Quantitative Stability Analysis of Stochastic Quasi-Variational Inequality Problems and Applications

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Abstract. We consider a parametric stochastic quasi-variational inequality problem (SQVIP for short) where the underlying normal cone is defined over the solution set of a parametric stochastic cone system. We investigate the impact of variation of the probability measure and the parameter on the solution of the SQVIP. By reformulating the SQVIP as a natural equation and treating the orthogonal projection over the solution set of the parametric stochastic cone system as an optimization problem, we effectively convert stability of the SQVIP into that of a one stage stochastic program with stochastic cone constraints. Under some moderate conditions, we derive Hölder outer semicontinuity and continuity of the solution set against the variation of the probability measure and the parameter. The stability results are applied to a mathematical program with stochastic semidefinite constraints and a mathematical program with SQVIP constraints.

Key words. Stochastic quasi-variational inequality, quantitative stability analysis, mathematical program with stochastic semidefinite constraints, mathematical program with SQVIP constraints

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1 Introduction

Let X, Y and Z be Banach spaces equipped with norm $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ respectively. For $x_0 \in X$ and a closed convex set $\mathcal{Y} \subset Y$, we consider the following stochastic quasi-variational inequality problem (SQVIP): for a fixed parameter $x_0 \in X$, find $y \in \mathcal{Y}$ such that

$$0 \in \mathbb{E}_P[F(x_0, y, \xi(\omega))] + \mathcal{N}_{\Gamma_P(x_0,y)}(y), \tag{1.1}$$

where $F : X \times Y \times \Xi \to Y$ is a continuous function, $\xi : \Omega \to \Xi$ is a random vector defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with support set $\Xi \subseteq \mathbb{R}^k$, $\mathbb{E}_P[\cdot]$ denotes the expected value with respect to the probability measure P and $\Gamma_P : X \times Y \to 2^Y$ is a closed set-valued mapping which depends on the random variable $\xi(\omega)$ and the probability measure P, and $\mathcal{N}_{\Gamma_P(x_0,y)}(z)$ is the normal cone of $\Gamma_P(x_0, y)$ at z in the sense of convex analysis. Throughout the paper, we assume that $\mathbb{E}_P[F(x_0, y, \xi)]$ is well defined for any $(x, y) \in X \times Y$. To ease notation, we will use ξ to denote either the random vector $\xi(\omega)$ or an element of $\mathbb{R}^k$ depending on the context.

In practice, Γ_P may take a closed form in the sense that it has an explicit structure such as Aumann's integral of a random set-valued mapping [3]. It could also be implicitly defined as the set of

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solutions to a stochastic system of equalities and inequalities. Here we consider the latter case but in a slightly more general fashion where $\Gamma_P(x,y)$ is the set of solutions to the following stochastic cone system:

$$\Gamma_P(x,y) = \{z \in \mathcal{Y} : \mathbb{E}_P[G(x,y,z,\xi)] \in \mathcal{K}\}. \tag{1.2}$$

Here $\mathcal{Y}$ and $\mathcal{K}$ are closed convex set and closed convex cone in the spaces of Y and Z respectively. To simplify discussion, we assume that $\Gamma_P(x,y)$ is a convex set for each (x,y). A sufficient condition is that for each $(x,y,\xi) \in X \times Y \times \Xi$, $G(x,y,\cdot,\xi) : Y \to Z$ is a $(-\mathcal{K})$-convex function (see [18]) on $\mathcal{Y}$, that is, for $z_1, z_2 \in \mathcal{Y}$ and $t \in (0,1)$, we have

$$t\mathbb{E}_P[G(x,y,z_1,\xi)] + (1-t)\mathbb{E}_P[G(x,y,z_2,\xi)] - \mathcal{K} \subseteq \mathbb{E}_P[G(x,y,tz_1+(1-t)z_2,\xi)] - \mathcal{K}.$$

The SQVIP model (1.1) is an extension of deterministic QVIPs ([7, 17, 16]) to address underlying uncertainties in various equilibrium problems in economics, transportation, structural balance and electrical power flow networks [11].

In a particular case when $\Gamma_P(x,y)$ is independent of $\mathbb{E}_P[\cdot]$, x and y, (1.1) reduces to stochastic VIP (SVIP for short) which has been intensively studied over the past few years from numerical scheme such as stochastic approximation method and Monte Carlo method to the fundamental theory and applications; see for instance [11, 24, 25, 14, 19, 13] and the references therein. At this point, we should remind the readers that the SVIP models we are discussing here are expected value based and they should be distinguished from another class of SVIP models proposed by Chen and Fukushima [8] where a deterministic solution is sought through minimization of the expected residual of scenario based SVIP, see [9] for the recent development of the latter.

When the cone system (1.2) satisfies some appropriate constraint qualifications and G is continuously differentiable in z, it is possible to transform SQVIP (1.1) into a standard SVIP, see discussions by Pang and Fukushima [17] for deterministic case. To avoid overlap with the existing research on SVIP, here we concentrate our discussion on the case when it is difficult to transfer the SQVIP to a SVIP either because G is not continuously differentiable or it is difficult to represent $\mathcal{N}_{\Gamma_P(x_0,y)}(y)$ in terms of the normal cone of $\mathcal{K}$. Specifically we address the complications arising from the dependence of $\Gamma_P(x,y)$ on the underlying probability distribution, parameter x and variable y when we come down to numerical schemes and the underlying theory for the SQVIP.

One of the main challenges in dealing with stochastic optimization or stochastic equilibrium problems is to tackle the mathematical expectation because in practice it is often difficult to obtain a closed form of the expected value of a random function. Various approximation schemes such as sample average approximation or numerical integration have been proposed to address the challenge, see an excellent overview by Römisch [21]. Instead of looking into a particular numerical scheme for the SQVIP, here we consider variation of the probability measure P and its impact on the set of the solutions. This allows our results to cover a wide range of approximation schemes for the SQVIP and it is in alignment with standard stability analysis of stochastic programming.

Let Q denote a perturbation of the probability measure P. We consider the following perturbed stochastic quasi variational inequalities: given a parameter $x \in X$, find a decision vector $y \in \mathcal{Y}$ such that

$$0 \in \mathbb{E}_Q[F(x,y,\xi)] + \mathcal{N}_{\Gamma_Q(x,y)}(y), \tag{1.3}$$

where $\Gamma_Q : X \times Y \to 2^Y$ is defined by

$$\Gamma_Q(x,y) = \{z \in \mathcal{Y} : \mathbb{E}_Q[G(x,y,z,\xi)] \in \mathcal{K}\}. \tag{1.4}$$

Let $S_P(x_0)$ and $S_Q(x)$ denote the set of solutions to problems (1.1) and (1.3) respectively. We investigate the relationship between $S_Q(x)$ and $S_P(x_0)$ as Q approximates P under some appropriate metric and parameter x converges to x_0. To this end, we reformulate SQVIP (1.1) and its perturbation (1.3) respectively as

$$\mathbf{F}_P^{\text{nat}}(x_0, y) := \Pi_{\Gamma_P(x_0,y)}(y - \mathbb{E}_P[F(x_0, y, \xi)]) - y = 0 \tag{1.5}$$

and

$$\mathbf{F}_Q^{\text{nat}}(x, y) := \Pi_{\Gamma_Q(x,y)}(y - \mathbb{E}_Q[F(x, y, \xi)]) - y = 0, \tag{1.6}$$

where Π denotes the orthogonal projection, $\mathbf{F}_P^{\text{nat}}(x_0, y)$ and $\mathbf{F}_Q^{\text{nat}}(x, y)$ are *natural maps* associated with the respective SQVIPs.

One of the advantages of the reformulation is that the projection map $\Pi_{\Gamma_Q(x,y)}(y - \mathbb{E}_Q[F(x, y, \xi)])$ is single and finite valued and hence it is relatively easier to handle compared to the normal cone $\mathcal{N}_{\Gamma_Q(x,y)}(y)$ (which is set-valued and unbounded). A key step for the stability analysis is to quantify the gap between $\Pi_{\Gamma_Q(x,y)}(y-\mathbb{E}_Q[F(x, y, \xi)])$ and $\Pi_{\Gamma_P(x_0,y)}(y-\mathbb{E}_P[F(x_0, y, \xi)])$. Since the projection can be reformulated as the solution to a parametric stochastic minimization problem with cone constraints (1.2), we quantify the gap by investigating stability of the optimal solutions of the latter. As far as we are concerned, the main contributions of this paper can be summarized as follows.

- We carry out quantitative stability analysis for a deterministic parametric minimization problem with cone constraints. Under Slater constraint qualification, we derive Hölder continuity for the feasible solution set mapping (Lemma 2.2) and the optimal solution set mapping against variation of the parameter over Banach space (Theorem 2.1). In comparison with the existing stability results for parametric programming (see e.g [6, Chapter 4], [23], [22] and [2]), our results are established without any assumption on continuous differentiability of the underlying functions or reducibility of $\mathcal{K}$.

- We apply the stability result to quantify the discrepancy between $\Pi_{\Gamma_P(x_0,y)}(y - \mathbb{E}_P[F(x_0, y, \xi)])$ and $\Pi_{\Gamma_Q(x,y)}(y - \mathbb{E}_Q[F(x, y, \xi)])$ and consequently the gap between the solutions of (1.5) and (1.6) under the Slater constraint qualification of parametric cone system (1.2). Moreover, under some metric regularity conditions and Lipschitz homeomorphism of the natural mapping, we obtain a liner error bound for the solution set mapping with respect to variation of the probability measures under pseudometric, see Theorem 4.1. Prior to the stability analysis, we also investigated sufficient conditions for existence of solutions to perturbed SQVIP (1.3) through (1.5), see Theorem 3.1.

- We apply the stability results to a mathematical program with stochastic semidefinite constraints and a mathematical program with SQVIP constraints. The former provides a new stochastic model which has potential applications in portfolio optimization and the latter extends existing research on stochastic mathematical programs with equilibrium constraints, see details in Section 5.

The rest of the paper is organized as follows. Section 2 sets out some preliminary concepts and results in set-valued analysis and stochastic programming needed in this paper. It also presents some main quantitative stability results for a deterministic parametric minimization problem with cone constraints to be used for the stability analysis of SQVIP. Section 3 addresses existence of solutions to the perturbed SQVIP (1.3) followed by the main stability analysis in Section 4. Section 5 outlines applications of the established stability results in Section 4 to two new class of stochastic optimization models.

2 Preliminaries

2.1 Notation

Throughout this paper we use the following notation. By convention, $\mathbb{R}^n$ stands for n-dimensional Euclidean space. For a Banach space X, we write $\langle u, x\rangle$ for dual pairing of $x \in X$ and u from the dual space of X. In the case when X is finite dimensional, the dual pairing reduces to scalar product. For a set-valued mapping Φ, we write $\operatorname{gph}\Phi$ for its graph and by convention $\operatorname{int} S, \operatorname{cl} S$ $\operatorname{bd} S$ the interior, closure, and boundary of S. We use $B_X(x,\delta)$ to denote the closed ball with center $x \in X$ and radius δ and $\mathcal{B}_X$ specifically for the closed unit ball in X. We let $d(x,S) := \inf_{x'\in S}\|x-x'\|_X$ denote the distance from point $x \in X$ to a set $S \subset X$. For two sets $S_1, S_2 \subset X$,

$$\mathbb{D}(S_1,S_2) := \inf\{t \geq 0 : S_1 \subset S_2 + t\,\mathcal{B}_X\}$$

signifies the deviation of S_1 from S_2 and $\mathbb{H}(S_1,S_2) := \max(\mathbb{D}(S_1,S_2),\mathbb{D}(S_2,S_1))$ the Hausdorff distance between the two sets. Finally, for a sequence of subsets $\{S_k\}$ in a metric space, we follow the standard notation (see [4, Definition 1.1.1]) by using $\limsup_{k\to\infty} S_k$ to denote its upper limit, that is,

$$\limsup_{k\to\infty} S_k = \{x : \liminf_{k\to\infty} d(x,S_k) = 0\}$$

and $\liminf_{k\to\infty} S_k$ to denote its lower limit, that is,

$$\liminf_{k\to\infty} S_k = \{x : \lim_{k\to\infty} d(x,S_k) = 0\}.$$

It is well known that $\liminf_{k\to\infty} S_k \subseteq \limsup_{k\to\infty} S_k$, see [4].

2.2 Some basic concepts and results in set-valued and variational analysis

Let $\Psi : X \to 2^Y$ be a set-valued mapping. Recall that Ψ is said to be *closed* at $\bar{x}$ if $x_k \in X$, $x_k \to \bar{x}$, $y_k \in \Psi(x_k)$ and $y_k \to \bar{y}$ implies $\bar{y} \in \Psi(\bar{x})$. Ψ is said to be *upper semi-continuous* (usc for short) at $\bar{x} \in X$ if and only if for any neighborhood $\mathcal{U}$ of $\Psi(\bar{x})$, there exists a positive number $\delta > 0$ such that for any $x' \in B_X(x,\delta) \cap X$, $\Psi(x') \subset \mathcal{U}$. When Ψ is closed at $\bar{x}$ and $\Psi(\bar{x})$ is compact, Ψ is upper semicontinuous at $\bar{x}$ if and only if for every $\epsilon > 0$, there exists a constant $\delta > 0$ such that

$$\Psi(\bar{x} + \delta\mathcal{B}_X) \subset \Psi(\bar{x}) + \epsilon\mathcal{B}_Y.$$

Ψ is said to be *lower semi-continuous* (lsc for short) at $\bar{x} \in X$ if and only if for any $\bar{y} \in \Psi(\bar{x})$ and any sequence $\{x_k\} \subset X$ converging to $\bar{x}$, there exists a sequence $\{y_k\}$, where $y_k \in \Psi(x_k)$, converging to $\bar{y}$. The lower semicontinuity holds if and only if for any open set $\mathcal{U}$ with $\mathcal{U} \cap \Psi(\bar{x}) \neq \emptyset$, the set $\{x \in X : \mathcal{U} \cap \Psi(x) \neq \emptyset\}$ is a neighborhood of $\bar{x}$. Ψ is said to be *continuous* at $\bar{x}$ if it is both usc and lsc at the point; see [4] for details.

Note that in the literature of set-valued analysis, there are other notions describing semi-continuity of a set-valued mapping. For instances, Rockafellar and Wets [20, Definiton 5.4] introduce the concept of inner semicontinuity and the outer semicontinuity and assert that when a set-valued mapping is locally bounded, usc coincides with outer semicontinuity and lsc agrees with inner semicontinuity, see commentary of [20, Chapter 5].

One of the main tasks of set-valued and variational analysis is to detect the stability of a nonlinear system when perturbations of the data occur. For this purpose, we need the notion of metric regularity which is well established in the literature (see e.g. [6]).

Definition 2.1 Let $\Psi : X \to 2^Y$ be a set-valued mapping. Ψ is said to be *metrically regular* at a point $(x_0, y_0) \in \text{gph}(\Psi)$, with radius C, if for all (x, y) in a neighborhood of (x_0, y_0),

$$d(x, \Psi^{-1}(y)) \leq C d(y, \Psi(x)). \tag{2.1}$$

A powerful result arising from metric regularity is Robinson-Ursescu theorem which effectively relates the well known Slater type constraint qualification to metric regularity of a set-valued mapping.

Lemma 2.1 (Robinson-Ursescu stability theorem [6]) *Let $\Psi : X \to 2^Y$ be a closed convex set-valued mapping. Then Ψ is metrically regular at $(x_0, y_0) \in \text{gph}(\Psi)$ if and only if the regularity condition*

$$y_0 \in \text{int}(\text{range } \ \Psi). \tag{2.2}$$

More precisely, there exist positive constants ν and η such that (2.2) implies (2.1) with $C = 4\nu/\eta$ when

$$\|x - x_0\|_X \leq \frac{1}{2}\nu; \quad \|y - y_0\|_Y \leq \frac{1}{8}\eta.$$

When Ψ is single-valued, we need the following concept of locally Lipschitz Homeomorphism.

Definition 2.2 A continuous function $\Psi : \mathcal{O} \subseteq X \to X$ is said to be *locally Lipschitz invertible* near $x \in \mathcal{O}$ if there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{O}$ of x such that the restricted map $\Psi|_{\mathcal{N}} : \mathcal{N} \to \Psi(\mathcal{N})$ is bijective and its inverse is Lipschitz continuous. F is said to be a *locally Lipschitz homeomorphism* near x if Ψ is locally Lipschitz invertible near x and Ψ itself is locally Lipschitz continuous near x.

2.3 Quantity stability analysis for deterministic parametric programs

Let U be a metric space equipped with norm $\|\cdot\|_U$, let ϕ and ψ be continuous functions from $Y \times U$ to $\mathbb{R}$ and Z respectively. We consider the following parametric minimization problem:

$$(\mathcal{P}_u) \qquad \begin{array}{ll} \min\limits_{z \in Y} & \phi(z, u) \\ \text{s.t.} & \psi(z, u) \in \mathcal{K}, \\ & z \in \mathcal{Y}, \end{array} \tag{2.3}$$

where $u \in U$ is a fixed parameter, $\mathcal{Y} \subset Y$ is a compact set[4] and $\mathcal{K}$ is a closed convex cone in the space of Z. Unless specified otherwise, we do not assume differentiability of ψ or ϕ.

Let $\Psi(u)$, $\mathcal{Z}(u)$ and $\vartheta(u)$ denote respectively the feasible solution set, the set of the optimal solutions and the optimal value of $\mathcal{P}_u$. Assume $\Psi(u) \neq \emptyset$ for each u concerned. Since $\mathcal{Y}$ is a compact set, and $\phi(\cdot, u)$ and $\psi(\cdot, u)$ are continuous functions, then $\mathcal{Z}(u)$ is a nonempty compact set and $\vartheta(u)$ is finite valued.

We investigate stability of the parametric program by considering a perturbation of parameter u in a neighborhood of $\bar{u}$ and quantifying its impact on the optimal value and the optimal solutions. At this point, we note that quantitative stability analysis of parametric programs is well documented, see excellent monograph by Bonnans and Shapiro [6]. Here we concentrate on the case when the underlying functions are not necessarily differentiable.

To this end, we first study stability of the feasible set $\Psi(u)$. If we regard $\Psi(\cdot)$ as a set-valued mapping, then the research is in essence down to characterizing continuity of the set-valued mapping at $\bar{u}$ and quantifying the difference between $\Psi(u)$ and $\Psi(\bar{u})$ when u is sufficiently close to $\bar{u}$. Without loss of generality, we assume that $\bar{u}$ and its perturbation u are restricted to a compact set $\mathcal{U} \subset U$.

Lemma 2.2 *Assume: (a) $\psi(\cdot, u) : \mathcal{Y} \to Z$ is a continuous $(-\mathcal{K})$-convex function; (b) $\psi(z, u)$ is continuous in u at $\bar{u}$ uniformly w.r.t. z; (c) problem (2.3) satisfies the following Slater constraint qualification (SCQ for short) for $u = \bar{u}$:*

$$0 \in \text{int}\,\{\psi(\mathcal{Y}, \bar{u}) - \mathcal{K}\}, \tag{2.4}$$

where $\psi(\mathcal{Y}, \bar{u}) := \{\psi(z, \bar{u}) : z \in \mathcal{Y}\}$. Then

(i) $\Psi(\cdot)$ is continuous at $\bar{u}$;

(ii) if, in addition, (d) $\psi(z, u)$ is Hölder continuous in u at $\bar{u}$ uniformly w.r.t. z, i.e., there exist some positive constants $\sigma > 0$ and $\nu \in (0, 1)$ such that

$$\|\psi(z, u) - \psi(z, \bar{u})\|_Z \leq \sigma \|u - \bar{u}\|_U^\nu$$

for u close to $\bar{u}$, then

$$\mathbb{H}(\Psi(u), \Psi(\bar{u})) = O(\|u - \bar{u}\|_U^\nu) \tag{2.5}$$

for all u close to $\bar{u}$. Here and later on, we write $f(t) = O(t)$ for existence of some positive constant C such that $|f(t)| \leq C|t|$ when t is close to 0.

Proof. Part (i). Observe that (2.4) implies that $\Psi(\bar{u}) \neq \emptyset$, and under uniform continuity w.r.t. u further that $\Psi(u) \neq \emptyset$ when u is sufficiently close to $\bar{u}$.

It is well known (see [4, Definition 1.4.6]) that

$$\liminf_{u \to \bar{u}} \Psi(u) \subseteq \text{cl}\, \Psi(\bar{u}) \subseteq \limsup_{u \to \bar{u}} \Psi(u).$$

[4]Instead of using notation $\mathcal{Z}$, here we use $\mathcal{Y}$ in order to be consistent with the domain of variable z in the cone system (1.2).

Since $\Psi(\bar{u})$ is closed, to show continuity of $\Psi(\cdot)$ at $\bar{u}$, it suffices to show that the set-valued mapping is both upper and lower semicontinuous at $\bar{u}$, that is,

$$\limsup_{u\to\bar{u}} \Psi(u) \subseteq \Psi(\bar{u}) \subseteq \liminf_{u\to\bar{u}} \Psi(u). \tag{2.6}$$

The upper semicontinuity can be easily verified under the continuity of ψ and closeness of $\mathcal{K}$. Hence it suffices to show the lower semicontinuity. In the case when $\mathcal{Y}$ is a singleton, $\Psi(u)$ reduces to being single valued and consequently upper semicontinuity coincides with lower semicontinuity. In what follows, we consider the general case when $\mathcal{Y}$ is a set.

Observe that condition (2.4) and continuity of $\psi(\cdot,\bar{u})$ imply that $\text{int}\ \Psi(\bar{u}) \neq \emptyset$. Moreover, since $\psi(z,u)$ is uniformly continuous w.r.t. u under condition (b), it is easy to show that for any $z \in \text{int}\ \Psi(\bar{u})$, we can set u sufficiently close to $\bar{u}$ such that $z \in \Psi(u)$. This allows us to claim that

$$\text{int}\ \Psi(\bar{u}) \subseteq \liminf_{u\to\bar{u}} \Psi(u). \tag{2.7}$$

Moreover, since $\Psi(\bar{u})$ is convex and closed, we have

$$\Psi(\bar{u}) = \text{cl}\ (\text{int}\ \Psi(\bar{u})) \subseteq \text{cl}\ (\liminf_{u\to\bar{u}} \Psi(u)) \subseteq \text{cl}\ (\Psi(\bar{u})) = \Psi(\bar{u}).$$

Let $z^* \in \text{cl}\ (\liminf_{u\to\bar{u}} \Psi(u)) \backslash \liminf_{u\to\bar{u}} \Psi(u)$. Then $z^* \in \text{bd}\ \Psi(\bar{u})$. It is adequate to show that

$$z^* \in \liminf_{u\to\bar{u}} \Psi(u) \tag{2.8}$$

in that (2.8) means $\text{bd}\ \Psi(\bar{u}) \subseteq \liminf_{u\to\bar{u}} \Psi(u)$ and in a combination with (2.7), we effectively demonstrate equality holds in the second inclusion of (2.6).

Let $\{z_k\} \subset \text{int}\ \Psi(\bar{u})$ such that $z_k \to z^*$. Let $\{u_s\} \subset \mathcal{U}$ be any sequence converging to $\bar{u}$. For each fixed z_k, it follows from (2.7) that there exists a sequence $\{z_k^s\}$ converging to z_k with $z_k^s \in \Psi(u_s)$. This means that for any sequence $\{u_s\}$ converging to $\bar{u}$, we can find a sequence $\{z_{k_s}\}$ converging to z^* with $z_{k_s} \in \Psi(u_s)$, which is, by definition, the inclusion (2.8).

Part (ii). Since $\Psi(\bar{u})$ is compact and $\Psi(\cdot)$ is closed at $\bar{u}$, the continuity of $\Psi(\cdot)$ at $\bar{u}$ means that for any number $\delta > 0$, there exists $\epsilon > 0$ such that when $\|u-\bar{u}\|_U \leq \epsilon$,

$$\mathbb{H}(\Psi(u), \Psi(\bar{u})) \leq \delta$$

or equivalently

$$\Psi(u) \subseteq \Psi(\bar{u}) + \delta\mathcal{B}_Y \text{ and } \Psi(\bar{u}) \subseteq \Psi(u) + \delta\mathcal{B}_Y. \tag{2.9}$$

In what follows, we use Lemma 2.1 (Robinson-Ursescu's theorem) to derive an error bound for set-valued mapping $\Psi(\cdot)$ at $\bar{u}$ under condition (d) and Slater constraint qualification (2.4).

Let

$$\mathcal{F}_u(z) := \begin{cases} \psi(z,u) - \mathcal{K} & \text{for } z \in \mathcal{Y}, \\ \emptyset & \text{for } z \notin \mathcal{Y}. \end{cases}$$

Condition (a) ensures that $\mathcal{F}_u(z)$ is closed and convex set-valued over its domain $\mathcal{Y}$. Moreover,

$$\Psi(u) = \mathcal{F}_u^{-1}(0), \tag{2.10}$$

and $z \in \Psi(u)$ if and only if $(z, 0) \in \text{gph } \mathcal{F}_u$. Furthermore, it follows from (2.4)

$$0 \in \text{int}(\text{range } \mathcal{F}_{\bar{u}}). \tag{2.11}$$

Let $z \in \Psi(\bar{u})$ be fixed, that is, $(z, 0) \in \text{gph } \mathcal{F}_{\bar{u}}$. By Lemma 2.1, there exist positive numbers δ_z, η_z and c_z (depending on z) such that

$$d(z', \mathcal{F}_{\bar{u}}^{-1}(q)) \leq c_z d(q, \mathcal{F}_{\bar{u}}(z')). \tag{2.12}$$

for each $z' \in B_Y(z, \delta_z) \cap \mathcal{Y}$ and $q \in B_Z(0, \eta_z)$. In particular,

$$d(z', \Psi(\bar{u})) = d(z', \mathcal{F}_{\bar{u}}^{-1}(0)) \leq c_z d(\psi(z', \bar{u}), \mathcal{K}). \tag{2.13}$$

Under condition (d), we have

$$d(\psi(z', \bar{u}), \mathcal{K}) \leq \|\psi(z', \bar{u}) - \psi(z', u)\|_Z + d(\psi(z', u), \mathcal{K}) \leq \sigma \|u - \bar{u}\|_U^\nu. \tag{2.14}$$

for $z' \in \Psi(u)$.

On the other hand, since z is arbitrarily chosen from $\Psi(\bar{u})$, the set $\Psi(\bar{u})$ may be covered by the union of a collection of δ-balls, i.e.,

$$\Psi(\bar{u}) \subseteq \bigcup_{z \in \Psi(\bar{u})} \text{int } B_Y(z, \delta_z).$$

Let δ be a positive constant defined as in (2.9) and $\delta_z \leq 2\delta$. Since $\Psi(\bar{u})$ is a compact set, by the finite covering theorem, there exist a finite number of points $z_1, z_2, \cdots, z_k \in \Psi(\bar{u})$ and positive constants $\delta_{z_i}, c_{z_i}, i = 1, \cdots, k$ such that

$$\Psi(\bar{u}) + \delta \mathcal{B}_Y \subseteq \bigcup_{i=1}^{k} \text{int } B_Y(z_i, \delta_{z_i}) \tag{2.15}$$

and

$$d(z', \Psi(\bar{u})) \leq c_{z_i} d(\psi(z', \bar{u}), \mathcal{K}) \tag{2.16}$$

for any $z' \in \mathcal{B}_Y(z_i, \delta_{z_i}) \cap \mathcal{Y}$. Let : $C = \max\{c_{z_1}, \cdots, c_{z_k}\}$. Combining (2.15) and (2.16), we obtain

$$d(z', \Psi(\bar{u})) \leq C d(\psi(z', \bar{u}), \mathcal{K}) \tag{2.17}$$

for all $z' \in [\Psi(\bar{u}) + \delta \mathcal{B}_Y] \cap \mathcal{Y}$. Through (2.9) and (2.14), we arrive at

$$\mathbb{D}(\Psi(u), \Psi(\bar{u})) \leq C d(\psi(z, \bar{u}), \mathcal{K}) \leq C\sigma \|u - \bar{u}\|_U^\nu$$

for all $z' \in \Psi(u)$.

To complete the proof of (2.5), it suffices to show

$$\mathbb{D}(\Psi(\bar{u}), \Psi(u)) \leq C\sigma \|u - \bar{u}\|_U^\nu. \tag{2.18}$$

For any $z \in \mathcal{Y}$, let $q = \psi(z, \bar{u}) - \psi(z, u)$. Under condition (d), $\|q\|_Z \leq \sigma \|u - \bar{u}\|_U^\nu$. Moreover, it is easy to see that $\Psi(u) = \mathcal{F}_{\bar{u}}^{-1}(q)$. Let $z \in \Psi(\bar{u})$, δ_z be defined as in (2.12) and ϵ be defined at the beginning of the proof of this part with $\sigma\epsilon^\nu \leq \delta_z$. Under condition (d), we have $\|q\| \leq \delta_z$. It follows from (2.12) (by setting $z' = z$) that

$$d(z, \Psi(u)) = d(z, \mathcal{F}_{\bar{u}}^{-1}(q)) \leq c_z d(q, \mathcal{F}_{\bar{u}}(z)). \tag{2.19}$$

Since $z \in \Psi(\bar{u})$, then $0 \in \mathcal{F}_{\bar{u}}(z)$ and hence

$$d(q, \mathcal{F}_{\bar{u}}(z)) \leq \|q\|_Z \leq \sigma\|u - \bar{u}\|_U^\nu.$$

Combining the two inequalities above, we obtain

$$d(z, \Psi(u)) \leq c_z\sigma\|u - \bar{u}\|_U^\nu$$

for any $z \in \Psi(\bar{u})$ and $\|u - \bar{u}\|_U \leq \epsilon$. Utilizing the second inclusion in (2.9), we can apply the finite covering theorem to set $\Psi(\bar{u})$ (similar to previous discussion) and find a positive constant C such that (2.18) holds. The proof is complete. □

It is important to note that both convexity of $\psi(\cdot, u)$ (w.r.t. $-\mathcal{K}$) and Slater constraint qualification are essential for the continuity of $\Psi(\cdot)$. Indeed, we can easily find a counter example that continuity may fail in the absence of either condition.

Example 2.1 Consider the following simple parametric cone system

$$\psi(z, u) \in \mathcal{K}, \tag{2.20}$$

where

$$\psi(z, u) = \begin{cases} |z| - 1 & \text{for } z \leq 1, \\ u(z-1)^2 & \text{for } z \geq 1, \end{cases}$$

u is a parameter, and $\mathcal{K} = (-\infty, 0]$. It is easy to see that $\psi(\cdot, u)$ is not convex. Let $u \in \mathcal{U} = [0, 1]$ and z is restricted to take values from $\mathcal{Y} := [-2, 2]$. Then

$$\Psi(u) := \{z \in \mathcal{Y} : \psi(z, u) \leq 0\} = \begin{cases} [-1, 1] & \text{for } u > 0, \\ [-1, 2] & \text{for } u = 0. \end{cases} \tag{2.21}$$

It is easy to verify that $\Psi(\cdot)$ is usc at 0 but it is not lsc at the point. However, problem (2.20) satisfies SCQ.

To see the necessity of SCQ, let us change $\psi(z, u)$ to the following:

$$\psi(z, u) = \begin{cases} |z| - 1 & \text{for } z < -1, \\ 0 & \text{for } z \in [-1, 1), \\ u(z-1)^2 & \text{for } z \geq 1. \end{cases}$$

Consequently problem (2.20) fails to satisfy SCQ because its solution set does not have an interior. On the other hand, $\psi(\cdot, u)$ is convex. The solution set $\Psi(u)$, however, remains the same as defined in (2.21), which means $\Psi(\cdot)$ is not continuous at 0.

With Lemma 2.2, we are now ready to present quantitative stability results of parametric minimization problem (2.3). Let $\mathcal{Z}(u)$ and $\mathcal{Z}(\bar{u})$ denote the respective set of optimal solutions and $\vartheta(u)$ and $\vartheta(\bar{u})$ the corresponding optimal values.

Theorem 2.1 *Let conditions (a)-(d) in Lemma 2.2 hold. Assume the objective function ϕ satisfy the following: (a) $\phi(z,u)$ is Hölder continuous in u at $\bar{u}$ uniformly w.r.t. z, i.e., there exist some positive constants ϱ and $\gamma \in (0,1)$ such that*

$$|\phi(z,u) - \phi(z,\bar{u})| \leq \varrho \|u-\bar{u}\|_U^\gamma$$

for u close to $\bar{u}$; (b) $\phi(z,u)$ is globally Lipschitz continuous in z with modulus L; (c) $\phi(\cdot,\bar{u})$ satisfies second order growth condition at the optimal solution set $\mathcal{Z}(\bar{u})$, that is, there exists a positive constant $\alpha > 0$ such that

$$|\phi(z,\bar{u}) - \vartheta(\bar{u})| \geq \alpha d\left(z, \mathcal{Z}(\bar{u})\right)^2, \quad \forall\, z \in \Psi(\bar{u}).$$

Then

(i) there exists a positive constant c' such that

$$\mathbb{D}\left(\mathcal{Z}(u), \mathcal{Z}(\bar{u})\right) \leq c'\|u-\bar{u}\|_U^\beta \tag{2.22}$$

for u close to $\bar{u}$, where $\beta = \frac{1}{2}\min(\nu,\gamma)$, ν is defined as in Lemma 2.2.

(ii) If the second order growth condition holds uniformly for all u close to $\bar{u}$, then

$$\mathbb{H}\left(\mathcal{Z}(u), \mathcal{Z}(\bar{u})\right) \leq c'\|u-\bar{u}\|_U^\beta. \tag{2.23}$$

Proof. Part (i). First we show that

$$\mathbb{D}\left(\mathcal{Z}(u), \mathcal{Z}(\bar{u})\right) \leq \max_{z\in\mathcal{Z}(u)} \|z - \Pi_{\Psi(\bar{u})}(z)\|_Y + R(u), \tag{2.24}$$

where $\Pi_{\Psi(\bar{u})}(z)$ denotes orthogonal projection of z on $\Psi(\bar{u})$,

$$R(u) := \left(\frac{1}{\alpha}\left[2\varrho\|u-\bar{u}\|_U^\gamma + L\left(\max_{z\in\mathcal{Z}(u)}\|z-\Pi_{\Psi(\bar{u})}(z)\|_Y + \min_{z\in\mathcal{Z}(\bar{u})}\|z-\Pi_{\Psi(u)}(z)\|_Y\right)\right]\right)^{\frac{1}{2}}. \tag{2.25}$$

Let $z(u) \in \mathcal{Z}(u)$ and $z(\bar{u}) \in \mathcal{Z}(\bar{u})$. Under conditions (b) and (c)

$$\begin{aligned} \vartheta(u) - \phi(\Pi_{\Psi(\bar{u})}(z(u)),\bar{u}) &= \phi(z(u),u) - \phi(\Pi_{\Psi(\bar{u})}(z(u)),\bar{u}) \\ &\geq -L\|z(u) - \Pi_{\Psi(\bar{u})}(z(u))\|_Y - \varrho\|u-\bar{u}\|_U^\gamma \end{aligned} \tag{2.26}$$

for u close to $\bar{u}$. On the other hand, since $\Pi_{\Psi(u)}(z(\bar{u})) \in \Psi(u)$ and $z(u)$ is an optimal solution to $\mathcal{P}_u$, we have

$$\vartheta(u) \leq \phi\left(\Pi_{\Psi(u)}(z(\bar{u})), u\right).$$

Using this inequality and the growth condition (c), we have

$$\begin{aligned} \vartheta(u) - \phi(\Pi_{\Psi(\bar{u})}(z(u)),\bar{u}) &= \phi(z(u),u) - \phi(z(\bar{u}),u) + (\phi(z(\bar{u}),u) - \phi(z(\bar{u}),\bar{u})) \\ &\quad -(\phi(\Pi_{\Psi(\bar{u})}(z(u)),\bar{u}) - \phi(z(\bar{u}),\bar{u})) \\ &\leq \phi(\Pi_{\Psi(u)}(z(\bar{u})),u) - \phi(z(\bar{u}),u) + \varrho\|u-\bar{u}\|_U^\gamma \\ &\quad -\alpha d\left(\Pi_{\Psi(\bar{u})}(z(u)), \mathcal{Z}(\bar{u})\right)^2 \\ &\leq L\|\Pi_{\Psi(u)}(z(\bar{u})) - z(\bar{u})\|_Y + \varrho\|u-\bar{u}\|_U^\gamma \\ &\quad -\alpha d\left(\Pi_{\Psi(\bar{u})}(z(u)), \mathcal{Z}(\bar{u})\right)^2 \end{aligned} \tag{2.27}$$

for u close to $\bar{u}$. Combining (2.26) and (2.27), we obtain

$$\begin{aligned} & d\left(\Pi_{\Psi(\bar{u})}(z(u)), \mathcal{Z}(\bar{u})\right) \\ & \leq \left([L\|z(u)-\Pi_{\Psi(\bar{u})}(z(u))\|_Y + 2\varrho\|u-\bar{u}\|_U^{\gamma} + L\|z(\bar{u})-\Pi_{\Psi(u)}(z(\bar{u}))\|_Y]/\alpha\right)^{\frac{1}{2}}, \end{aligned} \tag{2.28}$$

which yields (2.24) through the triangle inequality below

$$d\left(z(u), \mathcal{Z}(\bar{u})\right) \leq \|z(u)-\Pi_{\Psi(\bar{u})}(z(u))\|_Y + d\left(\Pi_{\Psi(\bar{u})}(z(u)), \mathcal{Z}(\bar{u})\right) \tag{2.29}$$

because $z(u)$ and $z(\bar{u})$ are arbitrarily taken from the set of optimal solutions. Since $\mathcal{Z}(u) \subseteq \Psi(u)$, it follows by Lemma 2.2 that there exists a constant $\epsilon > 0$ such that

$$\max\left\{\max_{z\in\mathcal{Z}(u)} \|z-\Pi_{\Psi(\bar{u})}(z)\|_Y, \min_{z\in\mathcal{Z}(\bar{u})} \|z-\Pi_{\Psi(u)}(z)\|_Y\right\} = \mathcal{O}(\|u-\bar{u}\|_U^{\nu}) \tag{2.30}$$

when $\|u-\bar{u}\|_U \leq \epsilon$. Combining (2.24) and (2.30), we obtain (2.22).

Part (ii). We only need to show that

$$\mathbb{D}\left(\mathcal{Z}(\bar{u}), \mathcal{Z}(u)\right) \leq c'\|u-\bar{u}\|_U^{\beta}.$$

From the proof of Part (i), we can see that we can swap u with $\bar{u}$ except that the first inequality in formulae (2.27) requires second order growth condition of $\phi(\cdot, u)$ over the optimal solution set $\mathcal{Z}(u)$. □

Theorem 2.1 gives rise to quantitative stability analysis for the parametric minimization problem (2.3) under standard conditions (see Klatte [12]) including uniform Hölder continuity of the constraint function ψ w.r.t. parameter u, Slater constraint qualification and second order growth condition of the objective function. Specifically, it says that the set of the optimal solutions is locally upper Hölder continuous at $\bar{u}$ if $\psi(\cdot, u)$ is convex, and locally Hölder continuous at $\bar{u}$ if the second order growth condition is uniform w.r.t. parameter u. Compared to existing stability results (see e.g [6, Chapter 4], [23], [22] and [2]), the strength of this theorem is that it is established without any assumption on continuous differentiability of the underlying functions or reducibility of $\mathcal{K}$. The result has potential to be applied to semidefinite programming by setting Z to a matrix space and $\mathcal{K}$ to the cone of semidefinite matrices albeit it is not our primal goal in this paper.

Let us now consider a special case of problem (2.3)

$$\begin{aligned} \min_{z\in Y} \quad & \phi(z,u) := \tfrac{1}{2}\|z-g(u)\|_Y^2, \\ \text{s.t.} \quad & \psi(z,u) \in \mathcal{K}, \\ & z \in \mathcal{Y}, \end{aligned} \tag{2.31}$$

where $g(u)$ is a continuous function of z and u. It is easy to observe that the optimal solution to (2.31) is the orthogonal projection of $g(u)$ on the feasible set $\Psi(u)$, namely $\Pi_{\Psi(u)}(g(u))$.

Corollary 2.1 *Consider parametric minimization problem (2.31). Assume that conditions (a)-(d) in Lemma 2.2 hold. Assume also that $g : \mathcal{U} \to Y$ is Hölder continuous at $\bar{u}$, that is, there exist positive constants ϱ and $\gamma \in (0,1)$*

$$\|g(u)-g(\bar{u})\|_Y \leq \varrho\|u-\bar{u}\|_U^{\gamma}$$

for u close to $\bar{u}$. Then

$$\|\Pi_{\Psi(u)}(g(u)) - \Pi_{\Psi(\bar{u})}(g(\bar{u}))\|_Y = O(\|u-\bar{u}\|_U^{\beta}) \tag{2.32}$$

for any u close to $\bar{u}$, where $\beta = \frac{1}{2}\min(\nu,\gamma)$, ν is defined as in Lemma 2.2.

Proof. It suffices to verify conditions of Theorem 2.1. Observe first that the set of optimal solutions $Z(u)$ is a singleton because $\Psi(u)$ is a convex set and $Z(u) = \{\Pi_{\Psi(u)}(g(u))\}$. Since the conditions on the constraints are identical to those in Theorem 2.1, we only need to verify the conditions for the objective function ϕ. Here

$$\phi(z,u) = \frac{1}{2}(\|z\|_Y^2 - 2\langle z, g(u)\rangle + \|g(u)\|_Y^2).$$

It is easy to see that $\phi(\cdot,u)$ is Lipschitz continuous with modulus being bounded by $\sup_{z\in\mathcal{Y},u\in\mathcal{U}}(\|z\|+\|g(u)\|)$, where $\mathcal{U}$ is a compact neighborhood of $\bar{u}$. $\phi(z,\cdot)$ is Hölder continuous at $\bar{u}$, that is,

$$|\phi(z,u)-\phi(z,\bar{u})| \leq \varrho \sup_{z\in\mathcal{Y},u\in\mathcal{U}}(\|z\|_Y + \|g(u)\|_Y)\|u-\bar{u}\|_U^{\gamma}$$

when u close to $\bar{u}$. This verifies conditions (a) and (b) of Theorem 2.1. To see the growth condition, the orthogonal project of $g(u)$ over the convex set $\Psi(\bar{u})$ means that

$$\left\langle \Pi_{\Psi(\bar{u})}(g(\bar{u})) - g(\bar{u}), z - \Pi_{\Psi(\bar{u})}(g(\bar{u}))\right\rangle \geq 0,\ \forall z\in\Psi(\bar{u}).$$

With the inequality, we can obtain

$$\begin{aligned}\phi(z,\bar{u}) - \vartheta(\bar{u}) &= \frac{1}{2}\|z-g(\bar{u})\|_Y^2 - \frac{1}{2}\|\Pi_{\Psi(\bar{u})}(g(\bar{u})) - g(\bar{u})\|_Y^2 \\ &\geq \frac{1}{2}\|z-\Pi_{\Psi(\bar{u})}(g(\bar{u}))\|_Y^2\end{aligned}$$

Note that the second order growth holds when $\bar{u}$ is replaced by any u. The proof is complete. □

2.4 Pseudometric

Let $F(x,y,\xi)$ and $G(x,y,z,\xi)$ be defined as in (1.1) and (1.2). Let $\mathscr{B}$ denote the sigma algebra of all Borel subsets of Ξ (the support set of ξ) and $\mathscr{P}$ be the set of all probability measures of the measurable space $(\Xi,\mathscr{B})$. Let

$$\mathscr{F} := \{F(x,y,\xi(\cdot)) : x\in\mathcal{X}, y\in\mathcal{Y}\},$$
$$\mathscr{G} := \{G(x,y,z,\xi(\cdot)) : x\in\mathcal{X}, y\in\mathcal{Y}, z\in\mathcal{Y}\}.$$

For any probability measures $P,Q\in\mathscr{P}$, define the distances

$$\mathscr{D}_F(Q,P) := \sup_{f\in\mathscr{F}}\|\mathbb{E}_P[f(\xi)] - \mathbb{E}_Q[f(\xi)]\|_Y,$$

$$\mathscr{D}_G(Q,P) := \sup_{g\in\mathscr{G}}\|\mathbb{E}_P[g(\xi)] - \mathbb{E}_Q[g(\xi)]\|_Z,$$

and

$$\mathscr{D}_H(Q,P) := \max\{\mathscr{D}_F(Q,P), \mathscr{D}_G(Q,P)\}.$$

It is easy to see that $\mathscr{D}_F(Q,P) = 0$ if and only if $\mathbb{E}_P[f(\xi)] = \mathbb{E}_Q[f(\xi)]$ for all $f \in \mathscr{F}$ but it does not necessarily mean $P = Q$ unless the set $\mathscr{F}$ is sufficiently large. Similar comments apply to $\mathscr{D}_G(Q,P)$. This type of distance is widely used for stability analysis of stochastic programming and is known as *pseudometric* in that it satisfies all properties of a metric except that $\mathscr{D}_F(Q,P) = 0$ does not necessarily imply $P = Q$ unless the set of functions $\mathscr{F}$ is sufficiently large. For a comprehensive discussion of the concept and related issues, see [21, Sections 2.1-2.2].

3 Existence of a solution to the perturbed SQVIP

In this section, we discuss existence of solutions to the perturbed SQVIP (1.3) before we move on to stability analysis in the next section. To this end, we assume that $\mathcal{Y}$ is a compact set throughout this section and make the following assumptions.

Assumption 3.1 *Consider (1.2). There exists $x_0 \in X$ such that*

$$0 \in \text{int}\{\mathbb{E}_P[G(x_0, y, \mathcal{Y}, \xi)] - \mathcal{K}\}, \quad \forall y \in \mathcal{Y}, \tag{3.1}$$

where $\mathbb{E}_P[G(x_0, y, \mathcal{Y}, \xi)] := \{\mathbb{E}_P[G(x_0, y, z, \xi)] : z \in \mathcal{Y}\}$.

Condition (3.1) is a stochastic analogue of Slater constraint qualification which we use in Section 2.3. Note that the condition implicitly assumes $\Gamma_P(x_0, y) \neq \emptyset$ for any $y \in \mathcal{Y}$.

Assumption 3.2 *Let x_0 be defined as in Assumption 3.1 and $\hat{\mathcal{P}} \subset \mathscr{P}$ be a set of probability measures such that $P, Q \in \hat{\mathcal{P}}$. Let $G(x, y, z, \xi)$ be defined as in (1.2). The following hold.*

(a) For each fixed $\xi \in \Xi$, $G(x, y, z, \xi)$ is uniformly locally Lipschitz continuous in x at x_0 and globally Lipschitz continuous in z on $\mathcal{Y}$ with Lipschitz modulus being bounded by $\kappa(\xi)$, where $\sup_{P \in \hat{\mathcal{P}}} \mathbb{E}_P[\kappa(\xi)] < \infty$.

(b) For each fixed $(y, z) \in \mathcal{Y} \times \mathcal{Y}$, $\sup_{P \in \hat{\mathcal{P}}} \|\mathbb{E}_P[G(x_0, y, z, \xi)]\|_Z < \infty$.

(c) For each fixed $\xi \in \Xi$, $F(x, y, \xi)$ is uniformly locally Lipschitz continuous in x at x_0 with Lipschitz modulus being bounded by $\kappa(\xi)$ and continuous in y on $\mathcal{Y}$, where $\sup_{P \in \hat{\mathcal{P}}} \mathbb{E}_P[\kappa(\xi)] < \infty$.

(d) For each fixed $y \in \mathcal{Y}$, $\sup_{P \in \hat{\mathcal{P}}} \|\mathbb{E}_P[F(x_0, y, \xi)]\|_Y < \infty$.

Assumption 3.2 (b) ensures well definedness of $\mathbb{E}_Q[G(x_0, y, z, \xi)]$ for each $(y, z) \in \mathcal{Y} \times \mathcal{Y}$. Together with Assumption 3.2 (a), it also guarantees well definedness of $\mathbb{E}_Q[G(x, y, z, \xi)]$ for x near x_0. Moreover, through Lebesgue dominance convergence theorem, $\mathbb{E}_Q[G(x, y, \cdot, \xi)]$ is continuous on $\mathcal{Y}$ for each fixed Q close to P and $y \in \mathcal{Y}$. Similar comments apply to $\mathbb{E}_Q[G(x_0, y, z, \xi)]$ under Assumptions 3.2 (c) and (d). With Assumptions 3.1 and 3.2, we can easily obtain the following.

Proposition 3.1 *Let Assumptions 3.1 and 3.2 hold and $x_0 \in X$ be given as in Assumption 3.1. Then the following assertions hold.*

(i) Both $\mathbb{E}_Q[G(x,y,z,\xi)]$ *and* $\mathbb{E}_Q[F(x,y,z,\xi)]$ *are uniformly Lipschitz continuous in* Q *(independent of* x *and* y*) and uniformly continuous in* x *at* x_0 *(independent of* Q *and* y*), that is,*

$$\|\mathbb{E}_Q[G(x,y,z,\xi)] - \mathbb{E}_P[G(x_0,y,z,\xi)]\|_Z \le \mathscr{D}_G(P,Q) + \sup_{P\in\hat{\mathcal{P}}} \mathbb{E}_P[\kappa(\xi)]\|x-x_0\|_X \tag{3.2}$$

and

$$\|\mathbb{E}_Q[F(x,y,\xi)] - \mathbb{E}_P[F(x_0,y,\xi)]\|_Y \le \mathscr{D}_F(P,Q) + \sup_{P\in\hat{\mathcal{P}}} \mathbb{E}_P[\kappa(\xi)]\|x-x_0\|_X \tag{3.3}$$

for any $(y,z) \in \mathcal{Y}\times\mathcal{Y}$ *and* $x \in \mathcal{X}$ *close to* x_0*.*

(ii) The perturbed cone system $0 \in \mathbb{E}_Q[G(x_0,y,z,\xi)] - \mathcal{K}$ *satisfies Slater Constraint Qualification, that is,*

$$0 \in \operatorname{int}\{\mathbb{E}_Q[G(x_0,y,\mathcal{Y},\xi)] - \mathcal{K}\}, \quad \forall\, y \in \mathcal{Y}, \tag{3.4}$$

when Q *is close to* P*.*

(iii) $\Gamma_Q(x,y)$ *is nonempty for* Q *close to* P *and* x *close to* x_0*, and* $\Gamma_Q(x,\cdot)$ *is continuous over* $\mathcal{Y}$*.*

(iv) For any $y_0 \in \mathcal{Y}$*,*

$$\lim_{y\to y_0} \Pi_{\Gamma_P(x,y)}(z) = \Pi_{\Gamma_P(x,y_0)}(z).$$

Proof. Part (i). Let $\mathcal{X}$ be a compact neighborhood of x_0. Observe first that for each fixed $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $\mathbb{E}_P[G(x,y,z,\xi)]$ is continuous w.r.t. variation of probability measure P under the pseudometric $\mathscr{D}$ defined in Section 2. In fact, for any $P,Q \in \hat{\mathcal{P}}$

$$\|\mathbb{E}_P[G(x_0,y,z,\xi)] - \mathbb{E}_Q[G(x_0,y,z,\xi)]\|_Z \le \mathscr{D}_G(P,Q) \tag{3.5}$$

for any $(y,z) \in \mathcal{Y}\times\mathcal{Y}$. Under Assumption 3.2 and the compactness of $\mathcal{X}$, we have

$$\sup_{x\in\mathcal{X},P\in\hat{\mathcal{P}}} \|\mathbb{E}_P[G(x,y,z,\xi)]\|_Z < \infty$$

for any $(y,z) \in \mathcal{Y}\times\mathcal{Y}$. On the other hand, for any $x \in \mathcal{X}$ close to x_0,

$$\|\mathbb{E}_Q[G(x,y,z,\xi)] - \mathbb{E}_Q[G(x_0,y,z,\xi)]\|_Z \le \sup_{Q\in\hat{\mathcal{P}}} \mathbb{E}_Q[\kappa(\xi)]\|x-x_0\|_X \tag{3.6}$$

for any $(y,z) \in \mathcal{Y}\times\mathcal{Y}$. Combining (3.5) and (3.6), we obtain (3.2). We obtain (3.3) in the same way.

Part (ii). For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$-\mathbb{E}_P[G(x_0,y,z,\xi)] + \mathcal{K} \subset -\mathbb{E}_Q[G(x,y,z,\xi)] + \mathcal{K} + \epsilon\mathcal{B}_Z \tag{3.7}$$

for all $(y,z) \in \mathcal{Y}\times\mathcal{Y}$ when $\max\{\mathscr{D}_G(P,Q), \|x-x_0\|_X\} \le \delta$. This shows

$$-\mathbb{E}_P[G(x_0,y,\mathcal{Y},\xi)] + \mathcal{K} \subset -\mathbb{E}_Q[G(x,y,\mathcal{Y},\xi)] + \mathcal{K} + \epsilon\mathcal{B}_Z.$$

By Assumption 3.1, there exists $\tau > 0$ such that

$$\tau\mathcal{B}_Z \subseteq -\mathbb{E}_P[G(x_0,y,\mathcal{Y},\xi)] + \mathcal{K}.$$

Let $\epsilon := \tau/2$. By (3.7), there exists positive number $\delta' < \delta$ such that

$$\frac{\tau}{2}\mathcal{B}_Z \subseteq -\mathbb{E}_Q[G(x,y,\mathcal{Y},\xi)] + \mathcal{K} \tag{3.8}$$

for any $y \in \mathcal{Y}$ when $\max\{\mathscr{D}_G(P,Q), \|x - x_0\|_X\} \leq \delta'$.

Part (iii). The nonemptiness comes from Part (ii). We make use of Lemma 2.2 (i) to show the continuity by treating $(\mathbb{E}_Q[\cdot], x, y)$ as a parameter. The convexity of $\mathbb{E}_Q[G(x,y,z,\xi)]$ in z is obvious under our generic assumption that G is convex in z for each x, y, ξ in the definition of SQVIP in Section 1. Part (i) of this proposition addresses condition (b) of the lemma whereas the Slater condition is explicitly assumed here. Therefore all conditions for Lemma 2.2 (i) are verified.

Part (iv). The conclusion follows by Part (ii) and [20, Proposition 4.9]. □

Proposition 3.1 paves the way for us to state our main result in this section which asserts existence of solutions to the perturbed SQVIP (1.3).

Theorem 3.1 *Under Assumptions 3.1-3.2, $S_Q(x)$ is a nonempty and compact set for Q close to P under pseudometric $\mathscr{D}$ and x close to x_0.*

Proof. We prove the result by making use of [17, Theorem 1]. It therefore suffices to verify the conditions of the theorem. From Proposition 3.1, we know that for fixed Q close to P and x close to x_0, $\Gamma_Q(x,y)$ is a nonempty convex set for any $y \in \mathcal{Y}$ and $\Gamma_Q(x,\cdot)$ is continuous on $\mathcal{Y}$. The closedness of $\Gamma_Q(x,y)$ is due to continuity of $\mathbb{E}_Q[G(x,y,\cdot,\xi)]$ for fixed x and y and closeness of $\mathcal{K}$. □

In the literature of deterministic QVIP, conditions for existence have been well investigated. For instance, under compactness assumptions, Pang and Fukushima [17] established an existence result without assuming any special structure of $\Gamma_P(x,y)$. Existence results without compactness condition are obtained by Facchinei and Pang [10]. Our Theorem 3.1 may be regarded as an analogue of [17, Theorem 1] for the stochastic QVIP (1.3).

4 Stability of SQVIP

In this section, we return to the main topic of the paper, that is, quantitative stability analysis of SQVIP (1.1). Specifically we investigate the relationship between the set of solutions to the perturbed SQVIP (1.3) and its true counterpart when parameter x varies locally at certain fixed point x_0 and the probability measure Q is close to P under the pseudometric $\mathscr{D}_H(Q,P)$ defined in Section 2.3. As explained in the introduction, we plan to do so through the normal equations (1.6) and (1.5). A key step is to estimate the discrepancy between $\mathbf{F}_Q^{\text{nat}}(x,y)$ and $\mathbf{F}_P^{\text{nat}}(x_0,y)$ which can be quantified by

$$\|\Pi_{\Gamma_Q(x,y)}(y - \mathbb{E}_Q[F(x,y,\xi)]) - \Pi_{\Gamma_P(x,y)}(y - \mathbb{E}_P[F(x_0,y,\xi)])\|_Y. \tag{4.9}$$

Observe that for fixed $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $Q \in \hat{\mathcal{P}}$, $\Pi_{\Gamma_Q(x,y)}(y - \mathbb{E}_Q[F(x,y,\xi)])$ is the unique optimal solution to the following convex minimization problem

$$\begin{array}{ll} \min\limits_{z \in \mathcal{Y}} & \frac{1}{2}\|z - y + \mathbb{E}_Q[F(x,y,\xi)]\|_Y^2 \\ \text{s.t.} & \mathbb{E}_Q[G(x,y,z,\xi)] \in \mathcal{K}, \end{array} \tag{4.10}$$

where z is the only decision variable, whereas x and y are parameters. In other words, the quantity to be estimated in (4.9) is essentially the deviation of the optimal solution of problem (4.10) from its true counterpart if we regard Q and x as a perturbation from P and x_0.

Along this line, we may carry out stability analysis of SQVIP (1.1) through problem (4.10). On the other hand, the latter may be treated as a special case of parametric minimization problem (2.31) and this is indeed one of the underlying reasons for us to present the stability analysis for generic parametric minimization problems. To see this argument clearly, let $u = (\mathbb{E}_Q[\cdot,\cdot], x, y)$. The only thing we need to explain is that the mathematical expectation operation $\mathbb{E}_Q[\cdot]$ can indeed be treated as a parameter in the metric space of probability measure $\mathscr{P}$. Let

$$\mathscr{M} := \{Q \in \mathscr{P} : \mathbb{D}_H(Q,P) \leq \delta\}.$$

For any nonnegative measure μ defined over Ω, let

$$\langle \mu, h\rangle := \int_\Omega h(\xi(\omega))\mu(d\omega).$$

Then $\mathbb{E}_Q[F(x,y,\xi)]$ may be written as $\langle \mathbb{E}_Q, F(x,y,\xi)\rangle$. In doing so, we can effectively treat operation $\mathbb{E}_Q[\cdot]$ as a parameter in the space of probability measures $\mathscr{P}$ equipped with the pseudometric.

Using Corollary 2.1, we are able to present quantitative stability analysis for problem (4.10) which is one of the main technical results in this section.

Proposition 4.1 *Let Assumptions 3.1-3.2 hold. Then there exist positive constants ρ and $\beta \in (0,1)$ such that*

$$\|\mathbf{F}_Q^{\text{nat}}(x,y) - \mathbf{F}_P^{\text{nat}}(x_0,y)\|_Y \leq \rho(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta \tag{4.11}$$

for any $y \in \mathcal{Y}$, x close to x_0 and Q close to P (under pseudometric $\mathscr{D}_H$ defined in Section 2.4).

Proof. We use Corollary 2.1 to prove the result. Note that the corollary has two set of conditions: one for the objective function while the other for the constraints.

Let $g(u) := \mathbb{E}_Q[y - F(x,y,\xi)]$ and $\psi(z,u) := \mathbb{E}_Q[G(x,y,z,\xi)]$ where u is a parameter comprising three components: x, y and $\mathbb{E}_Q[\cdot]$. Let $g(\bar{u}) := \mathbb{E}_P[y - F(x_0,y,\xi)]$ and $\psi(z,\bar{u}) := \mathbb{E}_P[G(x_0,y,z,\xi)]$ with $\bar{u}$ representing x_0, y and $\mathbb{E}_P[\cdot]$.

Conditions concerning the constraints are essentially about $-\mathcal{K}$-convexity of $\psi(\cdot,u)$, uniform Hölder continuity ψ in u and Slater condition. In this context, convexity is straightforward, while the other two properties are either addressed in Proposition 3.1 (i) or explicitly assumed.

The condition concerning the objective function is even simpler because $g(u)$ is indeed Lipschitz continuous by Proposition 3.1 (i). □

With Proposition 4.1, we are ready to state the main stability result of this section.

Theorem 4.1 *Let $\mathcal{X} \subseteq X$ and $\mathcal{Y} \subseteq Y$ be compact set. Under Assumptions 3.1-3.2, the following assertions hold.*

(i) There exist $\delta > 0$, $\gamma > 0$ and $\beta \in (0,1)$ such that

$$\mathbb{D}(S_Q(x), S_P(x_0)) \le R^{-1}\big(2\gamma(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta\big) \tag{4.12}$$

for any $(x,Q) \in B_X(x_0,\delta) \times U(P,\delta)$, where for any small positive number $\varepsilon > 0$,

$$R(\varepsilon) = \inf_{y\in\mathcal{Y},\ d(y,S_P(x_0))\ge\varepsilon} \left\|\mathbf{F}_P^{\rm nat}(x_0,y)\right\|_Y,$$

$U(P,\delta) = \{Q \in \mathscr{P} : \mathscr{D}_H(Q,P) < \delta\}$, $R^{-1}(\varepsilon) = \min\{t \in \mathbb{R}_+ : R(t) = \varepsilon\}$ and $R^{-1}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

(ii) If $\mathbf{F}_P^{\rm nat}(x_0,y)$ is metrically regular at each $y^* \in S_P(x_0)$ for 0, then there exist positive constants c and δ such that

$$\mathbb{D}(S_Q(x), S_P(x_0)) \le c\big(\mathscr{D}_H(Q,P) + \|x - x_0\|_X\big)^\beta \tag{4.13}$$

for any $(x,Q) \in B_X(x_0,\delta) \times U(P,\delta)$.

(iii) If $\mathbf{F}_P^{\rm nat}(x_0,y)$ is a locally Lipschitz homeomorphism at $y^* \in S_P(x_0)$, then there exist positive constants c, δ and a neighborhood $\mathcal{V}$ of y^* such that

$$\|y - y^*\|_Y \le c\big(\mathscr{D}_H(Q,P) + \|x - x_0\|_X\big)^\beta \tag{4.14}$$

for any $y \in S_Q(x) \cap \mathcal{V}$ with $(x,Q) \in B_X(x_0,\delta) \times U(P,\delta)$.

Proof. Part (i). By the definition of $R(\varepsilon)$, $R(0) = 0$ and $R(\varepsilon)$ is nondecreasing on $[0,+\infty)$. We show that $R(\varepsilon) > 0$ for $\varepsilon > 0$. Assume for the sake of a contradiction that $R(\varepsilon) = 0$ for some $\varepsilon > 0$. Then there exists a sequence $\{y_k\} \subseteq \mathcal{Y}$ such that $d(y_k, S_P(x_0)) \ge \varepsilon$ and

$$\lim_{k\to\infty} \left\|\mathbf{F}_P^{\rm nat}(x_0,y_k)\right\|_Y = 0. \tag{4.15}$$

Since $\mathcal{Y}$ is a compact set, we may assume without loss of generality that $y_k \to \bar{y}$ as $k \to \infty$. Then $\bar{y} \in \mathcal{Y}$ and by Proposition 3.1 (iv)

$$\lim_{k\to\infty} \left\|\mathbf{F}_P^{\rm nat}(x_0,y_k) - \mathbf{F}_P^{\rm nat}(x_0,\bar{y})\right\|_Y = 0,$$

which, together with (4.15), yield $\mathbf{F}_P^{\rm nat}(x_0,\bar{y}) = 0$ and hence $\bar{y} \in S_P(x_0)$. This leads to a contradiction as desired because $d(\bar{y}, S_P(x_0)) \ge \varepsilon > 0$. The discussion manifests that $R^{-1}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

In what follows, we show (4.12). Let ϵ be a fixed small positive number and $y \in \mathcal{Y}$ with $d(y, S_P(x_0)) > \epsilon$. Then $\left\|\mathbf{F}_P^{\rm nat}(x_0,y)\right\|_Y \ge R(\epsilon)$. Moreover, by Proposition 4.1, there exist constants $\bar{\delta} > 0$, $\gamma > 0$ and $\beta \in (0,1)$ such that

$$\begin{aligned}\left\|\mathbf{F}_Q^{\rm nat}(x,y)\right\|_Y &\ge \left\|\mathbf{F}_P^{\rm nat}(x_0,y)\right\|_Y - \left\|\mathbf{F}_Q^{\rm nat}(x,y) - \mathbf{F}_P^{\rm nat}(x_0,y)\right\|_Y \\ &\ge R(\epsilon) - \rho(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta \end{aligned}\tag{4.16}$$

for any $Q \in \mathscr{P}$ with $\mathscr{D}_H(Q,P) \le \bar{\delta}$ and $x \in \mathcal{X}$ with $\|x - x_0\|_X < \bar{\delta}$. Let $\bar{\delta}$ be sufficiently small such that

$$\rho(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta \le R(\epsilon)/2.$$

Then we arrive at $\left\|\mathbf{F}_Q^{\text{nat}}(x,y)\right\|_Y \geq R(\epsilon)/2 > 0$ which implies $y \notin S_Q(x)$. This shows

$$\mathbb{D}_H(S_Q(x), S_P(x_0)) \leq \epsilon \tag{4.17}$$

for any $Q \in \mathscr{P}$ with $\mathscr{D}_H(Q,P) < \bar{\delta}$ and $x \in \mathcal{X}$ satisfying $\|x - x_0\|_X < \bar{\delta}$. The rest follows from the definition of R^{-1}.

Part (ii). Since $\mathbf{F}_P^{\text{nat}}(x_0, y)$ is metrically regular at y^* for 0 with modulus $\alpha > 0$ and from Part (i) $S_Q(x)$ is upper semicontinuous w.r.t. (Q,x) at (P, x_0), there exist positive constants δ_1, δ_2 and α such that

$$d(y, S_P(x_0)) \leq \alpha d(0, \mathbf{F}_P^{\text{nat}}(x_0, y)) = \alpha \|\mathbf{F}_P^{\text{nat}}(x_0, y)\|_Y \tag{4.18}$$

for all $y \in S_Q(x) \cap B_Y(y^*, \delta_1)$ with $x \in B_X(x_0, \delta_2)$ and Q close to P. Let $y \in S_Q(x)$. Then $\mathbf{F}_Q^{\text{nat}}(x,y) = 0$ and

$$\left\|\mathbf{F}_P^{\text{nat}}(x_0, y)\right\|_Y = \left\|\mathbf{F}_Q^{\text{nat}}(x,y) - \mathbf{F}_P^{\text{nat}}(x_0,y)\right\|_Y \tag{4.19}$$

Combining (4.18) and (4.19), and making use of Proposition 4.1, we obtain

$$d(y, S_P(x_0)) \leq \alpha \left\|\mathbf{F}_Q^{\text{nat}}(x,y) - \mathbf{F}_P^{\text{nat}}(x_0,y)\right\|_Y \leq \alpha\rho(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta$$

for any $y \in S_Q(x) \cap B_Y(y^*, \delta_1)$ with Q close to P and x close to x_0. Analogous to the second part of the proof of Lemma 2.2, we can show by exploiting the compactness of $S_P(x_0)$ that there exists $\varepsilon > 0$ such that

$$d(y, S_P(x_0)) \leq \alpha\rho(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta \tag{4.20}$$

for every $y \in S_Q(x) \cap (S_P(x_0) + \varepsilon\mathcal{B}_Y)$. The conclusion is apparent in that we have shown in Part (i) $S_Q(x) \subseteq S_P(x_0) + \varepsilon\mathcal{B}_Y$.

Part (iii). The locally Lipschitz homeomorphism of $\mathbf{F}_P^{\text{nat}}(x_0, y)$ at y^* implies that there exist positive constants δ, ε and a unique locally Lipschitz continuous function $\tilde{y}(\cdot) : B_Y(0,\delta) \to B_Y(y^*, \varepsilon)$ such that

$$\tilde{y}(0) = y^*, \ \{y^*\} = S_P(x_0) \bigcap B_Y(y^*, \varepsilon) \quad \text{and} \quad \mathbf{F}_P^{\text{nat}}(x_0, \cdot)^{-1}(q) \bigcap B_Y(y^*, \varepsilon) = \tilde{y}(q)$$

for every $q \in B_Y(0,\delta)$. For each Q and x, let

$$\Delta_Q(x,y) := \mathbf{F}_Q^{\text{nat}}(x,y) - \mathbf{F}_P^{\text{nat}}(x_0,y)$$

It is easy to see that $\Delta_Q(x,y)$ is a continuous vector-valued function of y parameterized by Q and x. By Proposition 4.1, there exists positive number δ_1 such that

$$\sup_{x \in B_X(x_0,\delta_1), Q \in U(P,\delta_1), y \in B_Y(y^*,\varepsilon)\cap\mathcal{Y}} \|\Delta_Q(x,y)\|_Y < \delta,$$

which implies that $\Delta_Q(x,y) \in B_Y(0,\delta)$. Thus for each $\hat{y} \in S_Q(x) \cap B_Y(y^*, \varepsilon)$ with $x \in B_X(x_0, \delta_1)$ and $Q \in U(P, \hat{\delta}_1)$, $\hat{y}$ satisfies $\mathbf{F}_P^{\text{nat}}(x, \hat{y}) = \Delta_Q(x, \hat{y})$, or equivalently, $\mathbf{F}_P^{\text{nat}}(x, \cdot)^{-1}(\Delta_Q(x,\hat{y})) = \hat{y}$. By the definition of $\tilde{y}(\cdot)$ and Proposition 4.1, we have $\tilde{y}(\Delta_Q(x,\hat{y})) = \hat{y}$ and there exists $K > 0$ and $\mu > 0$ such that

$$\begin{aligned}
\|\hat{y} - y^*\|_Y &= \|\tilde{y}(\Delta_Q(x,\hat{y})) - \tilde{y}(0)\|_Y \leq K\|\Delta_Q(x,y)\|_Y \\
&\leq K \sup_{x \in B_X(x_0,\delta_1), Q \in U(P,\delta_1), y \in B_Y(y^*,\varepsilon)\cap\mathcal{Y}} \|\Delta_Q(x,y)\|_Y \\
&\leq K\mu(\mathscr{D}_H(Q,P) + \|x - x_0\|_X)^\beta.
\end{aligned}$$

The proof is complete. □

5 Applications

In this section, we discuss application of the stability results for the SQVIP. Since the SQVIP can be used to represent first order optimality conditions of many one stage stochastic optimization or equilibrium problems with stochastic cone constraints, we restrict our discussion to problems where there is a clear advantage to use our results over other existing results on SVIP.

5.1 One stage stochastic programs with stochastic semidefinite constraints

Let us start with a mathematical program with matrix cone constraints:

$$\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \mathbb{E}_P[v(x,y,\xi)] \\ \text{s.t.} & (\mathbb{E}_P[g(x,y,\xi)]-\mu_0)\Sigma_0^{-1}(\mathbb{E}_P[g(x,y,\xi)]-\mu_0)\leq\gamma, \\ & \mathbb{E}_P\left[(g(x,y,\xi)-\mu_0)(g(x,y,\xi)-\mu_0)^T\right]\preceq\Sigma_0, \\ & y\in\mathcal{Y}, \end{array} \tag{5.21}$$

where v,g are continuous functions mapping from $\mathbb{R}^n\times\mathbb{R}^m\times\Xi$ to $\mathbb{R}$ and $\mathbb{R}^p$ respectively, ξ is a random variable and $\mathcal{Y}$ is a closed convex subset of $\mathbb{R}^m$, μ_0 is a fixed vector and Σ_0 is a symmetric positive definite matrix. Here and later, for a matrix M we write $M\preceq 0$ and $M\succeq 0$ for M being negative semidefinite and positive semidefinite and $M\prec 0$ and $M\succ 0$ for M being negative definite and positive definite.

In portfolio optimization, we may interpret $g(x,y,\xi)$ as a vector of returns from several portfolios, μ_0 as the targeted mean value and Σ_0 a specified level of covariance. The model differs from existing models in the literature of portfolio optimization where g is often a real valued function representing aggregate return from a single portfolio of investments. By considering g as a vector valued function, the model allows one to divide the portfolio into a number of *groups* according to the nature of the assets and the constraint on covariance restricts correlation between the groups of assets. Through a simple mathematical maneuver, we can reformulate (5.22) as

$$\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \mathbb{E}_P[v(x,y,\xi)] \\ \text{s.t.} & \mathbb{E}_P\left[\begin{array}{ll} \Sigma_0 & g(x,y,\xi)-\mu_0 \\ (g(x,y,\xi)-\mu_0)^T & \gamma \end{array}\right]\succeq 0, \\ & \mathbb{E}_P[(g(x,y,\xi)-\mu_0)(g(x,y,\xi)-\mu_0)^T]\preceq\Sigma_0, \\ & y\in\mathcal{Y}. \end{array} \tag{5.22}$$

Let $\mathcal{S}^p$ and $\mathcal{S}^p_+$ denote the space of $p\times p$ symmetric matrices and the cone of positive semidefinite matrices in $\mathcal{S}^p$ respectively. Problem (5.22) can be recast as a one stage stochastic semidefinite programming (SSDP):

$$\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \mathbb{E}_P[v(x,y,\xi)] \\ \text{s.t.} & \mathbb{E}_P[G(x,y,\xi)]\in\mathcal{K}, \\ & y\in\mathcal{Y}, \end{array} \tag{5.23}$$

where

$$G(x,y,\xi)=\left[\begin{array}{cc} \Sigma_0 & g(x,y,\xi)-\mu_0 \\ (g(x,y,\xi)-\mu_0)^T & \gamma \end{array}\right]\otimes\left[\Sigma_0-(g(x,y,\xi)-\mu_0)(g(x,y,\xi)-\mu_0)^T\right],$$

and $\mathcal{K} = \mathcal{S}_+^{2p} \otimes \mathcal{S}_+^p$, where "$\otimes$" denotes the Cartesian product. In the case when $g(x,\cdot,\xi)$ is linear function in y for any fixed $x \in \mathcal{X}$ and $\xi \in \Xi$, we can show that

$$\Gamma_P(x) := \{z \in \mathcal{Y} : \mathbb{E}_P[G(x,z,\xi)] \in \mathcal{K}\} \tag{5.24}$$

is a convex set for any fixed $x \in \mathcal{X}$. Consequently, the solution set of SSDP can be represented as the solution set of SQVIP (1.1) with $\Gamma_P(x,y) := \Gamma_P(x)$ being defined in (5.24) and $F(x,y,\xi) := \nabla_y v(x,y,\xi)$.

Evidently, if the integral involved in the mathematical expectation can be evaluated either analytically or numerically, then the SSDP can be solved by existing numerical methods. However, in many situations, exact evaluation of the expected value is either impossible or prohibitively expensive. Consequently, some approximation methods such as sample average approximation (SAA) are needed to deal with the mathematical expectation and it is necessary to assess the impact on the optimal value and optimization solution. Here we take a general step by looking into stability of the problem in terms of variation of probability measures which should cover abroad class of approximation schemes with practical interest. Specifically, we consider

$$\begin{array}{cl} \min\limits_{y\in\mathbb{R}^m} & \mathbb{E}_Q[v(x,y,\xi)] \\ \text{s.t.} & \mathbb{E}_Q[G(x,y,\xi)] \in \mathcal{K}, \\ & y \in \mathcal{Y}, \end{array} \tag{5.25}$$

where probability measure Q is an approximation of P under some appropriate metric.

Assumption 5.1 *Let $\hat{\mathcal{P}} \subset \mathscr{P}$ be a set of probability measures such that $P, Q \in \hat{\mathcal{P}}$. The following hold.*

(a) For each fixed $\xi \in \Xi$ and $x \in \mathcal{X}$, $\nabla_y v(x,y,\xi)$ is uniformly Lipschitz continuous in y with Lipschitz modulus being bounded by $\kappa(\xi)$, where $\sup_{P\in\hat{\mathcal{P}}} \mathbb{E}_P[\kappa(\xi)] < \infty$.

(b) For each fixed $\xi \in \Xi$ and $x \in \mathcal{X}, \nabla_y v(x,y,\xi)$ is θ-monotone on $\mathbb{R}^m$ for some $\theta > 1$, i.e., there exists $c(\xi) > 0$ such that

$$(\nabla_y v(x,y,\xi) - \nabla_y v(x,y',\xi))^T(y-y') \geq c(\xi)\|y-y'\|^\theta$$

holds for any $y, y' \in \mathcal{Y}$ and each fixed $x \in \mathcal{X}$ and $\xi \in \Xi$ with $\sup_{P\in\hat{\mathcal{P}}} \mathbb{E}_P[c(\xi)] < \infty$. Unless specified otherwise, $\|\cdot\|$ denotes the Euclidean norm in a finite dimensional space throughout this section.

Theorem 5.1 *Let $\widehat{S}_P(x)$ and $\widehat{S}_Q(x)$ denote the set of solutions to problems (5.23) and (5.25) respectively. Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be compact sets. Assume: (a) $g(x,\cdot,\xi)$ be a linear function in y for any fixed $x \in \mathcal{X}$ and $\xi \in \Xi$, (b) Assumption 3.2 holds for $F(x,y,\xi) \equiv \nabla_y v(x,y,\xi)$ and $G(x,y,\xi)$ defined in (5.23), (c) for a fixed $x_0 \in \mathcal{X}$, there exists $\hat{y} \in \mathcal{Y}$ such that*

$$\begin{bmatrix} \Sigma_0 & g(x_0,\hat{y},\xi)-\mu_0 \\ (g(x_0,\hat{y},\xi)-\mu_0)^T & \gamma \end{bmatrix} \succ 0 \text{ and } \mathbb{E}_P[(g(x_0,\hat{y},\xi)-\mu_0)(g(x_0,\hat{y},\xi)-\mu_0)^T] \prec \Sigma_0. \tag{5.26}$$

Then the following assertions hold.

(i) There exist $\delta > 0$, $\rho > 0$ and $\beta \in (0,1)$ such that

$$\mathbb{D}(\widehat{S}_Q(x), \widehat{S}_P(x_0)) \leq R^{-1}\big(2\rho(\mathscr{D}_H(Q,P) + \|x - x_0\|)^\beta\big)$$

for any $(x, Q) \in B(x_0, \delta) \times U(P, \delta)$, where for any small positive number $\varepsilon > 0$,

$$R(\varepsilon) = \inf_{y \in \mathcal{Y},\ d(y, \widehat{S}_P(x_0)) \geq \varepsilon} \left\| \Pi_{\Gamma_P(x_0)}(y - \mathbb{E}_P[\nabla_y v(x_0, y, \xi)]) - y \right\|,$$

$R^{-1}(\varepsilon) = \min\{t \in \mathbb{R}_+ : R(t) = \varepsilon\}$ and $R^{-1}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

(ii) If, in addition, $\nabla_y v(x, y, \xi)$ satisfies Assumption 5.1, then $\widehat{S}_Q(x)$ and $\widehat{S}_P(x_0)$ reduce to singleton and there exist positive constants α, ρ and β such that

$$\mathbb{D}(\widehat{S}_Q(x), \widehat{S}_P(x_0)) \leq \rho(\mathscr{D}_H(Q,P) + \|x - x_0\|)^\alpha$$

for any $(x, Q) \in B(x_0, \delta) \times U(P, \delta)$,

where $\mathscr{D}_H(Q,P)$ is defined as in Section 2.3 with $F = \nabla_y v$ and G being defined as in this theorem.

Proof. The thrust of the proof is to apply Theorem 4.1. Let X, Y and Z in Theorem 4.1 be defined as $\mathbb{R}^n, \mathbb{R}^m$ and $\mathcal{S}^{2p} \otimes \mathcal{S}^p$ equipped with Euclidean norm for X and Y and Frobenius norm for Z.

Recall that $\mathcal{K} = \mathcal{S}^{2p}_+ \otimes \mathcal{S}^p_+$. It is easy to see that int $\mathcal{K}$ is nonempty and $\mathbb{E}_P[G(x_0, \cdot, \xi)]$ is $-\mathcal{K}$-convex. Moreover, condition (c) is equivalent to $\mathbb{E}_P[G(x_0, \hat{y}, \xi)] \in \textbf{int}\, \mathcal{S}^{2p}_+ \otimes \mathcal{S}^p_+$, and by [6, Proposition 2.106], it coincides with the Slater condition (3.1). The conclusion follows from Theorem 4.1 (i).

Part (ii). Assumption 5.1 (b) implies that $\mathbb{E}_P[\nabla_y v(x_0, y, \xi)]$ is θ-monotone over $\mathcal{Y}$. By [10, Theorem 2.3.3], $\widehat{S}_P(x_0)$ is a singleton, denoted by $\{y^*\}$. Moreover, there exists constant $\kappa > 0$ such that

$$\|y - y^*\| \leq \kappa \left\| \Pi_{\Gamma_P(x_0)}(y - \mathbb{E}_P[\nabla_y v(x_0, y, \xi)]) - y \right\|^{\frac{1}{\theta - 1}}, \forall y \in \mathcal{Y}.$$

Thus for any $y \in \widehat{S}_Q(x)$,

$$\|y - y^*\| \leq \kappa \left\| \Pi_{\Gamma_Q(x)}(y - \mathbb{E}_Q[\nabla_y v(x, y, \xi)]) - \Pi_{\Gamma_P(x_0)}(y - \mathbb{E}_P[\nabla_y v(x_0, y, \xi)]) \right\|^{\frac{1}{\theta - 1}}. \tag{5.27}$$

Furthermore, under Assumptions 3.2 and 3.2, it follows by Proposition 4.1 that there exist positive constants L and $\beta \in (0,1)$ such that

$$\left\| \Pi_{\Gamma_Q(x)}(y - \mathbb{E}_Q[\nabla_y v(x, y, \xi)]) - \Pi_{\Gamma_P(x_0)}(y - \mathbb{E}_P[\nabla_y v(x_0, y, \xi)]) \right\| \leq L(\mathscr{D}_H(Q,P) + \|x - x_0\|)^\beta.$$

The conclusion follows. □

Remark 5.1 *A sufficient condition for $\mathbb{E}_P[(g(x_0, \hat{y}, \xi) - \mu_0)(g(x_0, \hat{y}, \xi) - \mu_0)^T] \prec \Sigma_0$ is the smallest eigenvalue of Σ_0 being greater than $\|\mathbb{E}_P[g(x_0, \hat{y}, \xi)] - \mu_0\|^2$.*

5.2 Mathematical programs with SQVIP constraints

Let us now consider a mathematical program with SQVIP constraint:

$$\begin{array}{ll} \min\limits_{x,y} & f(x,y) \\ \text{s.t.} & 0 \in \mathbb{E}_P[F(x,y,\xi)] + \mathcal{N}_{\Gamma_P(x,y)}(y), \\ & (x,y) \in \mathcal{X} \times \mathcal{Y}, \end{array} \tag{5.28}$$

where f, F are continuous functions mapping from $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ to $\mathbb{R}$ and $\mathbb{R}^m$ respectively, ξ is a random variable and $\Gamma_P(x,y)$ is defined as in (1.2). In the case when $\Gamma_P(x,y)$ is a constant convex set, (5.28) is known as a one stage stochastic mathematical program with equilibrium constraint which has been well investigated over the past decade, see for instance [5].

Our interest here is in the case when $\Gamma_P(x,y)$ does depend on either P, x, y or at least one of them so that our discussion does not overlap with the existing research in the literature. Indeed, the novelty and challenge of model (5.28) lie in the SQVIP which may represent an equilibrium arising from a generalized stochastic game or the first order optimality conditions of a stochastic programming problem. Stochastic leader multiple followers problems, stochastic bilevel programming problems can all be put in this framework.

To maximize the coverage of our results, we consider $\Gamma_P(x,y)$ to be defined as in (1.2), that is,

$$\Gamma_P(x,y) := \{z \in \mathcal{Y} : \mathbb{E}_P[G(x,y,z,\xi)] \in \mathcal{K}\}$$

but leave $F(x,y,\xi)$ unspecified as to whether it arises from optimality of a single stochastic decision making problem or a stochastic game. In what follows, we present a quantitative stability result for (5.28) by exploiting the established stability results for SQVIP in the preceding sections. The result has a potential to provide a theoretical foundation for various numerical schemes for a range of stochastic optimization and equilibrium problems where either G is not continuously differentiable or it is complicated to characterize $\mathcal{N}_{\Gamma_P(x,y)}(y)$ in terms of the derivatives of G and normal cone of $\mathcal{K}$.

Theorem 5.2 *Let ϑ_P and ϑ_Q denote the optimal value of MPSQVIP (5.28) and its perturbation respectively. Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be compact sets. Assume: (a) f is a Lipschitz function in (x,y) with a constant $\kappa > 0$, (b) Assumption 3.1 holds at $x_0 \in \mathcal{X}$ and Assumptions 3.2 holds for $F(x,y,\xi)$ and $G(x,y,z,\xi)$ respectively, (c) for $x_0 \in \mathcal{X}$, $\mathbf{F}_P^{\text{nat}}(x_0,y)$ is a locally Lipschitz homeomorphism at every $y^* \in S_P(x_0)$. Then there exist positive constants L and β such that*

$$\|\vartheta_P - \vartheta_Q\| \leq L\mathscr{D}_H(Q,P)^\beta$$

for any Q close to P.

Proof. Let (x_P^*, y_P^*) and (x_Q^*, y_Q^*) denote the optimal solutions of (5.28) and its perturbation respectively and ϑ_P and ϑ_Q the corresponding optimal values. By Theorem 4.1 (iii), there exist constants $c_1 > 0$, $\beta > 0$ and a neighborhood $\mathcal{V}$ of y_P^* such that

$$\|y - y_P^*\| \leq c_1(\mathscr{D}_H(Q,P) + \|x - x_P^*\|)^\beta \tag{5.29}$$

for any $y \in S_Q(x) \cap \mathcal{V}$ with Q close to P. In particular, for $z_P^* \in S_Q(x_P^*) \cap \mathcal{V}$,

$$\|z_P^* - y_P^*\| \leq c_1\mathscr{D}_H(Q,P).$$

Consequently

$$\begin{array}{rcl}\vartheta_Q & \leq & f(x_P^*, z_P^*) \leq f(x_P^*, y_P^*) + \|f(x_P^*, z_P^*) - f(x_P^*, y_P^*)\| \\ & \leq & \vartheta_P + \kappa\|z_P^* - y_P^*\| \leq \vartheta_P + \kappa c_1 \mathscr{D}_H(Q, P),\end{array} \tag{5.30}$$

where c_1 is the Lipschitz modulus of f.

Likewise, for $z_Q^* \in S_P(x_Q^*)$, since $\mathbf{F}_P^{\text{nat}}(x_Q^*, y)$ is a locally Lipschitz homeomorphism at z_Q^*, by Theorem 4.1 (iii), there exist $c_2 > 0$, $\beta > 0$ and a neighborhood $\mathcal{V}$ of z_Q^* such that

$$\|y - z_Q^*\| \leq c_2(\mathscr{D}_H(Q, P) + \|x - x_Q^*\|)^\beta \tag{5.31}$$

for $y \in S_Q(x) \cap \mathcal{V}$ with Q close to P. In particular, for $y_Q^* \in S_Q(x_Q^*) \cap \mathcal{V}$,

$$\|y_Q^* - z_Q^*\| \leq c_2 \mathscr{D}_H(Q, P).$$

Consequently we have

$$\begin{array}{rcl}\vartheta_P & \leq & f(x_Q^*, z_Q^*) \leq f(x_Q^*, y_Q^*) + \|f(x_Q^*, z_Q^*) - f(x_Q^*, y_Q^*)\| \\ & \leq & \vartheta_Q + \kappa\|z_Q^* - y_Q^*\| \leq \vartheta_Q + \kappa c_2 \mathscr{D}_H(Q, P).\end{array} \tag{5.32}$$

Combining (5.30) and (5.32), we obtain the conclusion. □

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