogonal in general to $|\mathcal{F}_1\rangle$. Building the covariance matrix from this leading-order expansion of the state vector demonstrates that the following factors influence the degree to which the "pure" dipole $|f_1\rangle$ arises as an EOF of the variability: (i) the skewness of jet position fluctuations, (ii) the dependence of position fluctuations on either width or strength fluctuations, and (iii) the relative strength of position and width fluctuations. This analysis also indicates that asymmetric dipole EOFs can arise due to either asymmetries in the jet shape $f(x)$ or through mixing of $|f_1\rangle$ with other basis functions in the EOFs. In particular, a symmetric jet can generate asymmetric dipole EOFs if fluctuations in position are skewed or coupled to fluctuations in strength or width.

9 Conclusions

This study generalises the analysis of Monahan and Fyfe (2006), providing an analytic characterisation of the leading EOFs of a localised jet of arbitrary (smooth and localised) structure $f(x)$ with fluctuations in strength, position, and width of arbitrary distribution. Generalisations of the central conclusions of Monahan and Fyfe (2006) listed in the Introduction have been obtained:

1. A small number of basic shapes contribute to the leading order EOFs, corresponding to successive derivatives of the jet shape function $d^j f/dx^j$ and products $x^j\ d^j f/dx^j$. These basis functions and the EOFs are not generally either symmetric or antisymmetric around the jet axis. Basis functions produced by even and odd derivatives are orthogonal, but the even derivative basis functions are not mutually orthogonal (and similarly for the odd derivative basis functions). No simple orthogonality relationships exist among the functions $x^j\ d^j f/dx^j$.

2. The leading EOF structures corresponding to a jet fluctuating in one of strength, position, or width individually can be computed and for unskewed fluctuations are respectively $f(x)$, $f'(x)$, and $xf'(x)$. These EOF structures will be modified if the fluctuations in position or width are skewed, but are insensitive to the shape of the pdf of strength fluctuations.

3. If the jet fluctuates in more than one kinematic degree of freedom, the “dipole” structure $f'(x)$ arises as a distinct EOF mode as a result of fluctuations in jet position (as

the leading EOF if fluctuations in position are sufficiently large compared to those in strength and width), provided the position fluctuations are not strongly skewed or dependent on strength or width fluctuations. However, the associated Principal Component time series mixes together variability in strength, position, and width: the "zonal index" mode cannot be uniquely associated with a single kinematic jet degree of freedom.

4. The EOFs associated with individual degrees of freedom are not generally orthogonal, and may be mixed when more than one degree of freedom is active.

Furthermore, it is clear that asymmetric jet EOFs can arise as a consequence of an asymmetric jet shape, skewed position or width fluctuations, or coupling of position fluctuations with other kinematic degrees of freedom.

Returning to the question posed in the title, this analysis has demonstrated that to the extent that a variable jet can be described as a basic localised functional form $f(x)$ with a single extremum (so $f'(x)$ changes sign only once) that fluctuates in strength, position, and width, the factors which influence the extent to which a dipole-like structure will arise as an EOF are: (i) the skewness of position fluctuations, (ii) the dependence of position fluctuations on strength and width fluctuations, and (iii) the relative strength of position and width fluctuations. In particular, the leading EOF will be a dipole if jet position fluctuations are not strongly skewed, not strongly dependent on position and width fluctuations, and sufficiently large relative to strength and width fluctuations. That these conditions appear to be characteristic of the tropospheric zonal-mean eddy-driven jets in observations and models (e.g. Fyfe and Lorenz, 2005; Monahan and Fyfe, 2006) explains the ubiquity of dipolar zonal-mean zonal wind EOFs in these systems. As jets are generic features of flow on rotating spheres, to the extent that these jets can be characterised as a basic shape displaying fluctuations in strength, position, and width the results of this study are relevant to characterisation of variability in the middle atmosphere, in the ocean, and in the atmospheres of other planets. Finally, the present study reinforces in a more general context a central conclusion of Monahan and Fyfe (2006): in the troposphere, the dipole EOF arises due to variability in jet position, but its associated PC time series also carries information about strength and width fluctuations. The statistical analysis provides a picture of the jet dynamics, but an incomplete one: as through a PCA, darkly.

Acknowledgements

The authors would like to thank Bill Merryfield and John Scinocca for their helpful comments on this manuscript. Adam Monahan acknowledges support from the Natural Sciences and Engineering Research Council of Canada and from the Canadian Institute for Advanced Research Earth System Evolution Program.

Appendix: Notation

For notational convenience, we adopt the "bra-ket" notation for vectors common in quantum mechanics (e.g. Cohen-Tannoudji et al., 1977). A function $f(x)$ can be considered as a vector $|f\rangle$ (denoted the "ket") in a vector space $\mathcal{H}$. The "transpose" of this vector (technically the corresponding element in the dual space of linear functionals; Cohen-Tannoudji et al. (1977)) is written as $\langle f|$ (denoted the "bra"). The inner product of the vectors $|f\rangle$ and $|g\rangle$ is given by the "bracket" (thus the terms *bra* and *ket*)

$$\langle f|g\rangle = \int_{x_1}^{x_2} f(x)g(x)\;dx. \tag{67}$$

The "dyadic product" $A = |f\rangle\langle g|$ defines an operator: acting on any vector $|e\rangle$:

$$A\,|e\rangle = |f\rangle\langle g|e\rangle = f(x)\int_{x_1}^{x_2} g(x)e(x)\;dx. \tag{68}$$

References

Cohen-Tannoudji, C., B. Diu, and F. Laloë, 1977: *Quantum Mechanics. Volume 1*. Wiley Interscience, Paris, France, 898 pp.

Fyfe, J. C., G. J. Boer, and G. M. Flato, 1999: The Arctic and Antarctic Oscillations and their projected changes under global warming. *Geophys. Res. Lett.*, **26**, 1601–1604.

Fyfe, J. C. and D. J. Lorenz, 2005: Characterizing zonal wind variability: Lessons from a simple GCM. *J. Climate*, **18**, 3400–3404.

Lorenz, D. J. and D. L. Hartmann, 2001: Eddy-zonal flow feedback in the southern hemisphere. *J. Atmos. Sci.*, **58**, 3312–3327.

Monahan, A. H., 2006: The probability distribution of sea surface wind speeds. Part I: Theory and SeaWinds observations. *J. Climate*, **19**, 497–520.

Monahan, A. H. and J. C. Fyfe, 2006: On the nature of zonal jet EOFs. *J. Climate*, **19**, 6409–6424.

— 2008: On annular modes and zonal jets. *J. Climate*, **21**, 1964–1979.

Vallis, G. K., E. P. Gerber, P. J. Kushner, and B. A. Cash, 2004: A mechanism and simple dynamical model of the North Atlantic Oscillation and annular modes. *J. Atmos. Sci.*, **61**, 264–280.

Wittman, M. A., A. J. Charlton, and L. M. Polvani, 2005: On the meridional structure of annular modes. *J. Climate*, **18**, 2199–2122.

Figure Captions

Figure 1: Jet shape function $f(x)$ for spatial skewness (Eqn. 28) $\mathcal{S} = 0$ (black curve) and $\mathcal{S} = 1$ (gray curve).

Figure 2: EOFs of a jet with Gaussian fluctuations in position alone (grey curves: predicted; black curves: numerically simulated) with $h = 0.5$ for spatial skewness $\mathcal{S} = 0$ and $\mathcal{S} = 1$. Leftmost two panels: first EOF $\left|E^{(1)}\right\rangle$. Rightmost two panels: second EOF $\left|E^{(2)}\right\rangle$. The dashed lines in the second and fourth panels are respectively the first and second EOF patterns for the symmetric jet ($\mathcal{S} = 0$).

Figure 3: Leading EOF $\left|E^{(1)}\right\rangle$ of a jet with Gaussian fluctuations in width alone (grey curves: predicted; black curves: numerically simulated) with $v = 0.15$ for spatial skewness $\mathcal{S} = 0$ and $\mathcal{S} = 1$. In the right-hand panel, the dashed line is the simulated EOF structure for $\mathcal{S} = 0$.

Figure 4: EOFs of a symmetric jet with Gaussian strength fluctuations ($l = 0.185$) and skewed position fluctuations ($h = 0.3$) (grey curves: predicted; black curves: numerically simulated) for skew(λ) = 0, 0.75, and 1.5. In the second and third columns, the dashed line is the simulated EOF pattern for skew(λ) = 0. Upper panels: first EOF $\left|E^{(1)}\right\rangle$. Lower panels: second EOF $\left|E^{(2)}\right\rangle$.

Figure 5: Leading EOF $\left|E^{(1)}\right\rangle$ of an asymmetric jet (with spatial skewness $\mathcal{S} = 1$) with Gaussian fluctuations in both width ($v = 0.1$) and position ($h = 0.1, 0.2, 0.3$); (grey curves: predicted; black curves: numerically simulated) In the second and third panels, the dashed line is the simulated EOF structure for $h = 0.1$.

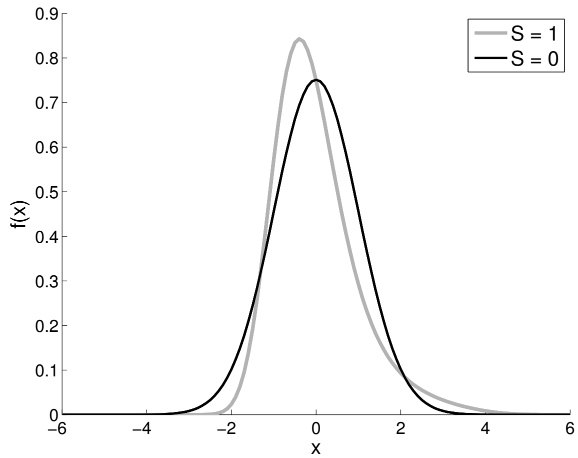

Figure 1: Jet shape function $f(x)$ for spatial skewness (Eqn. 28) $\mathcal{S} = 0$ (black curve) and $\mathcal{S} = 1$ (gray curve).

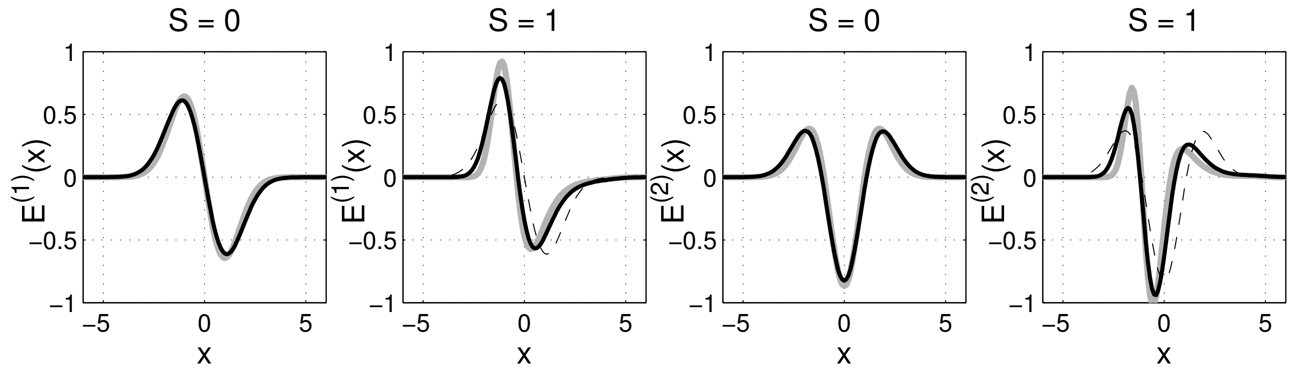

Figure 2: EOFs of a jet with Gaussian fluctuations in position alone (grey curves: predicted; black curves: numerically simulated) with $h = 0.5$ for spatial skewness $\mathcal{S} = 0$ and $\mathcal{S} = 1$. Leftmost two panels: first EOF $\left|E^{(1)}\right\rangle$. Rightmost two panels: second EOF $\left|E^{(2)}\right\rangle$. The dashed lines in the second and fourth panels are respectively the first and second EOF patterns for the symmetric jet ($\mathcal{S} = 0$).

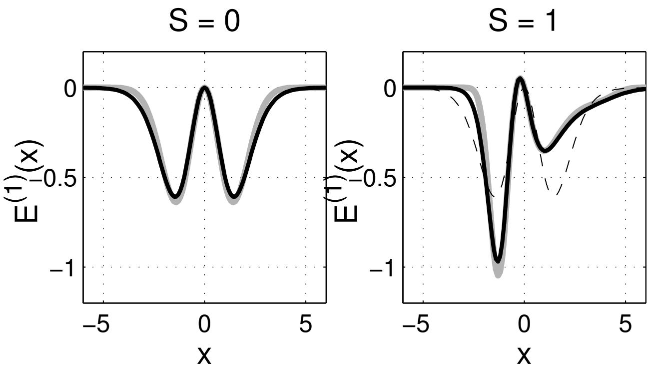

Figure 3: Leading EOF $\left|E^{(1)}\right\rangle$ of a jet with Gaussian fluctuations in width alone (grey curves: predicted; black curves: numerically simulated) with $v = 0.15$ for spatial skewness $\mathcal{S} = 0$ and $\mathcal{S} = 1$. In the right-hand panel, the dashed line is the simulated EOF structure for $\mathcal{S} = 0$.

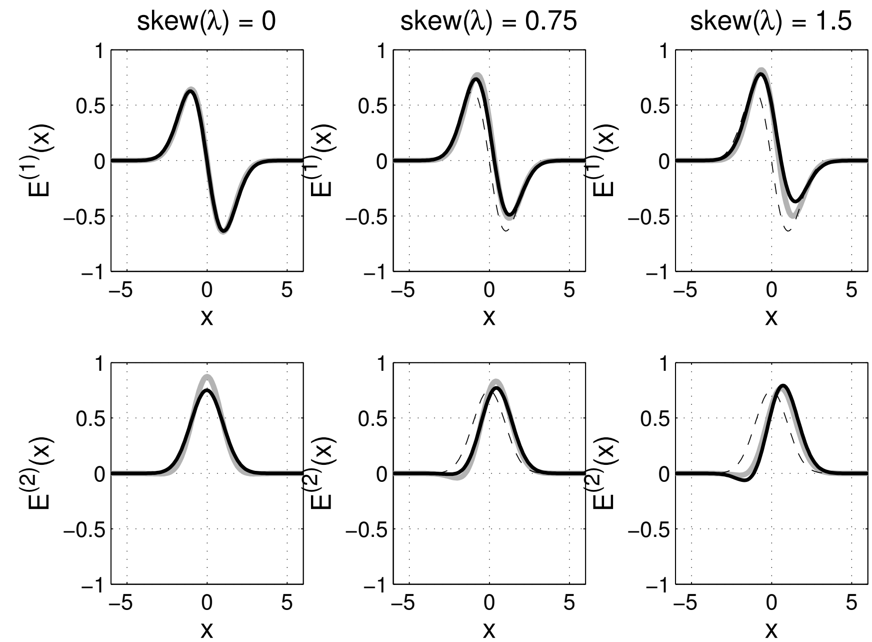

Figure 4: EOFs of a symmetric jet with Gaussian strength fluctuations ($l = 0.185$) and skewed position fluctuations ($h = 0.3$) (grey curves: predicted; black curves: numerically simulated) for skew(λ) = 0, 0.75, and 1.5. In the second and third columns, the dashed line is the simulated EOF pattern for skew(λ) = 0. Upper panels: first EOF $\left|E^{(1)}\right\rangle$. Lower panels: second EOF $\left|E^{(2)}\right\rangle$.

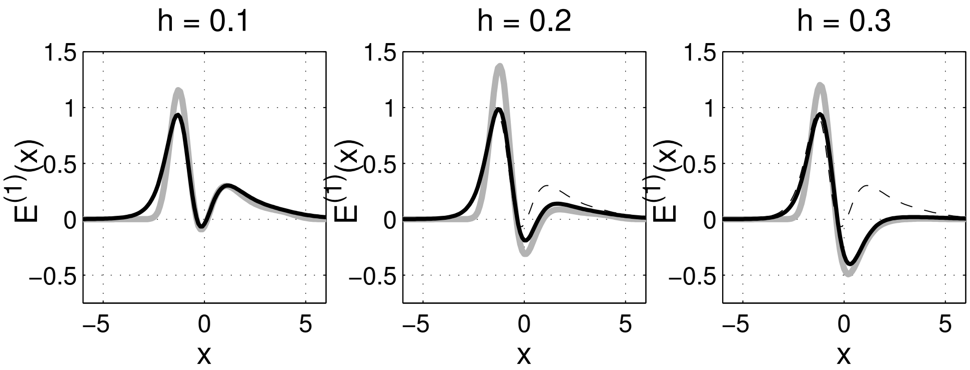

Figure 5: Leading EOF $\left|E^{(1)}\right\rangle$ of an asymmetric jet (with spatial skewness $\mathcal{S} = 1$) with Gaussian fluctuations in both width ($v = 0.1$) and position ($h = 0.1, 0.2, 0.3$); (grey curves: predicted; black curves: numerically simulated) In the second and third panels, the dashed line is the simulated EOF structure for $h = 0.1$.