Efficient winning strategies in random-turn Maker-Breaker games

Asaf Ferber [*] Michael Krivelevich [†] Gal Kronenberg [‡]

June 1, 2016

Abstract

We consider random-turn positional games, introduced by Peres, Schramm, Sheffield and Wilson in 2007. A p-random-turn positional game is a two-player game, played the same as an ordinary positional game, except that instead of alternating turns, a coin is being tossed before each turn to decide the identity of the next player to move (the probability of Player I to move is p). We analyze the random-turn version of several classical Maker-Breaker games such as the game Box (introduced by Chvátal and Erdős in 1987), the Hamilton cycle game and the k-vertex-connectivity game (both played on the edge set of K_n). For each of these games we provide each of the players with a (randomized) efficient strategy which typically ensures his win in the asymptotic order of the minimum value of p for which he typically wins the game, assuming optimal strategies of both players.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the $(a : b)$ Maker-Breaker game $\mathcal{F}$, two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Breaker going first. The set X is called the *board* of the game and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker claims a board elements per turn, whereas Breaker claims b elements. The parameters a and b are called the *bias* of Maker and of Breaker, respectively. We assume that Breaker moves first. Maker wins the game as soon as he occupies all elements of some winning set. If Maker does not fully occupy any winning set by the time every board element is claimed by either of the players, then Breaker wins the game. We say that the $(a : b)$ game $\mathcal{F}$ is *Maker's win* if Maker has a strategy that ensures his victory against any strategy of Breaker, otherwise the game is *Breaker's win*. The most basic case is $a = b = 1$, the so-called *unbiased* game, while for all other choices of a and b the game is called *biased*. Let $F \in 2^X$ and let $f : 2^X \to \mathbb{R}$ be a function. We call $f(F)$ the *payoff* of the game if Player 1 (Maker in our case) claimed the elements of F. In Maker-Breaker games, f is a {-1,1} function and $f(F) = 1$ if and only if $F \in \mathcal{F}$ (that is, if and only if Maker wins).

[*]Department of Mathematics, Yale University, and Department of Mathematics, MIT. Emails: asaf.ferber@yale.edu, and ferbera@mit.edu.

[†]School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 6997801, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by USA-Israel BSF Grant 2010115 and by grant 912/12 from the Israel Science Foundation.

[‡]School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 6997801, Israel. Email: galkrone@mail.tau.ac.il.

It is natural to play Maker-Breaker games on the edge set of a graph $G = (V, E)$. In this case, $X = E$ and the winning sets are all the edge sets of subgraphs of G which possess some given graph property $\mathcal{P}$. In this case, we refer to this game as the $(a : b)$ $\mathcal{P}$-game. In the special case where $G = K_n$ we denote $\mathcal{P}^n := \mathcal{P}(K_n)$. In the *connectivity game*, Maker wins if and only if his edges contain a spanning tree of G. In the *perfect matching* game the winning sets are all sets of $\lfloor |V(G)|/2 \rfloor$ independent edges of G. Note that if $|V(G)|$ is odd, then such a matching covers all vertices of G but one. In the *Hamiltonicity game* the winning sets are all edge sets of Hamilton cycles of G. Given a positive integer k, in the *k-connectivity game* the winning sets are all edge sets of k-vertex-connected spanning subgraphs of G. Given a graph H, in the *H-game* played on G, the winning sets are all edge sets of copies of H in G.

Playing unbiased Maker-Breaker games on the edge set of K_n is frequently in a favor of Maker. For example, it is easy to see (and also follows from [19]) that for every $n \geq 4$, Maker can win the unbiased connectivity game in $n-1$ moves (which is clearly also the fastest possible strategy). Other unbiased games played on $E(K_n)$ like the perfect matching game, the Hamiltonicity game, the k-vertex-connectivity game and the T-game where T is a given spanning tree with bounded maximum degree, are also known to be an easy win for Maker (see e.g, [8, 9, 12]). It is thus natural to give Breaker more power by allowing him to claim $b > 1$ elements in each turn.

Given a monotone increasing graph property $\mathcal{P}$, it is easy to see that the Maker-Breaker game $\mathcal{P}(G)$ is *bias monotone*. That is, none of the players can be harmed by claiming more elements. Therefore, it makes sense to study $(1 : b)$ games and the parameter b^* which is the *critical bias* of the game, that is, b^* is the maximal bias b for which Maker wins the corresponding $(1 : b)$ game $\mathcal{F}$.

As expected, the parameter b^* in various biased Maker-Breaker is well studied. For example, Chvátal and Erdős [7] showed that for every $\varepsilon > 0$, playing with bias $b = \frac{(1+\varepsilon)n}{\ln n}$, Breaker can isolate a vertex in Maker's graph while playing on the board $E(K_n)$. It thus follows that with this bias, Breaker wins every game for which the winning sets consist of subgraphs of K_n with positive minimum degree, and therefore, for each such game we have that $b^* \leq \frac{(1+o(1))n}{\ln n}$. Later on, Gebauer and Szabó showed in [10] that the critical bias for the connectivity game played on $E(K_n)$ is indeed asymptotically equal to $\frac{n}{\ln n}$. In a relevant development, the second author of this paper proved in [17] that the critical bias for the Hamiltonicity game is asymptotically equal to $\frac{n}{\ln n}$ as well. We refer the reader to [2, 13] for more background on positional games in general and on Maker-Breaker games in particular.

In this paper we consider a random-turn variant of Maker-Breaker games. A *p-random-turn Maker-Breaker game* is the same as ordinary $(1 : 1)$ Maker-Breaker game, except that instead of alternating turns, before each turn a biased coin is being tossed and Maker plays this turn with probability p independently of all other turns. Maker-Breaker games under this setting were initially considered by Peres, Schramm, Sheffield and Wilson in [20], where, among other games, they studied the $1/2$-random-turn version of the so called game HEX which was introduced by Piet Hein in 1942 [14], and later discussed by John Nash in 1948 [18].

The game of HEX is played on a rhombus of hexagons of size $n \times n$, where every player has two opposite sides and his goal is to connect these two sides. At a first glance, HEX does not fit the general framework of a Maker-Breaker game, but there is a legitimate way to cast it as such a game. Although ordinary HEX is notoriously difficult to analyze, Peres et al. showed that the optimal strategy for 1/2-random-turn HEX turns out to be very simple.

To be more precise, Player 1 (respectively, Player 2) will play *optimally* if in every turn he chooses an unclaimed element s which maximizes $S(T_1 \cup \{s\}, T_2)$ (respectively, minimizes $S(T_1, T_2 \cup \{s\})$). The function $S(T_1, T_2)$, the *expected payoff* after the sets $T_1, T_2 \subseteq X$ were claimed by Player 1 and Player 2 (respectively), is computed in the following way. Using a backward analysis, let $S(T_1, X \setminus T_1) = f(T_1)$ where f is the payoff function of the game. Assume that we have computed $S(T_1, T_2)$ for $k+1 \leq |T_1 \cup T_2| \leq |X|$, then for $|T_1 \cup T_2| = k$, $S(T_1, T_2) = \max_{s \in X \setminus (T_1 \cup T_2)} S(T_1 \cup \{s\}, T_2)$ if it is the turn of Player 1, and the optimal strategy for him is to claim an element s that maximizes $S(T_1 \cup \{s\}, T_2)$, and $S(T_1, T_2) = \min_{s \in X \setminus (T_1 \cup T_2)} S(T_1, T_2 \cup \{s\})$ if it is the turn of Player 2, and the optimal strategy for him is to claim an element s that minimizes $S(T_1, T_2 \cup \{s\})$.

More generally, Peres et al. showed that the outcome of a p-random-turn game which is played by two *optimal* players is *exactly* the same as the outcome of this game played by two *random* players.

In particular, one can easily deduce the following theorem from their arguments.

Theorem 1.1 *Let $0 \leq p \leq 1$, and let $\mathcal{P}$ be any graph property. Then if both players play according to their optimal strategies, the probability for Maker to win the p-random-turn game $\mathcal{P}^n$ is the same as the probability that a graph $G \sim \mathcal{G}(n,p)$ satisfies $\mathcal{P}$.*

Proof [Sketch] Let S_B be any strategy of Breaker and denote by G_M (respectively G_B) the graph which Maker (respectively Breaker) builds by the end of the game. Assume that Maker plays according to S_B as well. That is, before the i^{th} turn of the game, Maker aims to claim the same edge e_i as Breaker should claim playing according to S_B. It thus follows that throughout the game, both players, Maker and Breaker want to claim the same edge e_i. Therefore, Maker claims e_i with probability p, and Breaker with probability $1-p$, for every element of the board. Thus, playing according to the suggested strategy, $G_M \sim \mathcal{G}(n,p)$. However, a symmetric argument applied on Breaker implies that if Breaker follows S_M, where S_M is the strategy of Maker, then $G_B \sim \mathcal{G}(n, 1-p)$. All in all, if both players play according to their optimal strategies, Maker's graph, G_M, satisfies $G_M \sim \mathcal{G}(n,p)$ and thus the probability for Maker to win the p-random turn game $\mathcal{P}^n$ is the same as the probability of $\mathcal{G}(n,p)$ to satisfy the property $\mathcal{P}$. $\square$

Note that Theorem 1.1 does not provide any of the players with an optimal strategy. However, as mentioned above (and in [20]), the set of possible optimal strategies for perfect information games can be found by computing the expected payoff after every turn. Using this fact and Theorem 1.1, we can deduce the following optimal strategy for the p-random-turn game $\mathcal{P}^n$. Let $\mathcal{F} \subseteq 2^X$ be a monotone increasing family of sets, and consider the p-random-turn Maker-Breaker game $\mathcal{F}$. Let X_M and X_B denote the sets chosen by Maker and Breaker by the end of the game, respectively. Let $F \in 2^X$, and let f be a boolean function such that $f(F) = 1$ if $F \in \mathcal{F}$, and $f(F) = -1$ otherwise. Then $f(X_M)$ is the *payoff* of the game. Assume we are in the middle of a game. Let T_M and T_B be the elements Maker and Breaker have claimed (respectively) so far during the game, and let $S(T_M, T_B)$ be the expected payoff for Maker at this stage of the game. In this case, $S(T_M, T_B) = \mathbb{E}(f(T_M \cup Z))$, where Z denotes a random subset of $X \setminus (T_M \cup T_B)$, chosen by including each element with probability p, independently at random. Maker will play *optimally* if in each turn he claims an element $s \in X \setminus (T_M \cup T_B)$

for which $S(T_M \cup \{s\}, T_B)$ is **maximal**, and an optimal strategy for Breaker is to **minimize** $S(T_M, T_B \cup \{s\})$ at each turn.

These general and optimal strategies are far away from being efficient. Indeed, consider the p-random-turn game where $X = E(K_n)$ and the winning sets are all the edge sets of subgraphs of K_n satisfying some property $\mathcal{P}$. Before each turn, the player should run over all the possibilities for his next move and simulate the game while calculating the payoff for each possible subgraph. In particular, after the k^{th} turn of the game, there are $\binom{n}{2} - k$ edges left, so each player has to run over all $\binom{n}{2} - k$ options for edges, and to calculate the expected payoff for every option. This amounts to calculating the payoff of $2^{\binom{n}{2}-k-1}$ subgraphs for each possible edge. So we have that in the $(k+1)^{st}$ turn, the player to move should check $(\binom{n}{2} - k) \cdot 2^{\binom{n}{2}-k-1}$ possible subgraphs. Therefore, even if the calculation time for the outcome of f is $O(1)$, the number of simulations each player should run before each turn makes the total calculation exponential (actually, with the exponent being quadratic in n most of the time). The main goal of this paper is to present better (polynomial-time) strategies for various natural games. We say that a strategy of a player is a *polynomial time strategy* (or an *efficient strategy*), if in every round, the calculation time for the player's next move is polynomial in the size of the board. We say that the strategy is *random* if the player chooses his elements randomly (according to some distribution) in some of the turns.

Given a monotone graph property $\mathcal{P}$, a p-random-turn game $\mathcal{P}$ is monotone with respect to p. That is, if Maker has a strategy to win with high probability (w.h.p.) the p-random-turn game $\mathcal{P}$, then Maker also has a strategy to win (w.h.p.) the q-random-turn game $\mathcal{P}$ for each $q \geq p$. It is thus natural to define the *probability threshold* of the game, p^*, to be such that if $p = o(p^*)$ then w.h.p. Maker loses the game, and if $p = \omega(p^*)$, then Maker has a strategy that w.h.p. ensures his victory. Note that, by Theorem 1.1, it follows that p^* is also the threshold probability that a $G \sim \mathcal{G}(n,p)$ satisfies $\mathcal{P}$. Since the problem of finding the probability threshold of a game is a purely random graph theoretical problem, and since Theorem 1.1 does not provide either player with an efficient strategy that typically ensures his victory, a natural research direction under this setting is to find such (possibly randomized) strategies. Here we make a progress in this direction by finding polynomial time (randomized) strategies for both players in several natural games.

One could expect that for small values of p, there is a connection between the outcome of a deterministic $(1:b)$ game $\mathcal{F}$ and its corresponding random turn version $\mathcal{F}_p$ where $p = \Theta\left(\frac{1}{b}\right)$. Indeed, as follows from Theorem 1.1 together with known results for $\mathcal{G}(n,p)$ (see, i.e. [6]), in many cases this is the correct order of magnitude. For example, the second author proved in [17] that the critical bias for the deterministic Hamiltonicity game is $b^* = (1+o(1))\frac{n}{\ln n}$, and therefore, recalling the fact that the threshold for Hamiltonicity in $\mathcal{G}(n,p)$ is $\frac{\ln n + \ln\ln n}{n}$ we have that $p^* = \Theta\left(\frac{1}{b^*}\right)$. However, in many other games, for example in the game of building a fixed graph in K_n, this is not the case, as Theorem 1.1 states. It is well known that the threshold function for the appearance of a triangle is $\frac{1}{n}$ (see, e.g. [6]). Therefore, for instance, in the random-turn game on $E(K_n)$ where Maker's goal is to build a triangle, Theorem 1.1 implies that if, say, $p = n^{-2/3}$, it is typically Maker's win. But as shown by [7] and [3], in the corresponding deterministic game, for $b = \Theta\left(\frac{1}{p}\right)$ with the above value of $p = p(n)$, Breaker is the winner of the game.

The most basic (and extremely useful) Maker-Breaker game is the so called game *Box* due to Chvátal and Erdős [7]. The game $Box(a_1, \ldots, a_n; m)$ is an $(m : 1)$ Maker-Breaker game (the players are also referred to as BoxMaker and BoxBreaker, respectively), where there are n disjoint winning sets $\{F_1, \ldots, F_n\}$ (referred to as *boxes*) such that $|F_i| = a_i$ for every i. In their paper [7], Chvátal and Erdős used this game as an auxiliary game to provide Breaker with a strategy in the minimum degree game played on $E(K_n)$. As it turns out the game Box is extremely useful as an auxiliary game in much more complicated settings. We first analyze it under the random-turn setting.

The *p-random-turn game Box*, denoted by $Box_p(a_1, \ldots, a_n)$ is similar to the ordinary $(1 : 1)$ game Box (in the sense that in each turn, exactly one element on the board is claimed by a player), except of the fact that before each turn, the identity of the current player (to pick exactly one element) is decided by tossing a biased coin, where BoxBreaker plays with probability p independently at random. Similar to the deterministic version of game, in this paper we also use the game $Box_p(a_1, \ldots, a_n)$ as an auxiliary game where BoxBreaker plays the role of Maker. For that reason, this setting is different from the standard setting, and it is BoxBreaker (rather than BoxMaker) who plays with probability p. In the proofs of the following theorems, we provide explicit polynomial (possibly randomized) strategies for the player to win – where the identity of a typical winner is given by an analogous version of Theorem 1.1.

In the first theorem we show that if the boxes are large enough as a function of p, then BoxBreaker has an efficient strategy typically winning for him:

Theorem 1.2 *There exists $C > 0$ such that for every sufficiently large integer n, the following holds. Suppose that:*

(i) $0 < p := p(n) \leq 1$, *and*

(ii) $a_1, \ldots, a_n$ *are integers such that* $a_i \geq \frac{C \ln n}{p}$ *for every* $1 \leq i \leq n$.

Then BoxBreaker has a polynomial-time strategy for the game $Box_p(a_1, \ldots, a_n)$ that w.h.p. leads him to win the game.

In the following theorem we show that if the boxes are not that large, then BoxMaker wins.

Theorem 1.3 *Let $\varepsilon > 0$. Then for a sufficiently large integer n, the following holds. Suppose that:*

(i) $0 < p := p(n) < 1$, *and*

(ii) $a_1, \ldots, a_n$ *are integers such that* $a_i \leq \frac{(1-\varepsilon) \ln n}{-\ln(1-p)}$ *for every* $1 \leq i \leq n$.

Then BoxMaker has a polynomial-time strategy for the game $Box_p(a_1, \ldots, a_n)$ that w.h.p. leads him to win the game.

Remark 1.4 *For $p = o(1)$ we can reduce the second assumption of Theorem 1.3 to $a_i \leq \frac{(1-\varepsilon) \ln n}{p}$ for every $1 \leq i \leq n$ and for every $\varepsilon > 0$.*

Using Theorems 1.2 and 1.3 as auxiliary games, we analyze various natural games played on graphs. It follows from Theorem 1.1 and from well known facts about random graphs (see e.g, [6]) that the critical p for the p-random-turn game on $E(K_n)$ where Breaker's goal is to isolate a vertex in Maker's graph is $p^* = \frac{(1-o(1))\ln n}{n}$. In the following theorem, analogously to Chvátal and Erdős [7], we show that playing a p-random-turn game on $E(K_n)$, Breaker has an efficient strategy that typically allows him to isolate a vertex in Maker's graph, provided that $p = \frac{(1-\varepsilon)\ln n}{n}$. It thus follows that for this range of p, Breaker typically wins every game whose winning sets consist of spanning subgraphs with a positive minimum degree (such as the Hamiltonicity game, the perfect matching game, the k-connectivity game, etc.).

Theorem 1.5 *Let $\varepsilon > 0$. For every $p \le \frac{(1-\varepsilon)\ln n}{n}$ and a sufficiently large n, in the p-random-turn game played on $E(K_n)$, Breaker has an efficient strategy that w.h.p. allows him to isolate a vertex in Maker's graph.*

Our next theorem shows that for $p = \Omega(\frac{\ln n}{n})$, Maker has a polynomial time randomized strategy that is typically a winning strategy for the p-random-turn Hamiltonicity game, $\mathcal{H}_p^n$, played on $E(K_n)$. Recall that by Theorem 1.1, the probability threshold of the Hamiltonicity game is the same as the probability threshold of $\mathcal{G}(n,p)$ to become Hamiltonian, which is $p^* = \frac{\ln n + \ln\ln n}{n}$ (see e.g, [5], [16]). Therefore, together with Theorem 1.5, we provide both of the players with efficient strategies which typically are winning strategies, for p's which are of the same order of magnitude as the probability threshold.

Theorem 1.6 *There exists $C_2 > 0$, such that for sufficiently large integer n, the following holds. Suppose that $p \ge \frac{C_2 \ln n}{n}$, then in the p-random-turn Hamiltonicity game played on $E(K_n)$ Maker has a polynomial time randomized strategy which is w.h.p. a winning strategy.*

Let $\mathcal{C}_p^k$ be the *p-random-turn k-vertex-connectivity game* played on the edge set of K_n, where Maker's goal is to build a spanning subgraph which is k-vertex-connected. According to [6], we can deduce using Theorem 1.1 that the critical p for the k-connectivity game is $p^* = \frac{\ln n + (k-1)\ln\ln n}{n}$. In the following theorem, we announce an efficient strategy for Maker for the $\mathcal{C}_p^k$ game. Again, together with Theorem 1.5, we provide strategies for both payers, which are typically winning strategies for appropriate p of the same order of magnitude as the probability threshold.

Theorem 1.7 *Let k be a positive integer. There exists a constant $C_3 > 0$, such that for every $p \ge \frac{C_3 \ln n}{n}$ and a sufficiently large integer n, Maker has an efficient strategy for the game $\mathcal{C}_p^k$ played on $E(K_n)$ which is w.h.p. a winning strategy.*

1.1 Notation and terminology

Our graph-theoretic notation is standard and follows that of [21]. In particular we use the following:

For a graph G, let $V = V(G)$ and $E = E(G)$ denote its set of vertices and edges, respectively. For subsets $U, W \subseteq V$ we denote by $E_G(U)$ all the edges $e \in E$ with both endpoints in U, and by $E_G(U,W)$ (where $U \cap W = \emptyset$) all the edges $e \in E$ with both endpoints in $U \cup W$ for which $e \cap U \neq \emptyset$ and $e \cap W \neq \emptyset$. We also denote by $E_M(U,W)$ (respectively, $E_B(U,W)$) all such edges claimed by Maker (respectively, Breaker). For a subset $U \subset V$, we denote $N_G(U) = \{v \in V \setminus U : \exists u \in U \ s.t. \ uv \in E(G)\}$ and $N_M(U) = \{v \in V \setminus U : \exists u \in U \ s.t. \ uv \in E(M)\}$ (or $N_B(U)$), where M (or B) is the subgraph claimed by Maker (or Breaker).

We assume that n is large enough where needed. We say that an event holds *with high probability* (w.h.p.) if its probability tends to one as n tends to infinity. For the sake of simplicity and clarity of presentation, and in order to shorten some of the proofs, no real effort is made to optimize the constants appearing in our results. We also sometimes omit floor and ceiling signs whenever these are not crucial.

We can look at the turns of a p-random-turn game as a binary sequence, where the number of bits, denoted by ℓ, is the same as the number of turns in the entire game. For every random-turn game, we define the *sequence of turns*, denoted by $\vec{t}$, to be a binary sequence $\vec{t} \in \{0,1\}^\ell$ where there is 1 in its i^{th} place if and only if it is the Maker's turn to play. That is, every "bit" of the binary sequence is 1 with probability p, where p is the probability for Maker to play.

Define a *streak* of Breaker (respectively, Maker) as a consecutive subsequence containing only turns of Breaker (Maker). A *move* of Maker is a subsequence of Maker's turns between two consecutive turns of Breaker. The *length* of Maker's move is the number of turns of this move (can be also 0). We define an *interval* of the game as a subsequence of turns the players made during the game, i.e., a consecutive subsequence of bits from $\vec{t}$. The *length* of an interval I, denoted by $|I|$, is the number of turns taken by both Maker and Breaker in this interval, i.e., the number of bits in the subsequence. For an interval I, let M_I and B_I denote the number of turns Maker and Breaker have in the interval I, respectively. For a partition $\vec{t} = \cup I_i$ of the turns of the game into disjoint intervals, we sometimes denote $M_i := M_{I_i}$ and $B_i := B_{I_i}$, for every i.

A *list* L is a sequence of numbers, where $x_i \in L$ is referred to as the i^{th} element of L. The *size* of a list L, denoted by $|L|$, is the length of L. For a sub(multi)set $\{x_{i_1}, \ldots, x_{i_k}\} \in L$, we define $L' := L \setminus \{x_{i_1}, \ldots, x_{i_k}\}$ to be the list obtained from L by removing the elements $x_{i_1}, \ldots, x_{i_k}$ and re-enumerating the elements in the natural way.

For every non-negative integer j, we denote the j^{th} *harmonic number* by H_j. That is, $H_0 = 0$, and $H_j = \sum_{i=1}^{j} \frac{1}{i}$, for every $j \geq 1$.

We also write $x \in y \pm z$ for $x \in [y-z, y+z]$.

2 Auxiliary results

In this section we present some auxiliary results that will be used throughout the paper.

2.1 Binomial distribution bounds

We use extensively the following standard bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see, e.g., [1], [15]):

Lemma 2.1 *Let $X_1, \ldots, X_n$ be independent random variables, $X_i \in \{0,1\}$ for each i. Let $X = \sum_{i=1}^{n} X_i$ and write $\mu = \mathbb{E}(X)$, then*

- $\mathbb{P}\left(X < (1-a)\mu\right) < \exp\left(-\frac{a^2\mu}{2}\right)$ *for every $a > 0$.*
- $\mathbb{P}\left(X > (1+a)\mu\right) < \exp\left(-\frac{a^2\mu}{3}\right)$ *for every $0 < a < 1$.*

2.2 Properties of random sequences

Since any p-random-turn game is determined by the sequence of turns which is a random binary sequence, it might be useful to collect a few properties of such sequences. In the following lemma, we show that binomially distributed random variables with suitable parameters that serve us in later proofs are well concentrated around their means.

Lemma 2.2 *Let $0 < \delta < 1$, $\gamma > 0$ and $C > \frac{3}{\gamma\delta^2}$ be constants. Then, for sufficiently large integer n, the following holds. Suppose that:*

(i) $0 < p = p(n) \leq 1$, *and*

(ii) $s > \frac{C \ln n}{p}$.

Then w.h.p. the following properties hold:

(1) Let $X \sim \mathrm{Bin}(\gamma s, p)$, then $\mathbb{P}\left(X \notin (1 \pm \delta)\gamma sp\right) = o\left(\frac{1}{n}\right)$,

(2) Let $X \sim \mathrm{Bin}(\gamma s, 1-p)$, then $\mathbb{P}\left(X \notin (\frac{1}{p} - 1 \pm \delta)\gamma sp\right) = o\left(\frac{1}{n}\right)$.

Proof For proving (1), applying Lemma 2.1 and using the fact that $C > \frac{3}{\gamma\delta^2}$, we obtain that

$$\mathbb{P}\left[\mathrm{Bin}(\gamma s, p) < (1-\delta)\gamma sp\right] \leq e^{-\frac{1}{2}\delta^2\gamma sp} < e^{-\frac{1}{2}\delta^2\gamma C \ln n} = o\left(\frac{1}{n}\right),$$

and

$$\mathbb{P}\left[\mathrm{Bin}(\gamma s, p) > (1+\delta)\gamma sp\right] \leq e^{-\frac{1}{3}\delta^2\gamma sp} < e^{-\frac{1}{3}\delta^2\gamma C \ln n} = o\left(\frac{1}{n}\right).$$

Therefore, $\mathbb{P}\left[\mathrm{Bin}(\gamma s, p) \notin (1 \pm \delta)\gamma sp\right] = o\left(\frac{1}{n}\right)$.

For (2), just note that it is the complement of (1).

□

2.3 Expanders

For positive constants R and c, we say that a graph $G = (V, E)$ is an $(R, c) -$ *expander* if $|N_G(U)| \geq c|U|$ holds for every $U \subseteq V$, provided $|U| \leq R$. When $c = 2$ we sometimes refer to an $(R, 2)$-expander as an R-expander. Given a graph G, a non-edge $e = uv$ of G is called a *booster* if adding e to G creates a graph G' which is Hamiltonian, or contains a path longer than a maximum length path in G.

The following lemma states that if G is a "good enough" expander, then it is also a k-vertex-connected graph.

Lemma 2.3 [Lemma 5.1 from [4]] *For every positive integer k, if $G = (V, E)$ is an $(R, c)-$expander with $c \geq k$ and $Rc \geq \frac{1}{2}(|V| + k)$, then G is k-vertex-connected.*

The next lemma due to Pósa (a proof can be found for example in [6]), shows that every connected and non-Hamiltonian expander (that is, an expander graph which does not contain a Hamilton cycle) has many boosters.

Lemma 2.4 *Let $G = (V, E)$ be a connected and non-Hamiltonian R-expander. Then G has at least $\frac{(R+1)^2}{2}$ boosters.*

The following (fairly obvious) lemma states that in expander graphs, the sizes of connected components cannot be too small.

Lemma 2.5 *Let $G = (V, E)$ be an (R, c)-expander. Then every connected component of G has size at least $R(c + 1)$.*

2.4 The game Box

In the proofs of our main results we make use of the following theorem about the ordinary game Box introduced by Chvátal and Erdős in [7] and its doubly biased version. In this general version there are n boxes, each of size s, where in each round BoxMaker claims m elements while BoxBreaker claims b elements. BoxBreaker's goal is to claim at least one element from each box. Assume that BoxMaker plays first. We denote this game by $Box(n \times s; m : b)$. The following theorem was proved in [11]. For completeness, we give here a slightly different proof, including BoxBreaker's strategy and the running time argument.

Theorem 2.6 *Assume that s, m, b and n are positive integers that satisfy $s > \frac{m}{b} \cdot (H_n + b)$. Then BoxBreaker has a winning polynomial time strategy for the game $Box(n \times s; m : b)$.*

Proof The proof is a straightforward adaptation of the argument in [13] (see Chapter 3.4.1) where the case $b = 1$ is handled. At any point during the game we say that a box a is *active* if it was not previously touched by BoxBreaker. Denote by A the set of the active boxes. A box $a \in A$ is *minimal* if the number of free elements in a is minimal in A. BoxBreaker's strategy goes as follows. Before each turn, BoxBreaker looks only at the active boxes and claims an element from the minimal box (breaking ties arbitrarily).We show now that this is a winning strategy for BoxBreaker.

Assume to the contrary that BoxMaker wins the game in the k^{th} round ($1 \leq k \leq \lfloor \frac{n}{b} \rfloor$). By relabeling the boxes, we can assume that for every $0 \leq i \leq k-2$, BoxBreaker claims elements from the boxes $ib+1, \dots (i+1)b$ in the $(i+1)^{th}$ round of the game and that in the k^{th} round BoxMaker fully claims box $(k-1)b+1$. For every $i \in A \cap \{1, \dots, (k-1)b+1\}$ denote by c_i the number of free elements in the box i at any point during the game. For $0 \leq j \leq k-1$, let

$$\varphi(j) := \frac{1}{(k-1-j)b+1} \sum_{i=jb+1}^{(k-1)b+1} c_i$$

denote the potential function of the game just before BoxMaker's $(j+1)^{th}$ move. Note that $\varphi(0) = s$ and $\varphi(k-1) = c_{(k-1)b+1} \leq m$. For every $0 \leq j \leq k-1$, in the $(j+1)^{th}$ move, BoxMaker decreases $\varphi(j)$ by at most $\frac{m}{(k-1-j)b+1}$ and in the following move, BoxBreaker claims elements from the b minimal boxes. Therefore, $\varphi(j+1) \geq \varphi(j) - \frac{m}{(k-1-j)b+1}$. It follows that

$$\begin{aligned}
\varphi(k-1) &\geq \varphi(k-2) - \frac{m}{b+1} \geq \cdots \geq \varphi(0) - m\left(\frac{1}{b+1} + \frac{1}{2b+1} + \cdots + \frac{1}{(k-1)b+1}\right) \\
&= \varphi(0) - m\sum_{i=1}^{k-1}\frac{1}{ib+1} \geq s - m\sum_{i=1}^{k-1}\frac{1}{ib+1} = s - m\left(\sum_{i=0}^{k-1}\frac{1}{ib+1} - 1\right) \\
&\geq s - m\left(\sum_{i=0}^{\lfloor \frac{n}{b} \rfloor - 1}\frac{1}{ib+1} - 1\right) \geq s - \frac{m}{b}\cdot H_n > m,
\end{aligned}$$

and this is clearly a contradiction.

Note that the suggested strategy is polynomial in n. Indeed, before every turn BoxBreaker only needs to find the boxes that are active (run over at most n elements), and then the minimal boxes (again, run over at most n elements). Since the number of turns is also polynomial in n, we have that in total the calculation time in polynomial in n. □

3 Proofs

In this section we prove Theorems 1.2 – 1.6.

3.1 Proof of Theorem 1.2

First, we prove Theorem 1.2.

Proof Since the property "BoxBreaker has a winning strategy in the game $Box_p(a_1, \dots, a_n)$" is monotone increasing with respect to the parameters $a_1, \dots, a_n$, it is enough to prove Theorem 1.2 for the case $a_i = s = \frac{C}{p} \ln n$ for every $1 \leq i \leq n$. In this case we write $Box_p(n \times s)$.

In the proposed strategy, BoxBreaker claims elements from different boxes by simulating a deterministic $Box(n \times s'; m : b)$ game for appropriate parameters. In this simulated game, the number of boxes is the same as in the original game. During the game, BoxBreaker divides the sequence of turns of the game into disjoint intervals, and simulates the $Box(n \times s'; m : b)$ game as follows: In each of his turns in the interval I_1, BoxBreaker claims an element from an arbitrary box. Assume now that it is BoxBreaker's turn to move in some interval I_i with $i > 1$. BoxBreaker considers all the turns of BoxMaker in the previous interval, together with all of his turns in the current interval as one round of the simulated $Box(n \times s'; m : b)$ game and follows the strategy from Theorem 2.6. A problem might occur if a box F is full and some of its elements have been claimed by BoxMaker in the current interval. This situation can cause a problem since in the simulated game BoxBreaker ignores moves of BoxMaker in the current interval and it might happen that playing the simulated game, BoxBreaker needs to claim an element from F (and he can not!). In order to overcome this difficulty we take a suitable $s' < s$ (to be defined later) that in particular satisfies that $s - s'$ is at least the number of turns that BoxMaker makes (w.h.p.) in such an interval; such s' should not be very small to allow BoxBreaker to win $Box(n \times s', m : b)$.

Let $C > 0$, let $\delta > 0$ and $\gamma > 0$ be such that $C(1 - \gamma(2 + \delta))(1 - \delta) > (1 + \delta)^2$ and $C > \frac{3}{\gamma\delta^2}$ (for example, we can take $\delta = \frac{1}{2}$, $\gamma = \frac{1}{4}$ and $C = 49$). Let ℓ be the length of the game and let $T = \{0, 1\}^\ell$ denotes the set of all binary sequences of length ℓ (that is, T is the set of all potential turn-sequences that determine the game $Box_p(a_1, \dots, a_n)$), and divide the interval $[1, \ell]$ into disjoint subintervals $I_1, \dots, I_r$, such that $r = \lceil \frac{\ell}{\gamma s} \rceil$, $I_i = [\gamma s(i - 1) + 1, \gamma s i]$ for every $1 \le i \le r - 1$ and $I_r = [\ell] \setminus (\bigcup_i I_i)$. Observe that $|I_i| = \gamma s$ for each $1 \le i \le r - 1$ and that $0 \le |I_r| \le \gamma s$.

Using Lemma 2.2 and the union bound, we have that w.h.p. in each interval (except possibly the last one), BoxBreaker plays at least $(1 - \delta)\gamma s p$ turns, and BoxMaker plays at most $(1 + \delta)\gamma s(1 - p)$ turns. Since playing extra turns can not harm BoxBreaker, we can assume that in each interval (except possibly the last one) BoxMaker played exactly $(1 + \delta)\gamma s(1 - p)$ turns, and BoxBreaker played exactly $(1 - \delta)\gamma s p$ turns.

Let S_B be a winning strategy for BoxBreaker in the deterministic game $Box(n \times s'; m : b)$ where $s' = (1 - \gamma)s$, $m = (1 + \delta)\gamma s(1 - p)$ and $b = (1 - \delta)\gamma s p$. The existence of such a strategy follows from Theorem 2.6 and the fact that $s' = (1 - \gamma)s = \frac{(1-\gamma)C \ln n}{p} > \frac{m}{b} \cdot (H_n + b)$ (the last inequality follows from the assumptions on δ and γ).

Strategy S': In each of his turns in the intervals I_1 and I_r BoxBreaker claims elements from arbitrary free boxes. For every $2 \le i \le r - 1$, BoxBreaker plays his j^{th} turn in I_i as follows: Let S_B be the strategy proposed for BoxBreaker for the game $Box(n \times s'; m : b)$ as described in Theorem 2.6, with $m = (1 + \delta)\gamma s(1 - p)$, $b = (1 - \delta)\gamma s p$ and $s' = (1 - \gamma)s$. BoxBreaker simulates the game $Box(n \times s'; m : b)$ and pretends that all the turns of BoxMaker in I_{i-1} correspond to his $(i - 1)^{st}$ move in the simulated game and the turns of BoxBreaker in I_i correspond to his $(i - 1)^{st}$ move in the simulated game. Then, BoxBreaker plays according to the strategy S_B (at this point, BoxBreaker ignores BoxMaker's turns in I_i).

If at some point during the game BoxBreaker is unable to follow the proposed strategy then BoxBreaker forfeits the game.

Since following S_B BoxBreaker touches every box at least once, then S' is w.h.p. a winning strategy for BoxBreaker for the game $Box_p(a_1, \dots, a_n)$. It thus remains to prove that BoxBreaker w.h.p. can follow the strategy S'.

Indeed, following S_B, BoxBreaker can ensure that BoxMaker's largest box is of size less then s'. All in all, at any point during the game, the largest box that BoxMaker has been able to build is at most the maximal size he can build in the game $Box(n \times s'; m : b)$ plus the number of elements claimed in the current interval. That is, $s' + m < s$ and BoxBreaker is w.h.p. the winner of the game.

Before each turn, BoxBreaker first finds out in what interval he is playing now, which requires at most n^2 queries. After that, he plays exactly as in the strategy S_B, which we showed that is polynomial. Since the number of turns is also polynomial we have that in total the calculation time of the strategy S' is polynomial in n. This completes the proof. $\square$

The following Corollary is obtained by using Theorem 1.2. In this claim, we study the game $Box_p(n \times s, d)$, which is a version of the game Box. This game will be used later for proving Theorems 1.6 and 1.7. In the game $Box_p(n \times s, d)$ there are n boxes, each of size s, and two players, d-Maker and d-Breaker. In each turn d-Breaker plays with probability p independently at random and claims a previously unclaimed element. The goal of d-Breaker is to have exactly d elements of his in every box after exactly dn turns of d-Breaker.

Claim 3.1 *Let $d > 0$ be an integer and $C > 0$ be as in Theorem 1.2, and let n be a sufficiently large integer. Let $0 < p := p(n) \leq 1$ and let $s \geq \frac{Cd \ln dn}{p}$. Then there exists a polynomial time strategy that w.h.p. is a winning strategy for d-Breaker in the game $Box_p(n \times s, d)$ within dn turns.*

Proof At the beginning of the game, d-Breaker partitions each of the n boxes into d boxes, each of size $\frac{s}{d}$. Then, d-Breaker simulates the game $Box_p(dn \times \frac{s}{d})$ while pretending to be BoxBreaker. During the simulated game, a box F is called *free* if d-Breaker (as BoxBreaker) has not touched it yet, otherwise it is called *busy*. Note that since $\frac{s}{d} \geq \frac{Cd \ln dn}{dp} = \frac{C \ln dn}{p}$, it follows by Theorem 1.2 that there is a strategy that w.h.p. ensures d-Breaker's (as BoxBreaker) win in the game $Box_p(dn \times \frac{s}{d})$. Following an optimal strategy of BoxBreaker, it is clear that d-Breaker (as BoxBreaker) never touches busy boxes. All in all, by playing according to the strategy described above and by Theorem 1.2, it follows that w.h.p. d-Breaker wins the game $Box_p(dn \times \frac{s}{d})$ within dn turns. Hence, d-Breaker wins also the game $Box_p(n \times s, d)$. $\square$

3.2 Proof of Theorem 1.3

Next, we prove Theorem 1.3.

Proof BoxMaker's strategy goes as follows. After each turn of BoxBreaker, BoxMaker identifies a box F which has not been touched by BoxBreaker so far (if there is no such box then he forfeits the game), and tries to claim all the elements of F in his next consecutive turns (until the next turn of BoxBreaker). Clearly, this strategy has a polynomial calaulation time.

Note that there are n such trials (there are n boxes, so after n turns of BoxBreaker the game trivially ends) and all of them are independent. Moreover, the number of consecutive turns of BoxMaker in the i^{th} trial, X_i, is distributed according to the geometric distribution $X_i \sim Geo(p)$. It thus follows that the probability for BoxMaker, in the i^{th} trial, to claim all the elements of some box F is at least $(1-p)^{|F|} \geq (1-p)^{\frac{(1-\varepsilon)\ln n}{-\ln(1-p)}} = n^{-1+\varepsilon}$. All in all, the probability that BoxMaker loses the game (that is, the probability that BoxMaker fails to fill a box in his n attempts), is bounded from above by

$$(1 - n^{-1+\varepsilon})^n \leq e^{-n^{\varepsilon}} = o(1).$$

This completes the proof. $\square$

3.3 Proof of Theorem 1.5

In this subsection we prove Theorem 1.5.

Proof Let $\varepsilon' > 0$. It is enough to prove the theorem for $p = \frac{(1-\varepsilon')\ln n}{n}$. First we present a strategy for Breaker and then prove that w.h.p. this is indeed a winning strategy. At any point during the game, if Breaker is not able to follow the proposed strategy then he forfeits the game. Breaker's strategy is divided into the following two stages:

Stage I: In the first $\frac{1}{p^2}$ turns of the game, Breaker builds a clique C of size $k = \frac{1}{100p}$, and ensures that all the vertices in this clique are isolated in Maker's graph.

Stage II: In this stage, Breaker claims all edges between a vertex $v \in V(C)$ and $V(K_n) \setminus V(C)$.

It is evident that the proposed strategy is a winning strategy. It thus suffices to show that w.h.p. Breaker can follow the proposed strategy without forfeiting the game. We consider each stage separately.

Stage I: First we show that, throughout the first $\frac{1}{p^2}$ turns of the game, w.h.p. there are $n - o(n)$ vertices which are isolated in Maker's graph. Indeed, since Lemma 2.1 implies that for $X \sim \text{Bin}\left(\frac{1}{p^2}, p\right)$

$$\mathbb{P}\left(X > \frac{2}{p}\right) \leq e^{-\frac{1}{3p}} = e^{-\frac{n}{3(1-\varepsilon')\ln n}} = o(1),$$

it follows that w.h.p., throughout Stage I Maker plays at most $\frac{2}{p} = \frac{2n}{(1-\varepsilon')\ln n} = o(n)$ turns. Therefore, in total, w.h.p. Maker is able to touch at most $o(n)$ vertices.

Next, we show that w.h.p. Breaker can build the desired clique. Throughout Stage I, Breaker creates a clique C such that for every $v \in V(C)$, v is isolated in Maker's graph. Initially, $V(C) = \emptyset$. After each turn of Maker, Breaker updates $V(C) := V(C) \setminus \{x, y\}$, where xy is the edge that has just been claimed by Maker. Assume that Maker has just claimed an edge, that $|V(C)| < k$ and that it is now Breaker's turn. Let $v \in V(K_n) \setminus V(C)$ be a vertex which is isolated in Maker's graph (such a vertex exists since there are at least $n - o(n)$ vertices which are isolated in Maker's graph). In the following turns, until Maker's next move, Breaker tries to claim all edges vu with $u \in V(C)$. If Breaker has enough turns to do so, then he updates $V(C) := V(C) \cup \{v\}$. Otherwise, $V(C) := V(C)$. Note that in every turn Maker can decrease the size of C by at most one vertex, while in every streak of length $\frac{1}{100p}$, Breaker can increase the size of C by at least one. In order to show that Breaker can follow Stage I, in the following claim we show that w.h.p. Maker cannot stop Breaker from increasing the size of C up to k in the first $\frac{1}{p^2}$ turns.

Claim 3.2 *Breaker can follow (w.h.p.) the proposed strategy for Stage I, including the time limit.*

Proof As mentioned above, w.h.p. the number of Maker's turns in the first $\frac{1}{p^2}$ turns of the game is at most $\frac{2}{p}$. We wish to show that in the first $\frac{1}{p^2}$ turns of the game, Breaker can increase the size of the current clique C to the desired size. For this goal we wish to count the number of streaks of Breaker of length $\frac{1}{100p}$ and to show that w.h.p. there are more than $\frac{2}{p} + \frac{1}{100p}$ such streaks. Since in each such streak Breaker increases the size of C by at least one vertex, the claim will follow. Partition the sequence of the first $\frac{1}{p^2}$ turns of the game into disjoint intervals $I_1, \ldots, I_t$, $t = \frac{100}{p}$, such that for each $1 \leq i \leq t$, $|I_i| = \frac{1}{100p}$. Such an interval is called *successful* if all the turns in it belong to Breaker, and the probability for a successful interval is $(1-p)^{\frac{1}{100p}}$. Let X be a random variable which represents the number of successful intervals. Since $X \sim \mathrm{Bin}(t, (1-p)^{\frac{1}{100p}})$, it follows that

$$\mathbb{P}\left(X > \frac{3}{p}\right) = 1 - \mathbb{P}\left(X \leq \frac{3}{p}\right).$$

Using Lemma 2.1 and the fact that $(1-p)^{\frac{1}{p}} > e^{-2}$ for $p < \frac{1}{2}$, it follows that for $Y \sim \mathrm{Bin}(\frac{100}{p}, e^{-1/50})$

$$\mathbb{P}\left(X \leq \frac{3}{p}\right) \leq \mathbb{P}\left(Y \leq \frac{3}{p}\right) \leq \mathbb{P}\left(Y \leq \tfrac{1}{2} \cdot \tfrac{100}{p} \cdot e^{-1/50}\right) = o(1).$$

As mentioned before, after every streak of this length, Breaker adds one new vertex to his clique. Thus in total, Breaker w.h.p. adds more than $\frac{3}{p}$ new vertices to his clique. All in all, after $\frac{1}{p^2}$ turns of the game and since $\frac{3}{p} - \frac{2}{p} > \frac{1}{100p}$, w.h.p. Breaker is able to build a clique of size at least $\frac{1}{100p}$.

□

Stage II: For every $v \in V(C)$, let $F_v = \{vu : u \in V(K_n) \setminus V(C)\}$, and note that $|F_v| = n - k \leq n$. At this stage Breaker simulates the game $Box_p(k \times (n-k))$, where the boxes are F_v $(v \in V(C))$. Breaker plays this simulated game as BoxMaker according to a strategy that w.h.p. ensures BoxMaker's win. The existence of such a strategy follows from Theorem 1.3 and Remark 1.4 by showing that the assumptions are fulfilled. For this aim, observe that the number of boxes is $k = \frac{1}{100p} = \frac{n}{100(1-\varepsilon')\ln n}$, each of which is of size $n - k \leq n$, and therefore for $\varepsilon = \frac{\varepsilon'}{2}$ and sufficiently large n,

$$\frac{(1-\frac{\varepsilon'}{2})\ln k}{p} = \frac{(1-\frac{\varepsilon'}{2})(1-o(1))\ln n}{p} > n \geq n - k.$$

□

3.4 Proof of Theorem 1.6

In this subsection we prove Theorem 1.6. The proof of this theorem is based on ideas from [17], combined with techniques introduced in this paper which enable us to translate them to the p-random-turn setting. One main ingredient in the proof is the ability of Maker to build a "good" expander fast. This is shown in the following lemma:

Lemma 3.3 *For every positive integer k and a positive constant $\delta < (44ke)^{-8}$, there exists $C_1 > 0$ for which the following holds. Suppose that $p \geq \frac{C_1 \ln n}{n}$, then in the p-random-turn game played on the edge set of K_n, Maker has an efficient randomized strategy which w.h.p. enables him to create an $(R, 2k)$-expander (where $R = \delta n$), within $\frac{n^2}{\ln n}$ turns of the game.*

Proof Let $d = 16k$. Let $0 < \beta \le \frac{1}{5}$ be a constant and let $C > 0$ be as in Theorem 1.2. Let $C_1 = \frac{2Cd}{\beta}$.

At the beginning of the game, Maker assigns edges of K_n to vertices so that each vertex gets about $\frac{n}{2}$ edges incident to it. To do so, let D_n be any tournament on n vertices such that for every vertex $v \in V$, $|N^+(v)| = |N^-(v)| \pm 1$ if n is even and $|N^+(v)| = |N^-(v)|$ if n is odd. For each vertex $v \in V(D_n)$, define A_v to be the set of all edges (arcs) of D_n whose tail is v. Note that for every $v \in V(D_n)$ we have that $|A_v| = \lfloor \frac{n-1}{2} \rfloor$ or $|A_v| = \lceil \frac{n-1}{2} \rceil$ and that all the A_v's are pairwise disjoint.

Now, note that if G is an $(R, 2k)$-expander, then $G \cup \{e\}$ is also an $(R, 2k)$-expander for every edge $e \in E(K_n)$. Therefore, claiming extra edges can not harm Maker in his goal of creating an expander and we can assume $p = \frac{C_1 \ln n}{n}$ (if p is larger then it is only in favor of Maker).

Our goal is to provide Maker with a strategy for the p-random-turn game played on $E(K_n)$ such that by following this strategy, w.h.p. Maker's graph in the end of this game will be an $(R, 2k)$-expander. Moreover, we show that Maker can achieve this goal (w.h.p.) within $\frac{n^2}{\ln n}$ turns of the game. In this strategy, Maker pretends to be d-Breaker and simulates a game $Box_p(n \times s, d)$ with appropriate parameters, where the boxes are $\{F_v : v \in V(K_n)\}$ (that is, for each vertex $v \in V(K_n)$, there exists a corresponding box F_v). Every time that Breaker claims an edge in the graph, Maker looks at the corresponding arc in A_v and pretends that Breaker has claimed (as d-Maker) an element in the box F_v, in the simulated game $Box_p(n \times s, d)$. In case that Breaker claimed an arc in an A_v for which the corresponding F_v is already full, by faking moves, Maker pretends that d-Maker just claimed an element of some arbitrary free box F_w. By following a winning strategy for d-Breaker, in each turn Maker replies by claiming an element in some box F_v in the simulated game. He then translates this move to the set $E(K_n)$ by claiming an edge which corresponds to a **random** free arc in A_v. The game stops when the simulated game is over, and we then show that w.h.p., Maker's graph at the end of this procedure is an $(R, 2k)$-expander. Now we are ready to present Maker's strategy more formally.

Consider a game $Box_p(n \times s, d)$ for $d = 16k$ and $s = \beta n$, and let S' be a strategy for d-Breaker which w.h.p. ensures his win in this game $Box_p(n \times s, d)$, where the boxes of the simulated game $Box_p(n \times s, d)$ are $\{F_v : v \in V(K_n)\}$ (the existence of S' is guaranteed by Claim 3.1 and the fact that $s \ge \frac{2Cd \ln n}{p} \ge \frac{Cd \ln dn}{p}$). First, we present a strategy for Maker in the p-random-turn game and then prove that by following this strategy, w.h.p. Maker can build an $(R, 2k)$-expander. Maker's strategy is as follows:

Maker's strategy: Throughout the game, whenever Breaker claims an edge which corresponds to an arc $e \in A_v$, Maker pretends that d-Maker has claimed an element in the box F_v of the corresponding game $Box_p(n \times s, d)$. If the box F_v is already full, then Maker pretends that d-Maker has claimed an element in some arbitrary available box F_w. In his turns, Maker plays as follows. Assume that according to the strategy S', Maker (as d-Breaker) is to play in F_u. In this case, Maker pretends that he claims an element in F_u and claims a free element from A_u **at random**. The game ends when the simulated games ends. We denote this strategy (for creating an expander graph) by S_{exp}.

Note that $|A_v| \ge \frac{n-1}{2} - 1$ and $|F_v| \le \frac{n}{5}$ so it is evident that w.h.p. Maker can follow the proposed strategy. Therefore, following S', by the end of the simulated game, w.h.p. the number of Maker's elements in each box F_v is exactly $16k$. Then, following S_{exp} we have that

$|A_v| = 16k$. That is, w.h.p. Maker is able to build a graph with minimum degree at least $16k$. Using Lemma 2.1, one can see that w.h.p. after $\frac{n^2}{\ln n}$ turns of the game, Maker played more than $16kn$ turns. Therefore, the total number of turns in the simulated game is bounded by $\frac{n^2}{\ln n}$. Moreover, the total number of elements claimed from each A_v before Maker claimed his $16k$ elements, is at most the total number of elements claimed from each box in the simulated game, that is at most βn. Since $\beta \leq \frac{1}{5}$ and $|A_v| \geq \frac{n-1}{2}$, then $|A_v| - \beta n > \frac{n}{4}$ for every v. Thus, at any point in this stage, as long as F_v is still available, there are at least $\frac{n}{4}$ free elements in each box A_v. For some edge e in Maker's graph, we say that $e = \{u, v\}$ was *chosen* by the vertex v if according to the orientation D_n, $e \in A_v$ (as an arc).

We now prove that Maker's graph (after forgetting the orientation D_n) is w.h.p. an $(R, 2k)$-expander. Indeed, if we suppose that Maker's graph is not a $(R, 2k)$-expander, then there is a subset A, $|A| = a \leq R$ in Maker's graph M, such that $N_M(A) \subset B$, where $|B| = 2ka - 1$. Since the minimum degree in Maker's graph is $16k$ and $k \geq 1$, we can assume that $a \geq 5$ and there are at least $8ka$ of Maker's edges incident to A. Then at least $4ka$ of those edges were chosen by vertices from A – and all went into $A \cup B$, or at least $4ka$ of those edges were chosen by vertices from B – and all went into A. In the first case, assume that at some point during the game Maker chose an edge with one vertex $v \in A$ and whose second end point is in $A \cup B$. This means that the box F_v is still available, therefore at that point of the game, there are at least $\frac{n}{4}$ unclaimed edges incident to v. The probability that at that point Maker chose an edge at v whose second endpoint belongs to $A \cup B$ is thus at most $\frac{|A \cup B| - 1}{n/4}$. It follows that the probability that there are at least $4ka$ edges are chosen by vertices of A that end up in $A \cup B$ is at most $\left(\frac{(2k+1)a-2}{n/4}\right)^{4ka}$. For the second case, recall that at most $16k|B|$ edges of Maker were chosen by the vertices of B. Assume that at least $4ka$ of them are incident to A. For some vertex $u \in B$, the probability Maker chose its end point to be in A is at most $\frac{|A|}{n/4}$. Therefore, the probability that there are at least $4ka$ such edges is at most $\binom{16k|B|}{4ka}\left(\frac{a}{n/4}\right)^{4ka}$. Putting it all together, the probability that there are at least $8ka$ edges between A and $A \cup B$ is at most

$$\left(\frac{(2k+1)a-2}{n/4}\right)^{4ka} + \binom{16k|B|}{4ka}\left(\frac{a}{n/4}\right)^{4ka} < \left(\frac{44eka}{n}\right)^{4ka}.$$

Therefore the probability that there is such a pair of sets A, B as above is at most

$$\begin{aligned}\sum_{a=5}^{R}\binom{n}{a}\binom{n-a}{2ka-1}\left(\frac{44eka}{n}\right)^{4ka} &\leq \sum_{a=5}^{R}\left(\frac{ne}{a}\left(\frac{ne}{2ka}\right)^{2k}\left(\frac{44eka}{n}\right)^{4k}\right)^{a} \\ &= \sum_{a=5}^{R}\left(\frac{e^{6k+1}k^{2k}44^{4k}}{2^{2k}}\left(\frac{a}{n}\right)^{2k-1}\right)^{a} = o(1).\end{aligned}$$

The last equality is due to the fact that for $5 \leq a \leq \sqrt{n}$

$$\begin{aligned}\left(\frac{e^{6k+1}k^{2k}44^{4k}}{2^{2k}}\left(\frac{a}{n}\right)^{2k-1}\right)^{a} &\leq \left(\frac{e^{6k+1}k^{2k}44^{4k}}{2^{2k}}\left(\frac{\sqrt{n}}{n}\right)^{2k-1}\right)^{a} \\ &= \left(\Theta\left(\frac{1}{n^{k-0.5}}\right)\right)^{a} \leq \left(\Theta\left(\frac{1}{n^{k-0.5}}\right)\right)^{5} \\ &= o\left(\frac{1}{n}\right),\end{aligned}$$

and for $\sqrt{n} \leq a \leq R$ with $R = \delta n$ where $\delta < (44ke)^{-8}$ is a constant,

$$\begin{aligned}
\left(\frac{e^{6k+1}k^{2k}44^{4k}}{2^{2k}}\left(\frac{a}{n}\right)^{2k-1}\right)^{a} &\leq \left(\frac{e^{6k+1}k^{2k}44^{4k}}{2^{2k}}\left(\frac{R}{n}\right)^{2k-1}\right)^{\sqrt{n}} \\
&= \left(\frac{e^{6k+1}k^{2k}44^{4k}}{2^{2k}}\cdot\delta^{2k-1}\right)^{\sqrt{n}} \\
&< \left(e^{6k+1}k^{2k}44^{4k}\cdot(44ke)^{-16k+8}\right)^{\sqrt{n}} \\
&= o\left(\frac{1}{n}\right).
\end{aligned}$$

It follows that Maker is able to create an $(R, 2k)$-expander w.h.p. in at most $\frac{n^2}{\ln n}$ turns of the game. □

Using the above lemma, we show now that for $p := p(n) = \Omega\left(\frac{\ln n}{n}\right)$, Maker can build w.h.p. a Hamilton cycle playing on the edge set of K_n.

Proof of Theorem 1.6. Let $C > 0$ be as in Theorem 1.2. Let $\delta = (45e)^{-8}$, $\beta = \frac{1}{5}$, $d = 16$ and let $C_2 = \frac{2Cd}{\beta}$.

Let D_n be any tournament on n vertices such that for every vertex $v \in V$, $|N^+(v)| = |N^-(v)| \pm 1$ if n is even and $|N^+(v)| = |N^-(v)|$ if n is odd. For each vertex $v \in V(D_n)$, define A_v to be the set of all edges (arcs) of D_n whose tail is v. Note that for every $v \in V(D_n)$ we have that $|A_v| = \lfloor \frac{n-1}{2} \rfloor$ or $|A_v| = \lceil \frac{n-1}{2} \rceil$ and that all the A_v's are pairwise disjoint. Maker's strategy is composed of three stages:

Stage I – creating an expander: Following strategy S_{exp} from Lemma 3.3 for $k = 1$ and $R = \delta n$, Maker creates an R-expander before both players claimed in total $\frac{n^2}{\ln n}$ edges in the graph.

Stage II – creating a connected expander: Denote the graph that Maker built in Stage I by M. Then M is a R-expander, and by Lemma 2.5 the size of every connected component of M is at least $3R$. It follows that there are at most $\frac{n}{3R}$ connected components in M. In this stage, Maker will turn his graph M to a connected R-expander. Observe that there are at least $(3R)^2 = 9\delta^2 n^2$ edges of K_n between any two such components. To connect all the connected components into a connected graph, Maker will need at most $\frac{n}{3R} = \frac{1}{3\delta}$ turns. In each turn of this stage Maker finds two connected components and claims one free edge between them. Since finding connected components can be done in polynomial time, Maker has an efficient deterministic strategy to play this stage. Using Lemma 2.1, in the next $\frac{n^2}{\ln \ln n}$ turns Maker plays more than $n > \frac{1}{3\delta}$ turns. But until now, Breaker was able to claim at most $\frac{n^2}{\ln n} + \frac{n^2}{\ln \ln n} < \frac{2n^2}{\ln \ln n} \ll 9\delta^2 n^2$ edges. Therefore, Breaker cannot block Maker from achieving his goal. We denote the new graph Maker created by M'. It is evident that M' is still an R-expander, since adding extra edges to the graph can not harm this property.

Stage III – completing a Hamilton cycle: If M' contains a Hamilton cycle, then we are done. Otherwise, by Lemma 2.4, M' contains at least $\frac{(R+1)^2}{2}$ boosters. Observe that after adding a booster, the current graph is still an R-expander and therefore also contains at least $\frac{(R+1)^2}{2}$ boosters. Clearly, after adding at most n boosters, M' becomes Hamiltonian. We show now that also in this stage, Maker reaches his goal after less than $\frac{n^2}{\ln \ln n}$ turns of the game.

Since we are looking for a polynomial-time strategy, we need to find an efficient algorithm in order to follow Stage III. Observe that since the number of boosters during this stage is always quadratic in n, Maker can use a simple randomized strategy to add enough boosters. First, Lemma 2.1 implies that in the next $\frac{n^2}{\ln\ln n}$ turns of the game, Maker has at least $\frac{4n}{\delta^2}$ turns. Therefore, in order to complete a Hamilton cycle, in the next $\frac{4n}{\delta^2}$ turns of Maker, he claims a **random** unclaimed edge from the graph. We are now looking for the probability for such edge to be a booster. There are at most $\frac{n^2}{2} - \frac{n}{2}$ unclaimed edges, and according to Lemma 2.4 there are at least $\frac{(\delta n+1)^2}{2}$ boosters in M'. Since by the end of the game both players claimed at most $\frac{3n^2}{\ln\ln n}$ edges, the number of free boosters after each turn of Maker at any point of this stage is at least $\frac{(\delta n+1)^2}{2} - \frac{3n^2}{\ln\ln n}$. Therefore, in each turn of Maker, until he was able to claim n boosters, the probability for Maker to claim a booster is at least

$$\frac{\frac{(\delta n+1)^2}{2} - \frac{3n^2}{\ln\ln n}}{\binom{n}{2}} \geq \frac{(\delta n/2)^2}{\binom{n}{2}} \geq \frac{\delta^2}{3}.$$

Let Y be the number of boosters Maker claimed in $\frac{4n}{\delta^2}$ turns. Then by Lemma 2.1,

$$\mathbb{P}(Y < n) \leq \mathbb{P}\left(\mathrm{Bin}(\frac{4n}{\delta^2}, \frac{\delta^2}{3}) < n\right) \leq e^{-\left(\frac{1}{4}\right)^2 \frac{4}{3} n} = o\left(\frac{1}{n}\right).$$

Thus in the next $\frac{4n}{\delta^2}$ turns of Maker he is typically able to claim at least n boosters. All in all, in the next $\frac{n^2}{\ln\ln n}$ turns of the game, Maker was typically able to claim n boosters and thus build a Hamilton cycle. □

3.5 Proof of Theorem 1.7

Using Lemma 3.3 we show a strategy for Maker, which is typically a winning strategy, for the game $\mathcal{C}_p^k$.

Proof Let $C > 0$ be as in Theorem 1.2. Let $\delta = (45ke)^{-8}$, $\beta = \frac{1}{5}$, $d = 16k$ and let $C_3 = \frac{2Cd}{\beta}$. Maker's strategy goes as follows:

Stage I: Following strategy S_{exp} from Lemma 3.3 for $R = \delta n$, Maker creates an $(R, 2k)$-expander. According to Lemma 3.3 Maker is typically able to create a $(R, 2k)$-expander before both players claimed together $\frac{n^2}{\ln n}$ edges.

Stage II: Maker makes his graph an $(\frac{n+k}{4k}, 2k)$-expander in $\frac{n^2}{\ln\ln n}$ further turns of the game. During this stage, in every turn of Maker he claims a random edge from the graph (if the edge is already claimed, then Maker chooses an arbitrary free edge). By Lemma 2.1, during the next $\frac{n^2}{\ln\ln n}$ turns of the game, Maker has at least An turns where $A > \frac{3\delta - 2\delta\ln\delta}{-\ln\left(1-\frac{\delta^2}{3}\right)}$. It remains to prove that if Maker claims An edges randomly, then w.h.p. Maker's graph is a $(\frac{n+k}{4k}, 2k)$-expander. It is enough to prove that w.h.p. $E_M(U,W) \neq \emptyset$ for every two subsets $U, W \subseteq V$ such that $|U| = |W| = R$. Indeed, if there exists a subset $X \subseteq V$ of size $R \leq |X| \leq \frac{n+k}{4k}$ such that $|X \cup N_M(X)| < (2k+1)|X|$, then there are two subsets $U \subseteq X$ and $W \subseteq V \setminus (X \cup N_M(X))$ such that $|U| = |W| = R$ and $E_M(U,W) = \emptyset$. We now prove that w.h.p. $E_M(U,W) \neq \emptyset$ for every $|U| = |W| = R$ after An turns of Maker. Let U, W be two subsets such that $|U| = |W| = R$. Recall that in the entire game, Breaker claims at most $\frac{2n^2}{\ln\ln n}$ edges. Thus the number of free edges between U and W at any point throughout this stage is at least $|U||W| - \frac{2n^2}{\ln\ln n} > \frac{\delta^2 n^2}{3}$ for a large n. Then the probability that Maker claims an edge $e \in E(U,W) \setminus E_B(U,W)$ is at least $\frac{\delta^2 n^2/3}{\binom{n}{2}} \geq \frac{\delta^2}{3}$. So at the end of Stage II,

$$\mathbb{P}\left(E_M(U,W) = \emptyset\right) \leq \left(1 - \frac{\delta^2}{3}\right)^{An}.$$

Using the union bound, we get that the probability that there exist two subsets U, W, $|U| = |W| = R$ such that $E_M(U,W) = \emptyset$ is at most

$$\begin{aligned}
\binom{n}{\delta n}\binom{n}{\delta n}\left(1 - \frac{\delta^2}{3}\right)^{An} &\leq \left(\frac{en}{\delta n}\right)^{\delta n}\left(\frac{en}{\delta n}\right)^{\delta n}\left(1 - \frac{\delta^2}{3}\right)^{An} \\
&\leq \left(\frac{e}{\delta}\right)^{2\delta n}\left(1 - \frac{\delta^2}{3}\right)^{An} \\
&= e^{2\delta n - 2\delta n \ln \delta + An \ln\left(1 - \frac{\delta^2}{3}\right)} \\
&= o(1).
\end{aligned}$$

Then, w.h.p., by Lemma 2.3, since $(\frac{n+k}{4k}) \cdot 2k \geq \frac{1}{2}(|V| + k)$, Maker's graph is k-connected and w.h.p. he wins the game. $\square$

Acknowledgement. The authors wish to thank Noga Alon for insightful discussions. The authors are also grateful to the anonymous referees for careful reading of the first version of the paper and many helpful remarks.

References

[1] N. Alon and J. H. Spencer, **The Probabilistic Method**, 3rd ed., Wiley, 2008.

[2] J. Beck, **Combinatorial Games: Tic-Tac-Toe Theory**, Encyclopedia of Mathematics and Its Applications 114, Cambridge University Press, 2008.

[3] M. Bednarska and T. Łuczak. *Biased positional games for which random strategies are nearly optimal*, Combinatorica 20 (2000), 477–488.

[4] S. Ben-Shimon, A. Ferber, D. Hefetz and M. Krivelevich, *Hitting time results for Maker-Breaker games*, Random Structures & Algorithms 41 (2012), 23–46.

[5] B. Bollobás, *The evolution of sparse graphs*, Graph Theory and Combinatorics, Academic Press (1984), 35–57.

[6] B. Bollobás, **Random graphs**, Vol. 73, Cambridge University Press, 2001.

[7] V. Chvátal and P. Erdős, *Biased positional games*, Annals of Discrete Math. 2 (1978), 221–228.

[8] D. Clemens, A. Ferber, R. Glebov, D. Hefetz and A. Liebenau, *Building spanning trees quickly in Maker-Breaker games*, SIAM Journal on Discrete Mathematics 29 (2015), 1683–1705.

[9] A. Ferber and D. Hefetz. *Weak and strong k-connectivity games*, European Journal of Combinatorics 35 (2014), 169–183.

[10] H. Gebauer and T. Szabó, *Asymptotic Random Graph Intuition for the Biased Connectivity Game*, Random Structures & Algorithms 35 (2009), 431–443.

[11] Y. O. Hamidoune and M. Las Vergnas. *A solution to the box game*. Discrete Mathematics 65 (1987), 157–171.

[12] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, *Fast winning strategies in Maker-Breaker games*, Journal of Combinatorial Theory Series B 99 (2009), 39–47.

[13] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, **Positional Games (Oberwolfach Seminars)**, Birkhäuser, 2014.

[14] P. Hein. *Vil de laere Polygon*, article in Politiken newspaper 26 (1942).

[15] S. Janson, T. Łuczak and A. Ruciński. **Random graphs**, Wiley, 2000.

[16] J. Komlós and E. Szemerédi, *Limit distribution for the existence of Hamiltonian cycles in a random graph*, Discrete Mathematics 43 (1983), 55–63.

[17] M. Krivelevich, *The critical bias for the Hamiltonicity game is* $(1 + o(1))n/\ln n$, Journal of the American Mathematical Society 24 (2011), 125–131.

[18] H. W. Kuhn, ed. **The Essential John Nash**, Princeton University Press, 2002.

[19] A. Lehman. *A solution of the Shannon switching game*, Journal of the Society for Industrial and Applied Mathematics, 12 (1964), 687–725.

[20] Y. Peres, O. Schramm, S. Sheffield and D. B. Wilson, *Random-turn Hex and other selection games*, American Mathematical Monthly 114 (2007), 373–387.

[21] D. B. West, **Introduction to Graph Theory**, Prentice Hall, 2001.