A SPECTRAL METHOD FOR PANTOGRAPH-TYPE DELAY DIFFERENTIAL EQUATIONS AND ITS CONVERGENCE ANALYSIS*

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Abstract

We propose a novel numerical approach for delay differential equations with vanishing proportional delays based on spectral methods. A Legendre-collocation method is employed to obtain highly accurate numerical approximations to the exact solution. It is proved theoretically and demonstrated numerically that the proposed method converges exponentially provided that the data in the given pantograph delay differential equation are smooth.

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1. Introduction

In this paper we consider the delay differential equation:

$$u'(x) = a(x)u(qx), \quad 0 < x \le T, \tag{1.1}$$

$$u(0) = y_0, \tag{1.2}$$

where $0 < q < 1$ is a given constant and a is a smooth function on $[0, T]$. Eq. (1.1) belongs to the class of so-called pantograph delay differential equations; see [7, 10] for details on their theory and physical applications.

The existing numerical methods for solving (1.1)-(1.2) include Runge-Kutta type methods (see, e.g., the monograph [3]) and collocation methods (cf. [1,2,4,5]). The main difficulty in the application of Runge-Kutta methods to (1.1) is the lack of information at the grid points for the function on the right-hand-side of (1.1); these numerical data have to be generated by some local interpolation process. While collocation methods yield globally defined approximations, the collocation solutions are not globally smooth. Moreover, it has been shown in [5] that for

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arbitrarily smooth solutions of (1.1) the optimal order at the grid points of collocation methods using piecewise polynomials of degree m cannot exceed $p = m + 2$ when $m \geq 2$ (in contrast to their application to ordinary differential equations where collocation at the Gauss points leads to $\mathcal{O}(h^{2m})$-convergence).

If the function a is in $C^d[0, T]$, then the corresponding solution of the initial-value problem (1.1)-(1.2) lies in $C^{d+1}[0, T]$. In this case, it is more natural to employ spectral-type methods since they produce approximate solutions that are defined globally on $[0, T]$ and are globally smooth. Moreover, the resulting errors inherit the typical property of spectral method in that they decay exponentially.

For ease of notation we will describe and analyze the spectral method on the standard interval $I := [-1, 1]$. Hence, we employ the transformation

$$x = \frac{T}{2}(1 + t), \quad t = \frac{2x}{T} - 1. \tag{1.3}$$

Then problem the (1.1)-(1.2) becomes

$$y'(t) = b(t)y(qt + q_1), \quad -1 < t \leq 1, \tag{1.4}$$

$$y(-1) = y_0, \tag{1.5}$$

where

$$y(t) := u\left(\frac{T}{2}(1 + t)\right), \quad b(t) := \frac{T}{2}a\left(\frac{T}{2}(1 + t)\right), \quad q_1 := q - 1. \tag{1.6}$$

2. The Spectral Method

Let $\{t_k\}_{k=0}^N$ be the set of the $(N + 1)$-point Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto points in $[-1, 1]$, and denote by $\mathcal{P}_N$ the space of polynomials with degrees not exceeding N. Integration of (1.4) from $[-1, t_j]$ gives

$$y(t_j) = y_0 + \int_{-1}^{t_j} b(s)y(qs + q_1)ds, \quad j \geq 1, \tag{2.1}$$

and the linear transformation

$$s = \frac{t_j + 1}{2}v + \frac{t_j - 1}{2}$$

yields

$$y(t_j) = y_0 + \int_{-1}^{1} \tilde{b}(v; t_j) y\left(\frac{t_j + 1}{2}qv + q_{1j}\right)dv, \tag{2.2}$$

where

$$\tilde{b}(v; t_j) := \frac{1 + t_j}{2}b\left(\frac{t_j + 1}{2}v + \frac{t_j - 1}{2}\right), \quad q_{1j} := \frac{t_j + 1}{2}q - 1.$$

If we apply the $(N + 1)$-point Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto quadrature formula to (2.2) we obtain

$$y(t_j) \approx y_0 + \sum_{k=0}^{N} \omega_k \tilde{b}(v_k; t_j)y\left(\frac{t_j + 1}{2}qv_k + q_{1j}\right), \tag{2.3}$$

where $v_k = t_k$ and $\{\omega_k\}_{k=0}^N$ are the corresponding weights. Let $Y_j \approx y(t_j)$ and assume $Y \in \mathcal{P}_N$ is of the form

$$Y(t) = I_N(Y) := \sum_{j=0}^N Y_j F_j(t), \tag{2.4}$$

where $F_j(t)$ is the standard Lagrange interpolation polynomial associated with the Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto points $\{t_k\}_{k=0}^N$. It follows from (2.3) that the numerical scheme for solving (1.4)-(1.5) is given by

$$Y_j = y_0 + \sum_{k=0}^N \omega_k \tilde{b}(v_k; t_j) Y\left(\frac{t_j+1}{2} q v_k + q_{1j}\right), \quad 1 \le j \le N, \tag{2.5}$$

where $Y(\bullet)$ of the right-hand side in (2.5) is computed using (2.4). To evaluate $Y(\bullet)$ efficiently, it is necessary to evaluate $F_j(\bullet)$ fast. To this end, we expand $F_k(v)$ in terms of the Legendre polynomials:

$$F_k(v) = \sum_{m=0}^N c_{km} L_m(v). \tag{2.6}$$

If $\{x_s\}_{s=0}^N$ are the Legendre Gauss-Lobatto points, then the discrete Legendre coefficients c_{km} can be determined by the following relation (see, [6,12]):

$$c_{km} = \frac{2m+1}{N(N+1)} \sum_{s=0}^N F_k(x_s) \frac{L_m(x_s)}{[L_N(x_s)]^2} = \frac{2m+1}{N(N+1)} \frac{L_m(x_k)}{[L_N(x_k)]^2}. \tag{2.7}$$

Below we describe a more straightforward way to discretize Eq. (2.1). The idea is to approximate the whole integrand in (2.1) by an interpolation polynomial. More precisely, using a linear transformation to the integral in (2.1) gives

$$y(t_j) = y_0 + \frac{1}{q} \int_{-1}^{qt_j+q_1} b\left(\frac{v-q_1}{q}\right) y(v) dv, \quad j \ge 1. \tag{2.8}$$

Projecting the above integrand to $\mathcal{P}_N$, we have

$$b\left(\frac{v-q_1}{q}\right) y(v) \approx \sum_{k=0}^N b\left(\frac{t_k-q_1}{q}\right) y(t_k) F_k(v), \tag{2.9}$$

where F_k is the kth Lagrange basis function. The numerical scheme then follows from (2.8)-(2.9) together with (2.6)-(2.7); it is described by

$$Y_j = y_0 + \sum_{k=0}^N b_k Y_k w_{k,j}, \quad 1 \le j \le N, \tag{2.10}$$

where

$$b_k := b\left(\frac{t_k-q_1}{q}\right), \quad w_{k,j} := \frac{2m+1}{qN(N+1)} \sum_{m=0}^N \frac{L_m(x_k)}{[L_N(x_k)]^2} \int_{-1}^{qt_j+q_1} L_m(v) \, dv.$$

For the Legendre polynomials, we have $L_m(v) = \big(L'_{m+1}(v) - L'_{m-1}(v)\big)/(2m+1)$, (see, e.g., [6,12]) which yields

$$w_{k,j} = \frac{1}{qN(N+1)[L_N(x_k)]^2} \sum_{m=0}^N L_m(x_k)\big(L_{m+1}(qt_j+q_1) - L_{m-1}(qt_j+q_1)\big). \tag{2.11}$$

3. Some Useful Lemmas

Lemma 3.1. ([6]) *Assume that a $(N+1)$-point Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto quadrature formula relative to the Legendre weights is used to integrate the product $u\phi$, where $y \in H^m(I)$ with $I := (-1,1)$ and some $m \geq 1$, and $\phi \in \mathcal{P}_N$. Then there exists a constant C not depending on N such that*

$$\left| \int_{-1}^{1} y(x)\phi(x)dx - (y,\phi)_N \right| \leq CN^{-m}|y|_{\tilde{H}_{m,N}(I)}\|\phi\|_{L^2(I)}, \qquad (3.1)$$

where

$$|y|_{\tilde{H}_{m,N}(I)} = \left(\sum_{k=\min(m,N+1)}^{m} \|y^{(k)}\|_{L^2(I)}^2 \right)^{1/2}, \qquad (3.2)$$

$$(y,\phi)_N = \sum_{k=0}^{N} \omega_k y(x_k)\phi(x_k). \qquad (3.3)$$

Lemma 3.2. *Assume that $w \in H^m(I)$ and denote by $I_N w$ the interpolation polynomial associated with the $(N+1)$ Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto points $\{t_k\}_{k=0}^{N}$. Then*

$$\|w - I_N w\|_{L^2(I)} \leq CN^{-m}|w|_{\tilde{H}_{m,N}(I)}, \qquad (3.4)$$

$$\|w - I_N w\|_{L^\infty(I)} \leq CN^{1/2-m}|w|_{\tilde{H}_{m,N}(I)}. \qquad (3.5)$$

Proof. The estimate (3.4) is given on p. 289 of [6]. The estimate

$$\|y - I_N y\|_{H^1(I)} \leq CN^{1-m}|y|_{\tilde{H}_{m,N}(I)}, \quad 1 \leq s \leq m,$$

can also be found in [6]. Using the above estimate and the inequality

$$\|v\|_{L^\infty(a,b)} \leq \sqrt{\frac{1}{b-a} + 2}\|v\|_{L^2(a,b)}^{1/2}\|v\|_{H^1(a,b)}^{1/2}, \quad \forall v \in H^1(a,b),$$

we readily obtain (3.5). $\qquad \square$

From [11], we have the following result on the Lebesgue constant for Lagrange interpolation based on the zeros of the Legendre polynomials.

Lemma 3.3. *Assume that $\left\{F_j(x)\right\}_{j=0}^{N}$ are the Lagrange interpolation polynomials with respect to the Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto points $\{x_j\}$. Then*

$$\|I_N\|_\infty := \max_{x \in (-1,1)} \sum_{j=0}^{N} |F_j(x)| = \mathcal{O}(\sqrt{N}). \qquad (3.6)$$

Lemma 3.4 (Gronwall inequality) *If a non-negative continuous function $E(t)$ satisfies*

$$E(t) \leq C \int_{-1}^{t} E(qs + q_1)\,ds + G(t), \quad t \in [-1,1], \qquad (3.7)$$

where $0 < q < 1$ is a constant, $q_1 = q - 1$ and $G(t)$ is a given continuous function, then

$$\|E\|_{L^\infty(I)} \leq C\|G\|_{L^\infty(I)}. \qquad (3.8)$$

Proof. It follows from (3.7) and a simple change of variables that

$$E(t) \le Cq^{-1} \int_{-1}^{qt+q_1} E(s)\,ds + G(t). \tag{3.9}$$

Since $0 < q < 1$, we have $qt + q_1 = (q-1)(t+1) + t \le t$, for $t \in [-1,1]$. This, together with (3.9), leads to a standard Gronwall inequality for $E(t)$, and hence (3.8) follows. $\qquad\square$

4. Convergence Analysis

We only give the L^∞-error analysis for the method (2.5); its L^2-error analysis is similar with the help of (3.4). Moreover, two almost equivalent approximation methods are given in the last section, i.e., (2.5) and (2.10). The error analysis for the latter one is similar so we will not give a detail proof in this paper. A numerical comparison of the two methods for a simple case and a relatively more complicated case (i.e., systems of delay equations) will be provided in Section 5; see Tables 5.1 and 5.2 and Fig. 5.2.

Theorem 4.1. *Consider the problem* (1.4)-(1.5) *and its spectral approximation method* (2.5). *If the function b is sufficiently smooth (which also implies that the solution of* (1.4) *is smooth), then*

$$\begin{aligned}
\|Y - y\|_{L^\infty(I)} \le{}& CN^{-m-1/2}|b(\bullet)y(q\bullet + q_1)|_{\tilde{H}_{m,N}(I)} \\
&+ CN^{1/2-m}|b|_{\tilde{H}_{m,N}(I)}\|y\|_{L^2(I)},
\end{aligned} \tag{4.1}$$

provided that N is sufficiently large, where $Y \in \mathcal{P}_N$ is the spectral approximation given by (2.4)-(2.5), $q_1 = q - 1$, *and C is a constant that does not depend on N.*

Proof. It follows from (2.5) that

$$\begin{aligned}
Y_j &= y_0 + \int_{-1}^{1} \tilde{b}(v;t_j)Y\left(\frac{t_j+1}{2}q(v+1)-1\right)dv + J_1(t_j) \\
&= y_0 + \int_{-1}^{t_j} b(s)Y(qs+q_1)\,ds + J_1(t_j),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
J_1(t) :={}& \sum_{k=0}^{N} \omega_k \tilde{b}(v_k;t)Y\left(\frac{t+1}{2}q(v_k+1)-1\right) \\
&- \int_{-1}^{1} \tilde{b}(v;t)Y\left(\frac{t+1}{2}q(v+1)-1\right)dv.
\end{aligned} \tag{4.3}$$

By Lemma 3.1 we have

$$|J_1(t_j)| \le CN^{-m}|b|_{\tilde{H}_{m,N}(I)}\|Y\|_{L^2(I)}, \quad 0 \le j \le N. \tag{4.4}$$

Multiplying both sides of (4.2) by $F_j(t)$ and summing from 0 to N lead to

$$Y(t) = y_0 + \int_{-1}^{t} b(s)Y(qs+q_1)\,ds + J_1(t) + J_2(t) + J_3(t), \tag{4.5}$$

where $J_1(t) = I_N(J_1)$,

$$J_2(t) := I_N\left(\int_{-1}^t b(s)e(qs+q_1)\,ds\right) - \int_{-1}^t b(s)e(qs+q_1)\,ds, \qquad (4.6)$$

$$J_3(t) := I_N\left(\int_{-1}^t b(s)y(qs+q_1)\,ds\right) - \int_{-1}^t b(s)y(qs+q_1)\,ds. \qquad (4.7)$$

In (4.6), $e(t) := Y(t) - y(t)$, where $y(t)$ is the solution of (1.4)-(1.5). Using Lemma 3.2 with $m = 1$ and w being the integral function in (4.6) gives

$$\begin{aligned}\|J_2\|_{L^\infty(I)} &\leq CN^{-1/2}\|w'\|_{L^2(I)}\\&= CN^{-1/2}\|b(\bullet)e(q\bullet+q_1)\|_{L^2(I)}.\end{aligned} \qquad (4.8)$$

Similarly, using Lemma 3.2 with m being replaced by $m+1$ and w being the integral function in (4.7) yields

$$\begin{aligned}\|J_3\|_{L^\infty(I)} &\leq CN^{1/2-(m+1)}|w|_{\tilde{H}_{m+1,N}(I)}\\&= CN^{-m-1/2}|b(\bullet)y(q\bullet+q_1)|_{\tilde{H}_{m,N}(I)}.\end{aligned} \qquad (4.9)$$

Moreover, we have

$$\begin{aligned}\|J_1\|_{L^\infty(I)} &= \left\|\sum_{j=0}^N J_1(t_j)F_j(t)\right\|_{L^\infty(I)} \leq \max_j|J_1(t_j)|\left\|\sum_{j=0}^N F_j(t)\right\|_{L^\infty(I)}\\&\leq CN^{1/2}\max_j|J_1(t_j)| \leq CN^{1/2-m}|b|_{\tilde{H}_{m,N}(I)}\|Y\|_{L^2(I)},\end{aligned} \qquad (4.10)$$

where we have used Lemmas 3.3 and (4.4). It follows from (4.5) and (1.4) that

$$e(t) = \int_{-1}^t b(s)e(qs+q_1)\,ds + J_1(t) + J_2(t) + J_3(t), \qquad (4.11)$$

which gives

$$|e(t)| \leq C\int_{-1}^t |e(qs+q_1)|ds + |J_1(t)| + |J_2(t)| + |J_3(t)|. \qquad (4.12)$$

An application of Lemma 3.4 to the above inequality yields the estimate

$$\begin{aligned}\|e\|_{L^\infty(I)} &\leq C\left(\|J_1\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)} + \|J_3\|_{L^\infty(I)}\right)\\&\leq CN^{1/2-m}|b|_{\tilde{H}_{m,N}(I)}\|Y\|_{L^2(I)} + CN^{-1/2}\|e\|_{L^\infty(I)}\\&\quad + CN^{-m-1/2}|b(t)y(qt+q_1)|_{\tilde{H}_{m,N}(I)},\end{aligned} \qquad (4.13)$$

where in the last step we have used (4.8)-(4.10). Observe that

$$\|Y\|_{L^2(I)} \leq \|e\|_{L^2(I)} + \|y\|_{L^2(I)} \leq C\|e\|_{L^\infty(I)} + \|y\|_{L^2(I)}.$$

Combining the above results and (4.13) we arrive at the desired estimate (4.1) provided that N is sufficiently large. $\qquad\square$

Remark. We have based the description and analysis of the spectral method on the basic pantograph delay differential equation (1.1), in order to present the key ideas in a straightforward way. It is clear from the foregoing analysis (using a suitably adapted version of Lemma 3.4

and its proof) that the results of Theorem 4.1 remain valid for the more general pantograph equation

$$u'(x) = a(x)u(qx) + b(x)u(x) + g(x), \quad 0 < x \le T \quad (0 < q < 1),$$

with smooth a, b and g. We leave the details to the reader but will illustrate the results numerically in Section 5.

5. Numerical Tests

In the following, we use a number of numerical examples to illustrate the accuracy and efficiency of the spectral methods (2.5) and (2.10). In our computations, we use the Legendre Gauss quadrature with weights

$$\omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 < j \le N,$$

and the Legendre Gauss-Lobatto quadrature with weights

$$\omega_j = \frac{2}{N(N + 1)} \frac{1}{[L_N(x_j)]^2}, \quad 0 < j \le N.$$

It is found that the numerical results using the two quadratures are almost the same.

5.1. Linear pantograph equations

Example 5.1. Consider the delay equation (1.4) with

$$b(t) = -\frac{\sin(t)}{\cos(qt + q_1)}, \quad y_0 = 1.$$

The exact solution of the problem is then $y(t) = \cos(t)$.

In our computations, we choose $q = 0.5$ and $q = 0.99$. The maximum pointwise error between the numerical solution obtained by using the numerical scheme (2.5) and the exact solution is given in Table 5.1 and is shown by Fig. 5.1. For comparison, the maximum pointwise errors obtained by using the numerical scheme (2.10) is shown in Table 5.2. It is observed that both schemes (2.5) and (2.10) are of almost the same accuracy.

Example 5.2. Consider the more general pantograph equation with constant coefficients,

$$\begin{cases} y'(t) = \frac{1}{2}y(qt) - y(t) + g(t), \quad t \in (-1, 1], \\ y_0 = 1, \end{cases} \tag{5.1}$$

with $g(t) = -\frac{1}{2}e^{-qt}$. The exact solution of the problem is $y(t) = e^{-t}$.

The error behavior for $q = 0.5$ using the numerical scheme (2.5) is shown in Fig. 5.2. It is seen that with about 20 spectral collocation points the errors drop to the machine accuracy.

Table 5.1: Example 5.1: maximum point-wise error using (2.5).

N	Maximum error($q = 0.5$)	N	Maximum error($q = 0.99$)
6	6.410e-03	6	6.410e-03
8	6.145e-05	8	6.145e-05
10	3.055e-07	10	3.055e-07
14	1.890e-12	12	9.264e-10
16	2.220e-15	14	1.889e-12
18	1.110e-15	18	1.554e-15

Table 5.2: Example 5.1: maximum point-wise error using (2.10) .

N	Maximum error($q = 0.5$)	N	Maximum error($q = 0.99$)
6	1.351e-04	6	7.362e-07
8	1.102e-06	8	1.891e-09
10	5.662e-09	10	3.598e-12
14	1.854e-13	14	4.441e-16
16	5.551e-16	16	4.441e-16
18	3.331e-16	18	4.441e-16

5.2. A Nonlinear Pantograph Equation

Consider the non-linear version of (1.4)-(1.5) of the form

$$\begin{cases} y'(t) = f\big(y(qt + q_1)\big) + g(t), \quad t \in (-1, 1], \\ y(-1) = y_0. \end{cases} \tag{5.2}$$

Following the same procedure as for the linear case (1.4), we can obtain a spectral approximation for the nonlinear problem (5.2) by using

$$y(t_j) = y_0 + \int_{-1}^{1} \left[\tilde{f}\left(y\Big(\frac{t_j + 1}{2}qv + q_{1j}\Big) \right) + \tilde{g}\left(\frac{t_j + 1}{2}v + \frac{t_j - 1}{2} \right) \right] dv, \tag{5.3}$$

where $\tilde{f} = (t_j + 1)f/2$, $\tilde{g} = (t_j + 1)g/2$. This leads to a nonlinear algebraic system,

$$Y_j = y_0 + \sum_{k=0}^{N} \omega_k \left[\tilde{f}\left(Y\Big(\frac{t_j + 1}{2}qv_k + q_{1j}\Big) \right) + \tilde{g}\Big(\frac{t_j + 1}{2}v_k + \frac{t_j - 1}{2}\Big) \right], \tag{5.4}$$

for $1 \leq j \leq N$. Eq. (5.4) can be solved by some suitable nonlinear solvers. In particular, a spectral postprocessing technique based on the Gauss-Seidel iteration [13] can be used to handle the nonlinearity.

We point out that one can carry out the convergence analysis similar to that of Theorem 4.1 provided that f is Lipschitz continuous.

Example 5.3. Consider the nonlinear delay equation (5.2) in the form

$$y'(t) = y^2(qt + q_1) + g(t), \tag{5.5}$$

where $g(t) = -2t(1 + t^2)^{-2} - \big(1 + (qt + q_1)^2\big)^{-2}$.

The exact solution of the above problem is $y(t) = (1 + t^2)^{-1}$. The errors between the exact solution and the numerical solution for $q = 0.5$ using (5.4) is shown in Fig. 5.3. Again, the spectral rate of convergence is observed.
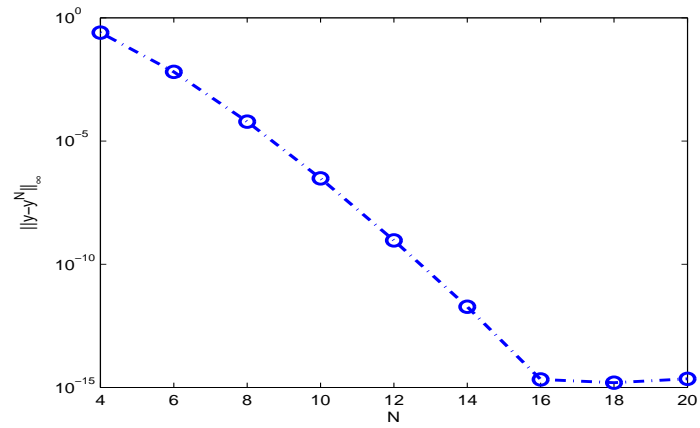
Fig. 5.1. Example 5.1: L^∞ error obtained by using (2.5).
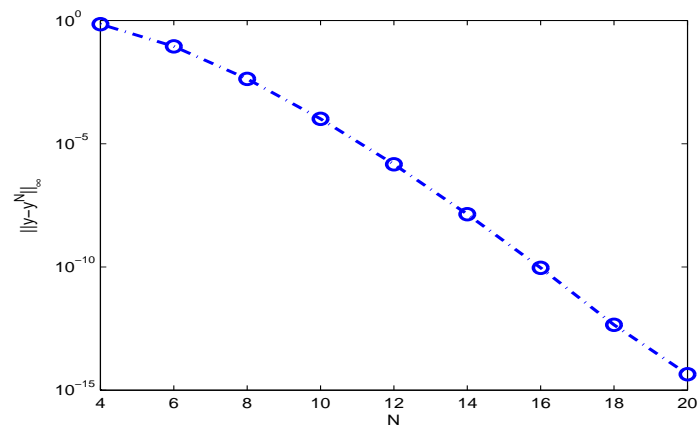
Fig. 5.2. Example 5.2: L^∞ error error obtained by using (2.5).
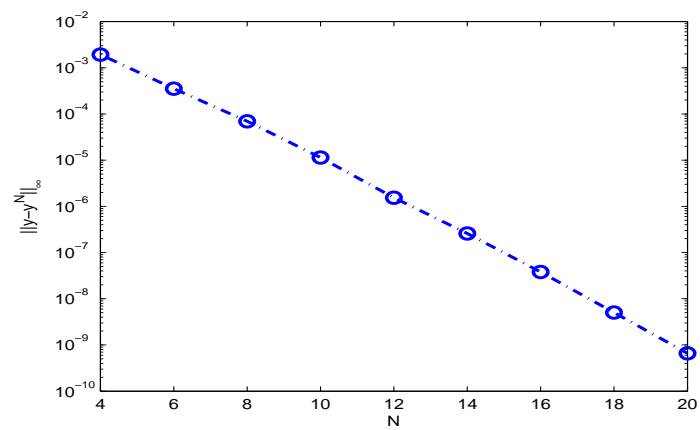
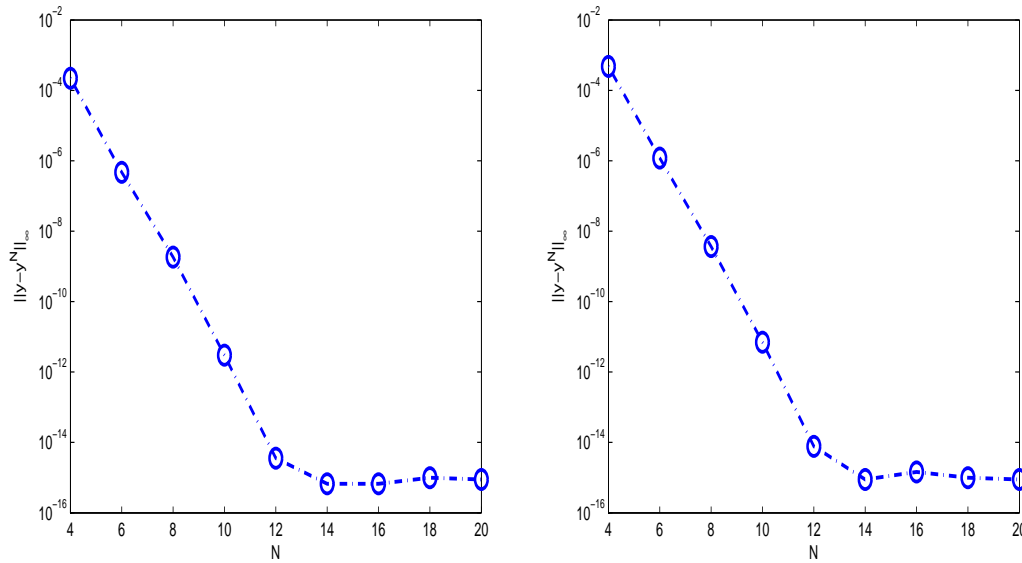Fig. 5.3. Example 5.3: L^∞ error obtained by using (5.4).

Fig. 5.4. Example 5.4: L^∞ errors obtained by using (a): (5.7), and (b): (5.8).

5.3. Systems of delay equations

Consider the system of delay differential equations of the form

$$\begin{cases} \dfrac{d\vec{y}}{dt}(t) = B(t)\vec{y}(qt) + \vec{g}(t), & t \in (-1, 1], \\ \vec{y}(-1) = \vec{y}_0, \end{cases} \tag{5.6}$$

where $B(t)$ is an $n \times n$ matrix, $\vec{y}$, $\vec{g}$ and $\vec{y}_0$ are n-dimensional vectors. Similar to (2.5), we obtain

$$\vec{Y}_j = \vec{y}_0 + \sum_{k=0}^{N} \tilde{B}(v; t_j) \omega_k \vec{Y}\left(\frac{t_j + 1}{2} q v_k + q_{1j}\right) + \tilde{g}(t_j), \quad 1 \le j \le N, \tag{5.7}$$

where

$$\tilde{B}(v; t_j) := B\left(\frac{t_j + 1}{2} v + \frac{t_j - 1}{2}\right), \quad \tilde{g}(t) = \int_{-1}^{t_j} \vec{g}(s)\, ds.$$

A numerical scheme analogous to (2.10) for (5.6) is given by

$$\vec{Y}_j = \vec{y}_0 + \sum_{k=0}^{N} B\left(\frac{t_k - q_1}{q}\right) \vec{Y}_k w_{k,j} + \tilde{g}(t_j), \quad 1 \le j \le N. \tag{5.8}$$

Example 5.4. Consider the system of delay differential equation (5.6) with

$$B(t) = \begin{pmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{pmatrix}.$$

The exact solution of the problem is $\vec{y} = [\sin(t), \cos(t)]^T$.

Fig. 5.4 shows the numerical errors obtained by using (5.7) and (5.8) for $q = 0.7$. As expected, the two methods produce almost the same results; both converge exponentially.
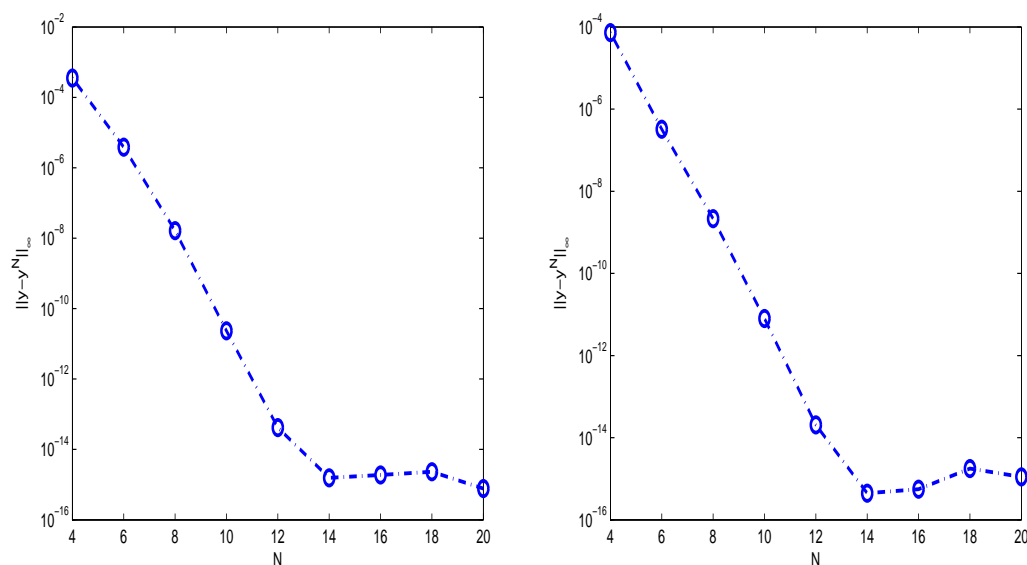
Fig. 5.5. Example 5.5: L^∞ errors for general pantograph equation with neutral term. (a): $q = 0.5$ and (b): $q = 0.99$.

5.4. A linear pantograph equation with neutral term

Example 5.5. Consider the general pantograph equation

$$\begin{cases} y'(t) = a(t)y(t) + b(t)y(qt) + c(t)y'(qt) + g(t), & t \in (-1, 1] \\ y_0 = 0. \end{cases} \tag{5.9}$$

with $a(t) = \sin(t), b(t) = \cos(qt), c(t) = -\sin(qt), g(t) = \cos(t) - \sin^2(t)$.

The exact solution of the problem is $y(t) = \sin(t)$. The maximum pointwise errors for $q = 0.5$ and $q = 0.99$ using (2.5) are shown in Fig. 5.5. It is seen that with about 12 spectral collocation points the error reaches the machine accuracy.

6. Conclusions

It has been the aim of this paper to show that it appears natural to approximate the (smooth) solution of a pantograph-type delay differential equation with smooth data by a spectral-type method, instead of a piecewise polynomial collocation or a Runge-Kutta method, since spectral approximations are globally smooth. They also exhibit the well-known exponential convergence property of spectral methods. Various numerical examples confirm the favorable convergence results.

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