Complete Submodularity Characterization in the Comparative Independent Cascade Model

Wei Chen[1] and Hanrui Zhang[2]

[1] Microsoft Research, Beijing, China
weic@microsoft.com
[2] Tsinghua University, Beijing, China
zhang-hr13@mails.tsinghua.edu.cn

Abstract. We study the propagation of comparative ideas in social network. A full characterization for submodularity in the comparative independent cascade (Com-IC) model of two-idea cascade is given, for competing ideas and complementary ideas respectively. We further introduce One-Shot model where agents show less patience toward ideas, and show that in One-Shot model, only the stronger idea spreads with submodularity.

1 Introduction

Propagation of information in social networks has been extensively studied over the past decades, along with its most prominent algorithmic aspect - influence maximization. The cascade procedure of ideas in a network is usually modeled by a stochastic process, and influence maximization seeks to maximize the expected influence of a certain idea by choosing k agents (the seed set) in the network to be early adopters of the idea. The seed set then initiates the propagation through the network structure.

Influence maximization is proven to be NP-hard [7] in almost any non-trivial setting. Most research therefore focuses on approximation algorithms, some particularly successful ones out of which are based on the celebrated $(1 - \frac{1}{e})$-approximate submodular maximization [11]. Submodularity of influence in the seed set therefore plays a central role in such optimization.[3]

Nevertheless, submodularity appears harder to tract when there are multiple ideas interacting with each other. Most prior work focuses on single-idea cascade, or completely competing propagation of ideas. These models somewhat fails in modeling real world behavior of agents. Lu et al. [9] introduce a general model called *comparative independent cascade (Com-IC) model*, which covers the entire spectrum of two item cascades from full competition to full complementarity. This full spectrum is crucially characterized by four probability parameters called *global adoption probabilities (GAP)*, and their space is called the GAP space. However, they only provide submodularity analysis in a few marginal cases of

[3] We say a function $f : 2^U \to \mathbb{R}$ is submodular, if for any $S \subseteq U$, $a, b \in U$, $f(S) + f(S \cup \{a, b\}) \leq f(S \cup \{a\}) + f(S \cup \{b\})$.

the entire GAP space, and a full submodularity characterization for the entire GAP space is left as an open problem discussed in their conclusion section.

Our contribution. In this paper, we provide a full characterization of the submodularity of the Com-IC model in both the mutually competing case and the mutually complementary case (Theorems 1, 2, and 3). Our results show that in the entire continuous GAP space, the parameters satisfying submodularity only has measure zero. Next, we introduce a slightly modified One-Shot model for the mutual competing case where agents are less patient: they would reject the second item if they get influenced but failed to adopt the first item. We provide the full submodularity characterization of the parameter space for this model (Theorem 4), which contains a nontrivial half space satisfying submodularity, constrasting the result for the Com-IC model. Our techniques for establishing these characterization results may draw separate interests from the technical aspect for the study of submodularity for various influence propagation models.

Related work. Single-idea models, where there is only one propagating entity for social network users to adopt, has been thoroughly studied. Some examples are the classic Independent Cascade (IC) and Linear Thresholds (LT) models [7]. Some other work studies pure competition between ideas. See, e.g. [1,2,3,4,6,8]. Beside competing settings, Datta et al. [5] study influence maximization of independently propagating ideas, and Narayanam et al. [10] discuss a perfectly complementary setting, which is extended in [9].

2 The Model

We first recapitulate the independent cascade model for comparative ideas (Com-IC).

First recall that in the classic Independent Cascade (IC) model, the social network is described by a directed graph $G = (V, E, p)$ with probabilities $p : E \to [0, 1]$ on each edge. Each vertex in V stands for an agent, an edge for a connection, whose strength is characterized by the associated probability. Cascading proceeds at each time step $0, 1, \ldots$. At time 0, only the seed set is active. At time t, each vertex u activated at time $t-1$ tries to activate its neighbor v, and succeeds with probability $p(u, v)$. The procedure ends when no new vertices are activated at some time step.

In comparative IC (Com-IC henceforth) model, there are two ideas, A and B, spreading simultaneously in the network, and therefore 9 states of each vertex:

$$\{A\text{-idle}, A\text{-adopted}, A\text{-rejected}\} \times \{B\text{-idle}, B\text{-adopted}, B\text{-rejected}\}.$$

When an A-proposal reaches an A-idle vertex u, if u is previously B-adopted, it adopts A w.p. $q_{A|B}$. Otherwise, it adopts A w.p. $q_{A|\emptyset}$. The rules for idea B is totally symmetric. The four probabilities, $q_{A|\emptyset}, q_{B|\emptyset}, q_{A|B}, q_{B|A}$, therefore fully characterize strengths of the two ideas and the relationship between them: when A and B are mutually competing ideas, $q_{A|\emptyset} \geq q_{A|B}$ and $q_{B|\emptyset} \geq q_{B|A}$; when they

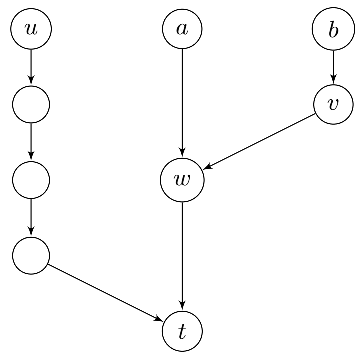

Fig. 1. counterexample used in the proofs of Theorem 1 and Theorem 2

are mutually complementary ideas, $q_{A|\emptyset} \leq q_{A|B}$ and $q_{B|\emptyset} \leq q_{B|A}$. These four probability parameters are referred as global adoption probabilities (GAP), and their space as the GAP space.

For tie-breaking, we generate a random ordering of all in-going edges for each vertex, and let proposals which reach at the same time try according to that order. If a vertex adopts two ideas at a same time step, it proposes the two ideas to its neighbors in the order adopted. We refer interested readers to [9] for more details of Com-IC model.

3 Notations

Let the set of possible worlds (the complete state of the network and vertices after fixing all randomness) be $\mathcal{W}$. For a possible world $W \in \mathcal{W}$, A-seed set S_A and B-seed set S_B (unless otherwise specified), let $\sigma_A(S_A, S_B, W)$ (resp. $\sigma_B(S_A, S_B, W)$) be the number of vertices which adopt A (resp. B) at the end of cascading in possible world W. $\sigma_A(S_A, S_B) = \mathbb{E}[\sigma_A(S_A, S_B, W)]$ (resp. $\sigma_B(S_A, S_B) = \mathbb{E}[\sigma_B(S_A, S_B, W)]$) then stands for the expected influence of A (resp. B) after cascading. Similarly, let $\sigma_A^u(S_A, S_B, W)$ be 1 if A affects u in W, and 0 if not, and $\sigma_A^u(S_A, S_B) = \mathbb{E}[\sigma_A^u(S_A, S_B, W)]$ the probability that A affects u. Parameters are ignored when in clear context.

4 Submodularity in the Mutually Competing Case

Recall that when the two ideas are competing, we have $q_{A|\emptyset} \geq q_{A|B}$, $q_{B|\emptyset} \geq q_{B|A}$. We are naturally interested in submodularity of $\sigma_A(S_A, S_B)$ in S_A fixing S_B. It turns out that this kind of submodularity is guaranteed only in a 0-measure subset of the parameter space. Formally, we have the following theorem:

Theorem 1 (Submodularity Characterization for the Mutually Competing Case). *When the two ideas are mutually competing, for a fixed S_B, σ_A is submodular in S_A whenever one of the following holds:*

- $q_{A|\emptyset} \in \{0, 1\}$,
- $q_{B|\emptyset} = 0$,
- $q_{A|\emptyset} = q_{A|B}$,
- $q_{B|\emptyset} = q_{B|A}$.

And when none of these conditions hold, submodularity is violated, i.e., there exists (G, S_A, S_B, u, v) *such that for each group of* $(q_{A|\emptyset}, q_{B|\emptyset}, q_{A|B}, q_{B|A})$,

$$\sigma_A(S_A, S_B) + \sigma_A(S_A \cup \{u, v\}, S_B) > \sigma_A(S_A \cup \{u\}, S_B) + \sigma_A(S_A \cup \{v\}, S_B).$$

Proof. First we prove the negative (non-submodular) half of the theorem by given an counterexample, illustrated in Figure 1. The basic seed sets for A and B are $S_A = \{a\}$ and $S_B = \{b\}$ respectively. In order to show non-submodularity, we consider the marginals of u at t when v is an A-seed and when v is not.

Note that considering submodularity at a single vertex suffices for establishing a global proof, since we could duplicate the vertex such that it dominates the expected influence. Also, we assume $p(u, v) = 1$ for each $(u, v) \in E$, since all positive (submodularity) proofs can be partially derandomized and done in each partial possible world, and for counterexamples, we simply set the probabilities to be 1.

Formally, define

$$M_1 = \sigma_A^t(S_A \cup \{u\}, S_B) - \sigma_A^t(S_A, S_B),$$
$$M_2 = \sigma_A^t(S_A \cup \{u, v\}, S_B) - \sigma_A^t(S_A \cup \{v\}, S_B).$$

Submodularity is violated if we show $M_1 < M_2$. We now calculate M_1 and M_2 separately. When v is not a seed, u has a marginal at t iff a fails to activate w and idea A succeeds in affecting t from u. That is,

$$M_1 = (1 - q_{A|\emptyset})[(1 - q_{B|\emptyset}^3)q_{A|\emptyset}^4 + q_{B|\emptyset}^3 q_{A|\emptyset}^3 q_{A|B}].$$

Similarly, when v is an A-seed, we have

$$M_2 = (1 - q_{A|\emptyset})[(1 - q_{B|\emptyset}q_{B|A}q_{B|\emptyset})q_{A|\emptyset}^4 + q_{B|\emptyset}q_{B|A}q_{B|\emptyset}q_{A|\emptyset}^3 q_{A|B}].$$

Taking the difference, we get

$$M_2 - M_1 = q_{A|\emptyset}^3 q_{B|\emptyset}^2 (1 - q_{A|\emptyset})(q_{A|B} - q_{A|\emptyset})(q_{B|A} - q_{B|\emptyset}).$$

It is easy to see, when none of the conditions listed in Theorem 1 hold, $M_2 - M_1 > 0$, and σ_A is not submodular in the seed set of A.

We now show case by case, that whenever one of the conditions holds, σ_A is submodular in the seed set of A.

- $q_{A|\emptyset} \in \{0, 1\}$. When $q_{A|\emptyset} = 0$, σ_A is always the size of S_A, so submodularity is obvious. Now suppose $q_{A|\emptyset} = 1$. Consider an equivalent formulation of the model: each vertex u draws two independent numbers uniformly at random from $[0, 1]$, denoted by $\alpha_A(u)$ and $\alpha_B(u)$ respectively. When an A-proposal

reaches an (A-idle, B-idle) or (A-idle, B-rejected) vertex u, if $\alpha_A(u) \le q_{A|\emptyset}$, u will accept A. When an A-proposal reaches an (A-idle, B-adopte) vertex u, if $\alpha_A(u) \le q_{A|B}$, u will accept A. The rules for B are symmetric.
After fixing all randomness, each vertex has two attributes for ideas A and B respectively. That is, each vertex u can be in exactly one state out of

$$\{\alpha_A(u) \le q_{A|B}, q_{A|B} < \alpha_A(u) \le q_{A|\emptyset}, q_{A|\emptyset} < \alpha_A(u)\} \times$$
$$\{\alpha_B(u) \le q_{B|A}, q_{B|A} < \alpha_B(u) \le q_{B|\emptyset}, q_{B|\emptyset} < \alpha_B(u)\}.$$

We show that in any possible world W, if $\sigma_A^t(S_A \cup \{u, v\}, S_B, W) = 1$, then $\sigma_A^t(S_A \cup \{u\}, S_B, W) + \sigma_A^t(S_A \cup \{v\}, S_B, W) \ge 1$. That is, if t is reachable by A when u and v are both A-seeds, then it is reachable by A when u or v alone is an A-seed. Submodularity then follows from monotonicity of $\sigma_A^t(S_A, S_B, W)$ in S_A and convex combination of possible worlds.
Let $p = (w_1, \ldots, w_k)$ be the A-path which reaches t when u and v are both A-seeds, where w_1 is an A-seed, and $w_k = t$. W.l.o.g. $v \notin p$. We argue that for each $w \in p$, if w is not B-adopted at the time A reaches it when u and v are A-seeds, then w is not B-adopted at the time A reaches it when only u is an A-seed, so p remains A-affected even if v is not an A-seed. Suppose not. Let w be the vertex closest to w_1 on p, which becomes affected by B when v is not a seed, p' be the B-path through which w is affected by B. Let x be the closest vertex to the B-seed on p', which is affected by A at the time the B-proposal reaches when v is an A-seed, and is affected by B when v is not a seed (such a vertex must exist). Then because $q_{A|\emptyset} = 1$, the subpath $[x, t] \subseteq p'$ must be completely A-affected when v is an A-seed, and reaches t earlier than p does, a contradiction.
Now since each vertex $w \in p$ which is not affected by B when v is an A-seed remains not affected when v is not, idea A can pass through the entire path p from some seed vertex to t just like when v is an A-seed, so t is still A-affected. In other words, w.l.o.g. $\sigma_A^t(S_A \cup \{u\}, S_B, W) = 1$.

- $q_{B|\emptyset} = 0$. B does not propagate at all. We simply remove S_B from the graph and consider the equivalent IC procedure of A alone. Submodularity is then easy.
- $q_{A|\emptyset} = q_{A|B}$. B does not affect the propagation of A. Again the propagation of A is equivalent as an IC procedure, and submodularity follows directly.
- $q_{B|\emptyset} = q_{B|A}$. We use the possible world model discussed in the first case, where $q_{A|\emptyset} \in \{0, 1\}$. Still, let $p = \{w_1, \ldots, w_k\}$ be the path through which t is affected by A when both u and v are A-seeds, and w.l.o.g. $v \notin p$. We apply induction on i to prove that A reaches w_i still at the $(i-1)$-th time slot when v is not an A-seed.
When $i = 1$, the statement holds evidently as w_1 is an A-seed. Assume at time $i-1$, w_i has just been reached by A and become A-adopted. Since the propagation of B is not affected by the A seed set, w_{i+1} is in the same state w.r.t. B as when v is a seed, so the A-proposal to w_{i+1} from w_i ends up just in the same way, and w_{i+1} becomes A-adopted at time i. And as a result, t is eventually A-adopted, i.e. $\sigma_A^t(S_A \cup \{u\}, S_B, W) = 1$.

5 Submodularity in the Mutually Complimentary Case

When the two ideas are complementary, i.e. when $q_{A|\emptyset} \leq q_{A|B}$ and $q_{B|\emptyset} \leq q_{B|A}$, enlarging the seed set of one idea helps the propagation of both the idea itself and that of the other idea. We discuss in this section the self and cross effect of the seed set of an idea.

5.1 Self Submodularity

Fixing S_B, we are interested in submodularity of σ_A in S_A, i.e., submodularity of the influence of some idea w.r.t. its own seed set, fixing the seed set of the other idea.

Theorem 2 (Self-Submodularity Characterization for the Mutually Complementary Case). *When the two ideas are complementary, for a fixed S_B, σ_A is submodular in S_A whenever one of the following holds:*

- $q_{A|\emptyset} \in \{0, 1\}$,
- $q_{B|\emptyset} = 0$,
- $q_{A|\emptyset} = q_{A|B}$,
- $q_{B|\emptyset} = q_{B|A}$.

And when none of these conditions hold, submodularity is violated, i.e., there exists (G, S_A, S_B, u, v) such that for each group of $(q_{A|\emptyset}, q_{B|\emptyset}, q_{A|B}, q_{B|A})$,

$$\sigma_A(S_A, S_B) + \sigma_A(S_A \cup \{u, v\}, S_B) > \sigma_A(S_A \cup \{u\}, S_B) + \sigma_A(S_A \cup \{v\}, S_B).$$

Proof. We first show the negative part. Recall that in the proof of Theorem 1, we calculate that for the graph in Figure 1,

$$M_2 - M_1 = q_{A|\emptyset}^3 q_{B|\emptyset}^2 (1 - q_{A|\emptyset})(q_{A|B} - q_{A|\emptyset})(q_{B|A} - q_{B|\emptyset}),$$

which remains exactly the same no matter whether A and B are competing or complementary. If none of the conditions in Theorem 2 hold, then $M_2 - M_1 > 0$, and σ_A^t is not submodular in the seed set of A.

Now we prove case by case the positive cases.

- $q_{A|\emptyset} \in \{0, 1\}$. When $q_{A|\emptyset} = 1$, σ_A is simply all vertices reachable from the A seed set, so submodularity is trivial. Now consider $q_{A|\emptyset} = 0$, and take again the possible world view. The fact that $q_{A|\emptyset} = 0$ gives us two messages: that A spreads only by following B, and consequently that A does not affect the propagation of B. We use the same notations as in the proof of Theorem 1. Assume that in possible world W, when both u and v are A-seeds, t is affected by A (or $\sigma_A^t(S_A \cup \{u, v\}, S_B, W) = 1$), and let $p = \{w_1, \ldots, w_k\}$ be the path through which A reaches t, where w.l.o.g. $v \notin p$. We prove by induction that when v is not an A-seed, A affects w_i exactly at time slot $i - 1$ for $i \in [k]$.

Then statement is trivial when $i = 1$. Assume that w_i becomes A-adopted at time $i-1$. Since the propagation of B is not interfered by the A seed set, w_{i+1} is in the same B-state as when v is an A-seed at time $i-1$, and the A-proposal from w_i gets the same reaction at w_{i+1}, i.e. acceptance. So t is eventually A-adopted, or $\sigma_A^t(S_A \cup \{u\}, S_B, W) = 1$.
- $q_{B|\emptyset} = 0$. That is, B spreads only through A-adopted vertices, and thus does not affect the propagation of A. The equivalent IC cascade procedure gives submodularity directly.
- $q_{A|\emptyset} = q_{A|B}$. Again, B does not affect A, and submodularity is trivial.
- $q_{B|\emptyset} = q_{B|A}$. The proof is totally similar to the case where $q_{A|\emptyset} = 0$.

Note 1. The conuterexample used in the proof of Theorem 2 is exactly the same as that used in the proof of Theorem 1. This versatility of the counterexample comes from the factor $(q_{A|\emptyset} - q_{A|B})(q_{B|\emptyset} - q_{B|A})$. In each case, $q_{A|\emptyset} - q_{A|B}$ and $q_{B|\emptyset} - q_{B|A}$ are of the same sign.

5.2 Cross Submodularity

Fixing S_A, because of the complementary nature of the two ideas, we are also curious about submodularity of σ_A in S_B, i.e., submodularity of the influence of some idea w.r.t. the seed set of the other idea, fixing its own seed set.

Theorem 3 (Cross-Submodularity Characterization for the Mutually Complementarity Case). *When the two ideas are complementary, for a fixed S_A, σ_A is submodular in S_B whenever one of the following holds:*

- $q_{A|\emptyset} \in \{0, 1\}$,
- $q_{B|\emptyset} = 1$,
- $q_{A|\emptyset} = q_{A|B}$.

And when none of these conditions hold, submodularity is violated, i.e., there exists (G, S_A, S_B, u, v) such that for each group of $(q_{A|\emptyset}, q_{B|\emptyset}, q_{A|B}, q_{B|A})$,

$$\sigma_A(S_A, S_B) + \sigma_A(S_A, S_B \cup \{u, v\}) > \sigma_A(S_A, S_B \cup \{u\}) + \sigma_A(S_A, S_B \cup \{v\}).$$

Proof. We prove the negative part first. Consider the counterexample presented in Figure 2, and let the basic seed sets of A and B be $S_A = \{a\}$, $S_B = \{b\}$. We consider the marginals of u as a B-seed when v is a B-seed and when v is not. Let

$$\begin{aligned} M_1 &= \sigma_A^t(S_A, S_B \cup \{u\}) - \sigma_A^t(S_A, S_B), \\ M_2 &= \sigma_A^t(S_A, S_B \cup \{u, v\}) - \sigma_A^t(S_A, S_B \cup \{v\}). \end{aligned}$$

When v is not a B-seed, u has a non-zero marginal iff it helps the propagation of A at t (while b does not). That is, a reaches w, b does not reach w, and u reaches t. Formally,

$$M_1 = q_{B|\emptyset}(q_{A|B} - q_{A|\emptyset})[(1 - q_{B|\emptyset})q_{A|\emptyset}^3 + q_{B|\emptyset}(1 - q_{B|\emptyset})q_{A|B}q_{A|\emptyset}^2].$$

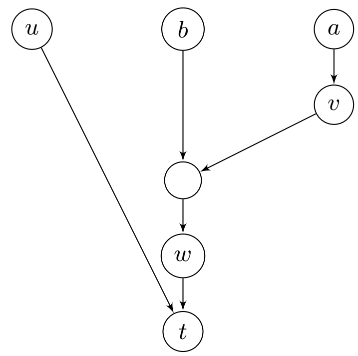

Fig. 2. counterexample used in the proof of Theorem 3

And when v is a B-seed, everything is the same except that it becomes easier for A to activate v. And therefore

$$M_2 = q_{B|\emptyset}(q_{A|B} - q_{A|\emptyset})[(1 - q_{B|\emptyset})q_{A|B}q_{A|\emptyset}^2 + q_{B|\emptyset}(1 - q_{B|\emptyset})q_{A|B}^2 q_{A|\emptyset}].$$

Taking the difference,

$$M_2 - M_1 = (q_{A|B} - q_{A|\emptyset})^2 q_{A|\emptyset}(1 - q_{A|\emptyset})(q_{A|\emptyset} + q_{B|\emptyset}q_{A|B}).$$

It is clear that when no conditions stated in Theorem 3 hold, $M_2 - M_1 > 0$ and submodularity fails.

Now we look at the positive cases.

- $q_{A|\emptyset} \in \{0, 1\}$. When $q_{A|\emptyset} = 1$, submodularity is trivial. Now suppose $q_{A|\emptyset} = 0$, i.e., A spreads only by following B and does not affect the propagation of B. We prove that for any possible world W, where $\sigma_A^t(S_A, S_B \cup \{u, v\}, W) = 1$, we have $\sigma_A^t(S_A, S_B \cup \{u\}, W) + \sigma_A^t(S_A, S_B \cup \{v\}, W) \geq 1$. That is, when t is A-adopted when both u and v are B-seeds, t will still be activated either when u alone is a B-seed or v alone is.
  When both u and v are seeds, let $p = \{w_1, \ldots, w_k\}$ be the path through which A reaches t, where w.l.o.g. $v \notin p$. Clearly in our possible world view, for any $w \in p$, $\alpha_B(w) \leq q_{B|\emptyset}$. When u alone is a B-seed, w_1 remains a B-seed, and can activate every vertex w on the path just as when v is also a B-seed. It follows that A still can reach t through p.
- $q_{B|\emptyset} = 1$. We follow the same manner as we do in the first bullet point. The same argument works in a sense that we still have $\alpha_B(w) \leq q_{B|\emptyset} = 1$ here for $w \in p$. Sobmodularity follows.
- $q_{A|\emptyset} = q_{A|B}$. That means b has nothing to do with the propagation of A. Submodularity of IC model then carries over directly.

6 The One-Shot Model

In foregoing sections, properties of a model with somewhat rational agents are discussed. The agents are rational, in a sense that when a first proposal of some idea fails, they still allow the other idea a chance to propose; and when a first proposal succeeds, they do not accept/reject the possible proposal from the other idea instantly. In this section, we look at a model where agents act more extremely.

6.1 The Model

As in the Com-IC model, there is a backbone network $G = (V, E, p)$. The model also has four parameters as the GAP parameters in Com-IC. We only consider the mutually competing case for the One-Shot model. The key difference here is that an idle vertex considers only the first proposal that reaches it. Each vertex has 4 possible states: idle, exhausted, A-adopted, B-adopted.

Cascading proceeds in the following fashion: when an A (resp. B) proposal reaches an idle vertex, the vertex adopts A (resp. B) w.p. $q_{A|\emptyset}$ (resp. $q_{B|\emptyset}$), and becomes exhausted w.p. $1 - q_{A|\emptyset}$ (resp. $1 - q_{B|\emptyset}$). Once a vertex becomes exhausted, it no longer considers any further proposals. Since A and B are competing ideas, an A-adopted (resp. B-adopted) vertex no longer considers proposals of B (resp. A). $(q_{A|\emptyset}, q_{B|\emptyset})$ therefore completely characterizes the strengths of the ideas.

6.2 Submodularity in One-Shot Model

The characterization of sumodularity in One-Shot model appears to be more interesting. It demonstrates a dichotomy over the GAP space of One-Shot model, i.e., only the stronger idea propagates with submodularity.

Theorem 4. *In One-Shot model, when $q_{A|\emptyset} \geq q_{B|\emptyset}$ or $q_{A|\emptyset} = 0$, σ_A is submodular in S_A; when $0 < q_{A|\emptyset} < q_{B|\emptyset}$, submodularity is violated. To be specific, when $0 < q_{A|\emptyset} < q_{B|\emptyset}$, there exists (G, S_A, S_B, u, v) such that*

$$\sigma_A(S_A, S_B) + \sigma_A(S_A \cup \{u, v\}, S_B) > \sigma_A(S_A \cup \{u\}, S_B) + \sigma_A(S_A \cup \{v\}, S_B).$$

Proof. We prove the negative part first. Consider the network shown in Figure 3, where the basic seed sets are $S_A = \emptyset$ and $S_B = \{b\}$. We calculate the marginals of u at t when v is an A-seed and when v is not. Formally, let

$$M_1 = \sigma_A^t(S_A \cup \{u\}, S_B) - \sigma_A^t(S_A, S_B),$$
$$M_2 = \sigma_A^t(S_A \cup \{u, v\}, S_B) - \sigma_A^t(S_A \cup \{v\}, S_B).$$

When v is not a seed, u has a positive marginal iff b fails to reach t and u successfully reaches t. That is,

$$M_1 = q_{A|\emptyset}^{k+3}(1 - q_{B|\emptyset}^{k+2}).$$

And when v is an A-seed, t has a positive marginal iff v fails to reach t and u scceeds. So,

$$M_2 = q_{A|\emptyset}^{k+3}(1 - q_{A|\emptyset}^{k+1}).$$

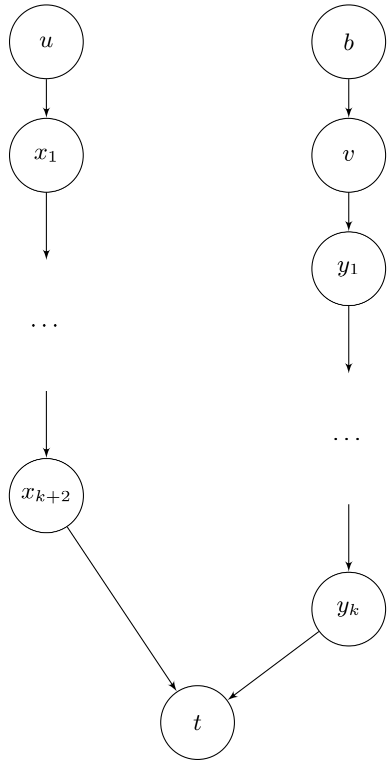

Fig. 3. counterexample used in the proof of Theorem 4

Taking the difference,

$$M_2 - M_1 = q_{A|\emptyset}^{k+3}(q_{B|\emptyset}^{k+2} - q_{A|\emptyset}^{k+1}).$$

As $q_{A|\emptyset} < q_{B|\emptyset}$,

$$\lim_{k\to\infty} \frac{q_{B|\emptyset}^{k+2}}{q_{A|\emptyset}^{k+1}} = \infty,$$

so when $q_{A|\emptyset} > 0$, there is some k such that $M_2 - M_1 > 0$, and submodularity is violated.

We prove the positive part now. When $q_{A|\emptyset} = 0$, $\sigma_A = |S_A|$ is clearly submodular in S_A. Now we consider the case where $q_{A|\emptyset} \geq q_{B|\emptyset}$. We take a different possible world view here, i.e., each vertex flips two independent coins and decide whether it accepts A-proposals and B-proposals. Each vertex has 4 possible realizations: A-only, B-only, susceptible and repudiating, indicating that the vertex accepts A-proposals only, B-proposals only, all proposals, and none respectively.

First we consider a partial realization of the world. We realize all susceptible and repudiating vertices first. To do so, for each vertex v, we flip a coin and determined with probability $q_{A|\emptyset}q_{B|\emptyset} + (1 - q_{A|\emptyset})(1 - q_{B|\emptyset})$ that v is eventually realized to be either susceptible or repudiating. If so, we then flip another coin to determine whether it is susceptible (w.p. $\frac{q_{A|\emptyset}q_{B|\emptyset}}{q_{A|\emptyset}q_{B|\emptyset}+(1-q_{A|\emptyset})(1-q_{B|\emptyset})}$) or repudiating (otherwise). For vertices remaining not realized, we flip a coin and decide it to be A-only w.p. $\frac{q_{A|\emptyset}(1-q_{B|\emptyset})-q_{B|\emptyset}(1-q_{A|\emptyset})}{q_{A|\emptyset}(1-q_{B|\emptyset})+q_{B|\emptyset}(1-q_{A|\emptyset})}$. Now upon full realization, each of the rest of vertices (which we call deferred vertices) is A-only exactly w.p. $\frac{1}{2}$ and B-only otherwise. The partial realization stops at this stage. We remove all repudiating vertices, leaving vertices in 3 possible states: susceptible, A-only, and deferred.

Now for S_A, S_B, t and a partial realization W_p, suppose there are k deferred vertices, $w_1, \ldots, w_k$. Define probability spaces $\Omega_0, \Omega_1, \ldots, \Omega_k$ in the following fashion: in Ω_i, deferred vertex w_j is realized such that:

- If $j > i$, w_i is A-only w.p. $\frac{1}{2}$ and B-only w.p. $\frac{1}{2}$.
- If $j \leq i$, w_i is susceptible w.p. $\frac{1}{2}$ and repudiating w.p. $\frac{1}{2}$.

We show that for all $i \in [k]$,

$$\mathbb{E}_{W_i \leftarrow \Omega_i}[\sigma_A^t(S_A, S_B, W_i)] = \mathbb{E}_{W_{i-1} \leftarrow \Omega_{i-1}}[\sigma_A^t(S_A, S_B, W_{i-1})].$$

Consider fixing randomness of $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_k$ in Ω_{i-1}. After doing so, we are able to determine the first proposal (if any) that reaches w_i, since that part of the propagation is fully deterministic. Say, the proposal is an A-proposal, then because w_i is A-only w.p. $\frac{1}{2}$ and B-only otherwise, it accepts the proposal w.p. $\frac{1}{2}$, and becomes exhausted otherwise. This is indeed equivalent w.r.t. propagation of A to making w_i susceptible w.p. $\frac{1}{2}$ and repudiating otherwise. Since for any partial realization of $\{w_1, \ldots, w_k\} \setminus \{w_i\}$, the above equivalence always holds, we may conclude that the two probability spaces are equivalent w.r.t. the influence of A. Formally,

$$\mathbb{E}_{W_i \leftarrow \Omega_i}[\sigma_A^t(S_A, S_B, W_i)] = \mathbb{E}_{W_{i-1} \leftarrow \Omega_{i-1}}[\sigma_A^t(S_A, S_B, W_{i-1})].$$

Now we only need to show submodularity in Ω_k. We fix all randomness, remove repudiating vertices, and establish submodularity in each possible world. In each possible world W_k drawn from Ω_k, there are possibly 2 types of vertices: susceptible ones and A-only ones. We show that for S_A, S_B, u, v, t,

$$\sigma_A^t(S_A \cup \{u, v\}, S_B, W_k) = 1 \Rightarrow \sigma_A^t(S_A \cup \{u\}, S_B, W_k) + \sigma_A^t(S_A \cup \{v\}, S_B, W_k) \geq 1.$$

Let $p = \{w_1, \ldots, w_k\}$ be the A-path through which A reaches t when both u and v are A-seeds. W.l.o.g. $v \notin p$. In the competing case, let w be the vertex closest to w_1 on p, which becomes not A-adopted (and in fact, B-adopted) when v is not a seed. w must be reachable from v. Let p' be the shortest path from v to w, and x the closest vertex to v on p' which becomes B-adopted when v is not a seed. Since v blocks B from affecting x through path $[v, x] \subseteq p'$, and when v is not a seed, x blocks w from being affected by A through path $[x, w] \subseteq p'$, clearly p' is a shorter A-path (recall that A can pass through every vertex in the world) from the A seed set to w than $[w_1, w] \subseteq p$ when v is an A-seed, a contradiction.

Note 2. Unlike in Theorem 1, Theorem 2 or Theorem 3, the counterexample needed for Theorem 4 has to be constructed after fixing $q_{A|\emptyset}$ and $q_{B|\emptyset}$.

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